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Positive Integer Solutions of a Class of Number-Theoretic Functional Equations Involving Euler's Totient Function

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Abstract—*This paper investigates the positive integer solutions of a number-theoretic functional equation involving Euler's totient function* $n_{\varphi(n)}$ *and the Smarandache LCM function* $SL(n)$*, namely* $a\varphi\left(\frac{n(n+1)}{2}\right)=\varphi(SL(n^3)), a,n\in Z^+$*.Using fundamental properties of Euler's totient function and the Smarandache LCM function, combined with techniques from elementary number theory such as case analysis, divisibility arguments, and valuation methods, the structure of the solutions is systematically discussed. The results show that the solutions are closely related to the existence of Fermat primes. Within the range of known Fermat primes, the equation has exactly seven positive integer solutions:* $(a,n)=(1,1),(2,2),(9,3),(8,4),(32,16),(512,256),(131072,65536)$

Keywords— *Euler's totient function; Smarandache LCM function; number-theoretic functional equation; Fermat prime; positive integer solution.*

I. Introduction

In recent years, scholars at home and abroad have carried out extensive research on composite equations involving the Euler function and the Smarandache function. Zhang Sibao et al. [4] discussed the positive integer solutions of a class of number-theoretic function equations related to hexagonal numbers; Gao Li [5] studied the solvability of the equation $k\varphi_2(n(n+1))+\varphi(n)=2S(SL(n^{11}))$; Zhu Shanshan et al. [6] explored the positive integer solutions of the equation $t\varphi(n)+\varphi_2(n)=S\left(SL(n^k)\right)$; Zhang Sibao and Chen Yingying [7] analyzed the solvability of a class of equations of the form $\gamma\left(\varphi_2(x)\right)^k=S(SL(x^l))$ Jiang Lianxia and Zhang Sibao [8][9] respectively studied the positive integer solution problems of the equation $\varphi_2(n)=2^{\omega(n)}P^{\Omega(n)}$ and the composite Euler function equation $\varphi(\varphi(n))=2^{\omega(n)}3^{\omega(n)}$.The above work provides an important reference for the research of this paper.

On the basis of previous work, this paper studies the equation $a\varphi\left(\frac{n(n+1)}{2}\right)=\varphi(SL(n^3)), a,n\in Z^+$. The innovation of this paper lies in: for the first time, it composes the Euler function of with the triangular number, systematically analyzes the structure of its solutions, and reveals its profound connection with Fermat primes.

II. Preliminary Knowledge

Lemma 1[4] When $n>2$, we have $2\mid\varphi(n)$.

Lemma 2[5] For any positive integer n, when $n\geqslant 2$, we have $\varphi(n)<n$.

Lemma 3[6] If the positive integer $n=p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}$, where $p_1<p_2<\cdots<p_k$ is a prime, then

$$\varphi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$$

$$SL(n)=max\left\{p_1^{l_1},p_2^{l_2}\cdots,p_k^{l_k}\right\}.$$

Lemma 4[6] For any positive integer n, there must be $\varphi(n)\geqslant\sqrt{\frac{n}{2}}$.

Lemma 5[6] For any positive integer m and n, we have

$$\varphi(mn)=\frac{\gcd(m,n)\varphi(m)\varphi(n)}{\varphi(\gcd(m,n))}=\varphi(nm).$$

In particular, $gcd(m,n)=1$时，有

$$\varphi(mn)=\varphi(m)\varphi(n).$$

Lemma 6[0] For any prime p, we have $\varphi(p)=p-1$.

Lemma 7[0] If p is a Fermat prime $F_t=2^{2^t}+1$, we have $\varphi(p)=p-1=2^{2^t}$.

III. Main Results

Theorem: The number-theoretic function equation

$$a\varphi\left(\frac{n(n+1)}{2}\right)=\varphi(SL(n^3)),a,n\in Z^+ \quad (1)$$

All positive integer solutions of:

$(a,n)=$
$(1,1),(2,2),(9,3),(8,4),(32,16),(512,256),(131072,65536).$

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If a larger Fermat prime exists $F_t = 2^{2^t}+1,(t\geqslant 5)$, then the corresponding solution is $(a,n)=(2^{2^t+1},2^{2^t})$。

IV. Proof of the Theorem

4.1 Verification of Small Values

When $n=1$, substituting into the equation gives: $\varphi(1)=1, SL(1)=1, \varphi(1)=1$, the equation (1) becomes $a\cdot 1=1$, so $a=1$, yielding the solution $(a,n)=(1,1)$.

When $n=2$, substituting into the equation gives: $\varphi(3)=2, SL(2^3)=2^3, \varphi(8)=4$ the equation (1) becomes $2a=4\Rightarrow a=2$, yielding the solution $(a,n)=(2,2)$.

In the following, assume $n>2$。

4.2 When n is a prime

assume $n=p$, By Lemma 3, we obtain $SL(p^3)=p^3$, Therefore

$$\varphi(SL(p^3))=p^2(p-1).$$

Since $gcd\left(p,\frac{p+1}{2}\right)=1$, by Lemma 5 we have

$$\varphi\left(\frac{p(p+1)}{2}\right)=\varphi(p)\varphi\left(\frac{p+1}{2}\right)=(p-1)\varphi\left(\frac{p+1}{2}\right).$$

Substituting into equation (1), we obtain:

$$a(p-1)\varphi\left(\frac{p+1}{2}\right)=p^2(p-1)\Rightarrow a\varphi\left(\frac{p+1}{2}\right)=p^2. \quad (2)$$

Since a is a positive integer, $\varphi\left(\frac{p+1}{2}\right)$ is a positive divisor of p^2, and its possible values are $1, p, p^2$.

①

If $\varphi\left(\frac{p+1}{2}\right)=p^2$, hen by Lemma 2 we obtain

$$\varphi\left(\frac{p+1}{2}\right)=p^2<\frac{p+1}{2}\Rightarrow 2p^2-p-1<0\Rightarrow p<1.$$

A contradiction.

②

If $\varphi\left(\frac{p+1}{2}\right)=p$, Similarly, we obtain

$$\varphi\left(\frac{p+1}{2}\right)=p<\frac{p+1}{2}\Rightarrow p<1.$$

A contradiction.

③If $\varphi\left(\frac{p+1}{2}\right)=1$, According to the properties of the Euler function，if and only if $\frac{p+1}{2}=1\Rightarrow p=1$（non-prime）。$\frac{p+1}{2}=2\Rightarrow p=3$, Substituting into (2), we obtain $a=9$, yielding the solution $(a,n)=(9,3)$.

4.3 When n is a power of 2

Let $n=2^k\ (k\geqslant 2)$, At this point $n^3=2^{3k}$, Then

$$SL(n^3)=2^{3k},\quad \varphi(SL(2^{3k}))=\varphi(2^{3k})=2^{3k-1}.$$

Also $\frac{n(n+1)}{2}=2^{k-1}(2^k+1)$,

$gcd(2^{k-1},2^k+1)=1$.

From Lemma 3 and Lemma 5, we can obtain

$$\varphi\left(\frac{n(n+1)}{2}\right)=\varphi(2^{k-1})\varphi(2^k+1)=2^{k-2}\varphi(2^k+1).$$

Substituting into equation (1), we obtain：

$$a\cdot 2^{k-2}\varphi(2^k+1)=2^{3k-1}\Rightarrow a\varphi(2^k+1)=2^{2k+1}. \quad (3)$$

Since $a,\varphi(2^k+1),2^{2k+1}$ are all positive integers，Then $\varphi(2^k+1)$ must be a power of 2。By a number-theoretic conclusion[0]：$\varphi(n)$ is a power of 2 if and only if $n=2^k$ or $n=p_1p_2\cdots p_r$, where each p_i is a distinct Fermat prime，Since 2^k+1 is odd，Therefore 2^k+1 must be a Fermat prime，that is $k=2^t$ and 2^k+1 is a prime。The known Fermat primes correspond to $t=0,1,2,3,4$ that is $k=2,4,8,16,(k\geqslant 2)$, By Lemma 7, we obtain:

$$\varphi(2^k+1)=2^k+1-1=2^k$$

Substituting into (3), we obtain:

$$a\cdot 2^k=2^{2k+1}\Rightarrow a=2^{k+1}.$$

We obtain the solution:

$(a,n)=(2^{k+1},2^k)=$

$(8,4),(32,16),(512,256),(131072,65536).$

4.4 n Containing odd prime factors and being neither a prime nor a power of 2

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Let the standard prime factorization of n be $n = p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k} = p^l \cdot m$,where $p_1 < p_2 < \cdots < p_k$ is a prime.Let p denote the largest prime factor,and let $p^l \square n$,Let $n = p^l \cdot m$,where $gcd(p^l, m) = 1$ .since n is neither a prime nor a power of 2,At least one of the following two cases holds: $l \geq 2$,or m contains a prime factor different from p $(m > 1)$.

4.4.1 Calculate $\varphi(SL(n^3))$

By Lemma 3, we can obtain :

$\varphi(SL(n^3)) = \varphi(p^{3l}) = p^{3l-1}(p-1)$ （Denote it as R）

4.4.2 Calculate $\varphi\left(\frac{n(n+1)}{2}\right)$ the exponent of the prime factor in p

(1) When n is odd

n this case, $n+1$ is even.let $n+1 = 2^{\alpha} u$, where $\alpha \geqslant 1$, u is odd, Therefore

$$\frac{n(n+1)}{2} = \frac{p^l m \cdot 2^{\alpha} u}{2} = p^l m \cdot 2^{\alpha-1} u.$$

and $gcd(p^l, u) = 1$, $gcd(2^{\alpha-1}, p^l mu) = 1$, By Lemma 5, we obtain

$$L = \varphi\left(\frac{n(n+1)}{2}\right) = \varphi(p^l m \cdot 2^{\alpha-1} u) = \varphi(p^l)\varphi(m)\varphi(2^{\alpha-1}$$

Therefore:

$$L = \varphi(p^l)\varphi(m)\varphi(2^{\alpha-1}u) = p^{l-1}(p-1)\varphi(m)\cdot\varphi(2^{\alpha-1})\varphi$$

Let the part of L that is coprime to p be denoted as C : $C = \varphi($

Then: $L = p^{l-1}(p-1)\cdot C$

Let $v_p(\cdot)$ denote the $p-$adic valuation [3], then

$v_p(L) = l - 1$.

(2) When n is even

n this case, $n+1$ is odd.let $n = 2^{\beta} v$, where $\beta \geqslant 1$, v is odd and $p \mid v$, Since p is the largest prime factor and is odd, can be written as $v = p^l m'$, where m' is odd and $gcd(m', p) = 1$. Then

$$\frac{n(n+1)}{2} = \frac{2^{\beta} u \cdot (n+1)}{2} = 2^{\beta-1} u \cdot (n+1)\cdot$$

and $gcd(2^{\beta-1}, v(n+1)) = 1$, $gcd(v, n+1) = 1$. Then

$$L = \varphi\left(\frac{n(n+1)}{2}\right) = \varphi(2^{\beta-1})\varphi(v)\varphi(n+1).$$

and $\varphi(v) = \varphi(p^l)\cdot\varphi(m') = p^{l-1}(p-1)\varphi(m').$

Therefore $L = 2^{\beta-2}\cdot p^{l-1}(p-1)\cdot\varphi(m')\cdot\varphi(n+1).$

Let the part of L that is coprime to p be denoted as C' :

Then $L = p^{l-1}(p-1)\cdot C'$.

Therefore $v_p(L) = l - 1$.

In summary: regardless of n whether is odd or even, as long as n is the largest prime factor of n , we have

$$v_p(L) = l - 1. \quad (7)$$

4.4.3 Deriving the relationship between a *and* p, l *from the equation* (4)

Substituting (4) and (5)(6) into equation (1), we obtain $a \cdot p^{l-1}(p-1)\cdot C = p^{3l-1}(p-1)$.

where C is a positive integer coprime to p (in the odd case it is C , and in the even case it is C'), then

$$a \cdot C = p^{2l}.$$

since $gcd(C, p) = 1$, there must be $C = 1$ and $a = p^{2l}$.

We prove below that $C = 1$ is the only possibility.

From (8), we know $p^{2l} \mid a$, Therefore $a \geq p^{2l}$ .Using Lemma 4 $L = \varphi(\frac{n(n+1)}{2}) \geq \sqrt{\frac{n(n+1)}{4}} \geq \sqrt{\frac{n^2}{4}} = \frac{n}{2} \geq \frac{p^l}{2}.$

At the same time, from (4), we obtain $R = \varphi(SL(n^3)) = p^{3l-1}(p-1)$ .Therefore

$$a = \frac{R}{L} \leqslant \frac{p^{3l-1}(p-1)}{\frac{p^l}{2}} = 2p^{2l-1}(p-1). \quad (9)$$

Combining $a \geq p^{2l}$ we obtain : $p^{2l} \leqslant 2p^{2l-1}(p-1) \Rightarrow p \geqslant 2.$

his holds automatically for odd primes, with no contradiction. Let

$a = p^{2l} \cdot t$, where t is a positive integer. Substituting into (9):

$$p^{2l} \cdot t \leq 2p^{2l-1}(p-1) \Rightarrow t \leq 2 - \frac{2}{p}.$$

since $p \geq 3$, we have

$\frac{2}{p} > 0$, Therefore $t < 2$. Also, since t is a positive integer, we must have

$t = 1$, Combining with (8), we obtain

$$a = p^{2l},\ C = 1. \quad (10)$$

4.4.4 Verifying $C = 1$ *whether holds*

(1) When n is odd

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we have $C=\varphi(m)\cdot\varphi\left(2^{\alpha-1}\right)\varphi(u)=1$, because $m\geq 1,\varphi(m)=1\Rightarrow m=1or2$, but m is odd and is coprime to p, therefore $m=1$. at the same time $\varphi\left(2^{\alpha-1}\right)\varphi(u)=1$,if $\alpha=1$, therefore $2^{\alpha-1}=1,\varphi(u)=1\Rightarrow u=1or2$, but u is odd, therefore $u=1$; if $\alpha\geq 2$, therefore $\varphi(2^{\alpha-1})\geq 2$, a contradiction. therefore $\alpha=1,u=1,n+1=2,n=1$, contradicts $n\geq 2$.

(2) When n is even

we have $C'=2^{\beta-2}\varphi(m')\cdot\varphi(n+1)=1$, because n **is even** and p is an odd prime factor, m' is odd and coprime to p. $C'=1$ requiring $2^{\beta-2}=1\Rightarrow\beta=2$, therefore $\varphi(m')\varphi=1,\varphi(m')=1\Rightarrow m'=1or2$, but m' is odd, therefore $m'=1$; and $\varphi(n+1)=1\Rightarrow n+1=1or2$, but $n\geq 2$, therefore $n+1=2\Rightarrow n=1$, contradiction.

In summary, under the condition that n contains an odd prime factor and is neither a prime nor a power of 2, there are no positive integer solutions satisfying the equation.

V. Conclusion

This paper completely solves all positive integer solutions of the equation $a\varphi\left(\frac{n(n+1)}{2}\right)=\varphi\left(SL(n^3)\right),a,n\in Z^+$ and proves that its solutions are only related to Fermat primes. The method in this paper can provide a reference for the study of other composite number-theoretic function equations.

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