

Influence of Damping on the Vibration of an Euler–Bernoulli Beam Due to Two Moving Concentrated Loads

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Abstract— This study examines the dynamic response of a damped beam subjected to two moving loads. The beam is modelled as a uniform and simply supported member at both ends, while the moving loads travel at a constant velocity. The governing fourth-order partial differential equation describing the beam's motion is initially reduced to a series of second-order ordinary differential equations using a finite Fourier transformation—a well-established analytical technique in structural mechanics. Subsequently, the reduced second-order equation is simplified using the Laplace transform technique, combined with convolution theory, to resolve problems involving travelling forces. Results presented in graphical form reveal the effects of the foundation modulus F_{st} , internal damping r_i , external damping r_e , the distance apart of forces M_a and M_b , x_0 , the first force M_a and the second force M_b . The study further shows that increases in the foundation modulus F_{st} lead to higher critical speeds, thereby reducing the likelihood of resonance and improving the structural safety and reliability.

Keywords— Vibration, internal damping, external damping, critical speed, resonance, foundation modulus, concentrated loads, Euler–Bernoulli beam.

I. INTRODUCTION

Structural beam elements are characterized by a longitudinal dimension that significantly exceeds their cross-sectional dimensions and by their ability to resist external loads. Besides the cross-sectional shape and length, the beam's mechanical performance is dictated by boundary conditions, damping, equilibrium conditions, and constitutive material properties. Consequently, these models are extensively used in civil engineering, transportation systems, and industry [1–4].

The supports are recognized as Winkler [11], viscoelastic [12], or Pasternak-viscoelastic foundations [13], while elastic structures are often represented as Euler-Bernoulli [5-8] or Timoshenko beams [9,10]. Researchers often analyze these systems in terms of load types: constant concentrated loads [14], harmonic excitations [15], and moving vehicles [16].

Problems on dynamic responses of beam structures under moving loads can be solved analytically and numerically. Saito and Terasawa [17] used the Fourier transform method to investigate the two-dimensional problem with a single distributed load on an infinitely long beam with the Pasternak-type foundation. Sun [6,18] extended the work and presented the inverse Fourier transform method to investigate an infinitely long beam structure under a more complex situation with harmonic line loads and a Winkler foundation. Kim and Roeset [19] carried out a dynamic analysis of an infinitely long beam under moving loads and analyzed the effects of high moving velocities and frequencies on the beam. In their study, the inverse Fourier transform method was used

Yu and Yuan [20] utilized the Laplace transform to enhance the Fourier transform method for investigating the dynamic vibration of a long beam on a viscoelastic foundation under a dynamic load. Froio et al. [21]. obtained an analytical steady-state solution of a two-dimensional infinite beam resting on a Pasternak elastic foundation subjected to a single moving load.

Furthermore, Yang et al. [22] integrated the Energy method and the Fourier sine series to formulate the ordinary differential equation of motion on an inclined bridge. They obtained the semi-analytical solution of the dynamic system. Mamaghani et al. [23] developed an exact mathematical model for the forced vibration of Euler–Bernoulli beams whose axial material parameters vary in gradient. Li et al. [24] introduced a semi-analytical method to determine the dynamical behavior of cantilever plates subjected to discrete harmonic excitations.

In some cases, most studies treat concentrated loads as independent, isolated phenomena. In the real world, structural systems are commonly subjected to simultaneous combinations of these load types. Common cases involve multi-axle vehicular loading on bridges, rail–wheel interactions where localized wheel forces act in tandem with distributed train body loads, and industrial conveyor systems carrying both bulk materials and localized mechanical equipment. Also, in reality, damping effects always exist in any dynamic system [25].

Moreover, most existing literature overlooks the simultaneous coupling between moving loads, flexible foundation stiffness, damping, and axial force. In practice, the response of a structural element under moving loads is primarily determined by load characteristics, boundary conditions, and foundation stiffness.

In this study, shortcomings in existing studies are addressed by investigating the influence of damping on an Euler–Bernoulli beam under two moving concentrated loads. The governing fourth-order partial differential equation describing the beam's motion is initially reduced to a series of second-order ordinary differential equations using a finite Fourier Sine Transform.

Nomenclatures

TABLE 1: Nomenclatures (standardized formatting)

Symbol	Description
$h(x, t)$ (m)	Transverse deflection of the beam
$\delta(x - ct)$	The Dirac delta function
$f(t)$ (m)	Instantaneous positions along the beam
M (kg)	Mass of the moving load
c (m/s)	uniform velocity
g (m/s ²)	Acceleration due to gravity
x (m)	Spatial coordinate along the beam
L (m)	Beam span
t (s)	Time coordinate
E (N/m ²)	Young's modulus
I (m ⁴)	Moment of inertia of the beam cross-section
μ (kg/m)	Mass per unit length of the beam
r_e (Ns/m ²)	coefficient of external damping of the beam
r_i (Ns/m ²)	coefficient of internal damping of the beam
F_{st} (N/m ³)	Foundation modulus (Winkler stiffness)
$N(x)$ (N)	Variable prestressed
$M_{Li}(x, t)$ (N)	Transverse moving loads
M_a (N)	is the first force
M_b (N)	is the second force
$P(x, t)$	Restoring force from the elastic foundation
x_0 (m)	Distance apart of forces M_a and M_b

II. MATHEMATICAL FORMULATION

Consider a slender beam subjected to two moving loads. The beam has a finite span, and the load is assumed to traverse along it with a constant velocity. Taking into account the effects of damping, the transverse displacement of the beam, expressed as a function of time t and the spatial coordinate x , is governed by the following fourth-order partial differential equation [26]:

$$EI \frac{\partial^4 h(x, t)}{\partial x^4} + \mu \frac{\partial^2 h(x, t)}{\partial t^2} = M_{Li}(x, t) - P(x, t) \quad (1)$$

Where EI is the rigidity of the beam, E is the Young's modulus of elasticity, $h(x, t)$ is the deflection of the structure, I is the moment of Inertia of the beam, c is the constant speed, x_0 is the distance apart of forces M_a and M_b , and μ is the mass per unit length of the beam. $M_{Li}(x, t)$ and $P(x, t)$ are the applied external loads and the re-establishing force from the foundation, respectively.

The Winkler foundation model assumes that the reactive pressure is proportional to the deflection of the beam, that is, $P(x, t) = F_{st}h(x, t)$. Here, F_{st} is called the modulus of subgrade reaction, and a constant F_{st} implies linear elasticity of the subgrade.

In reality, damping effects always exist in any dynamic system. If subgrade damping is considered, the restoring force becomes

$$P(x, t) = F_{st}h(x, t) + r_e \frac{\partial h(x, t)}{\partial t} + r_i \frac{\partial^5 h(x, t)}{\partial t \partial x^4} \quad (2)$$

r_e and r_i are the coefficients of external damping and internal damping of the beam, respectively. The external and internal damping of the beam are assumed to be proportional to the mass and stiffness of the beam. That is $r_e = A_1\mu$ and $r_i = A_2EI$ where A_1 and A_2 are proportionality constants. Substitution of the restoring force, equation (1a), into equation (1) gives

$$EI \frac{\partial^4 h(x, t)}{\partial x^4} + \mu \frac{\partial^2 h(x, t)}{\partial t^2} + F_{st}h(x, t) + r_e \frac{\partial h(x, t)}{\partial t} + r_i \frac{\partial^5 h(x, t)}{\partial t \partial x^4} = M_{Li}(x, t) \quad (3)$$

$$0 \leq t \leq \frac{L-d}{v}$$

2.1 End Conditions

A beam with Simply Supported Boundary conditions and Initial conditions is expressed as

$$h(x, t) = 0, \quad \frac{\partial^2 h(x, t)}{\partial t^2} = 0 \quad \text{at } x = 0 \text{ and } x = L$$

and

$$h(x, t)_{t=0} \text{ and } \frac{\partial h(x, t)}{\partial t}_{t=0} \quad (4)$$

2.2 Moving Load Representation

Consider a flexible Euler–Bernoulli beam of span (L) under two moving loads that traverse the structural domain from one end of the beam to the other. The combined moving loads on the structure at any spatial coordinate (x) and time (t) can generally be given as the superposition of the individual load contributions. Therefore, the resultant moving-load function is expressed by

$$M_{Li}(x, t) = M_a(x, t) + M_b(x, t) \quad (5)$$

Where $M_a(x, t)$ and $M_b(x, t)$ represent the spatial and temporal illustrations of the first and second moving loads, respectively.

For concentrated moving loads, the individual load functions are represented using the Dirac delta distribution as $M_a(x, t) = M_a\delta[x - f_a(t)]$ and $M_b(x, t) = M_b\delta[x - f_b(t)]$

(6)

Where $f_a(t)$ and $f_b(t)$ describe their instantaneous positions along the beam. The Dirac delta function ensures that each concentrated load is applied at its corresponding instantaneous location on the structural domain.

The general kinematic relation may initially be described by the trajectories of the two loads

$$f_a(t) = x_0 + ct + \frac{1}{2}at^2 \text{ and } f_b(t) = x_0 + ct + \frac{1}{2}at^2 \quad (7)$$

c represents their prescribed initial velocity, and a is the acceleration of the moving loads. In this formulation, both loads are assumed to travel along the beam with a constant velocity. Consequently, the acceleration a is zero. The initial position of the first load is taken as $x_0=0$, while the second load is initially located a finite distance x_0 from the first load.

Hence, the resultant load distribution becomes

$$M_{Li}(x, t) = M_a\delta(x - ct) + M_b\delta[x - (x_0 + ct)] \quad (8)$$

x_0 is therefore the distance apart of the loads M_{L1} and M_{L2} . Equation (8) indicates that both loads propagate along the beam with the same constant translational velocity (c), thereby maintaining a constant spatial separation throughout their motion.

III. METHOD OF SOLUTION

To solve equation (3), we use a finite Fourier transform called the finite sine transform, defined by

$$g(j, t) = \int_0^L h(x, t) \sin \frac{j\pi x}{L} dx \quad (9)$$

With the inverse

$$h(x, t) = \frac{2}{L} \sum_{j=1}^{\infty} g(j, t) \sin \frac{j\pi x}{L} \quad (10)$$

Applying the finite Fourier transform method, the solution to equation (6) takes the form

$$\begin{aligned} & \frac{2}{L} \sum_{j=1}^{\infty} \left[EI \left(\frac{j\pi}{L} \right)^4 g(j, t) + F_{st} g(j, t) + r_e \dot{g}(j, t) \right. \\ & \quad \left. + r_l \left(\frac{j\pi}{L} \right)^4 \dot{g}(j, t) + \mu \ddot{g}(j, t) \right] \sin \frac{j\pi x}{L} \\ & = M_a \delta(x - ct) + M_b \delta[x - (x_0 + ct)] \end{aligned} \quad (11)$$

Where $g(j, t)$ is the transform of the original $h(x, t)$

Multiplying each term in (11) by $\sin \frac{k\pi x}{L}$ and integrating with respect to x between 0 and L and taking into account the BC's one obtains

$$\begin{aligned} & \int_0^L \sum_{j=1}^{\infty} \left[\left\{ EI \left(\frac{j\pi}{L} \right)^4 + F_{st} \right\} g(j, t) + \left(r_e + r_l \left(\frac{j\pi}{L} \right)^4 \right) \dot{g}(j, t) \right. \\ & \quad \left. + \mu \ddot{g}(j, t) \right] \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L} \\ & = \int_0^L M_a \delta(x - ct) + M_b \delta[x - (x_0 + ct)] \sin \frac{k\pi x}{L} \end{aligned} \quad (12)$$

Equation (12), after some rearrangements and simplifications, yields

$$\begin{aligned} & a_1(j, k) \ddot{g}(j, t) + a_2(j, k) \dot{g}(j, t) + a_3(j, k) g(j, t) = \\ & M_a \sin \frac{k\pi ct}{L} + M_b \sin \frac{k\pi(ct+d)}{L} \end{aligned} \quad (13)$$

where

$$\begin{aligned} & a_1(j, k) = \mu \int_0^L \sin \frac{k\pi x}{L} \\ & a_2(j, k) = \left(r_e + r_l \left(\frac{j\pi}{L} \right)^4 \right) \int \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L} dx \\ & a_3(j, k) = \left(EI \left(\frac{j\pi}{L} \right)^4 + F_{st} \right) \int \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L} dx \end{aligned} \quad (14)$$

Equation (13) is a second-order ODE with constant coefficients

In what follows, we subject the system of ODE (13) to a Laplace transform defined as [In conjunction with the initial condition]

$$\mathcal{L}[a_1 \ddot{g}(j, t) + a_2 \dot{g}(j, t) + a_3 g(j, t)] = \mathcal{L} \left[M_a \sin \frac{k\pi ct}{L} + M_b \sin k\pi \frac{(ct+d)}{L} \right] \quad (15)$$

Further evaluation, we have

$$\begin{aligned} & [a_1 s^2 + a_2 s + a_3] g(i, s) = (M_a + M_b \cos \beta) \frac{\alpha_1}{s^2 + \alpha_1^2} + \\ & M_b \sin \beta \frac{s}{s^2 + \alpha_1^2} \end{aligned} \quad (16)$$

The above can be resolved, and we have

$$\begin{aligned} & g(i, s) = P_k \left\{ M_a \left(\frac{1}{d_1} \frac{d_1}{(s-d_1)} \frac{\alpha_1}{s^2 + \alpha_1^2} - \frac{1}{d_2} \frac{d_2}{(s-d_2)} \frac{\alpha_1}{s^2 + \alpha_1^2} \right) \right. \\ & \quad + M_b \cos \beta \left(\frac{1}{d_1} \frac{d_1}{(s-d_1)} \frac{\alpha_1}{s^2 + \alpha_1^2} - \frac{1}{d_2} \frac{d_2}{(s-d_2)} \frac{\alpha_1}{s^2 + \alpha_1^2} \right) \\ & \quad \left. + M_b \sin \beta \left(\frac{1}{d_1} \frac{d_1}{(s-d_1)} \frac{s}{s^2 + \alpha_1^2} - \frac{1}{d_2} \frac{d_2}{(s-d_2)} \frac{s}{s^2 + \alpha_1^2} \right) \right\} \end{aligned} \quad (17)$$

Where

$$\begin{aligned} & \alpha_1 = \frac{k\pi c}{L}, \beta = \frac{k\pi x_0}{L}, P_k = \frac{1}{d_1 - d_2} \\ & d_1 = \frac{-a_2 + \sqrt{a_2^2 - 4a_1 a_3}}{2a_1} \text{ and } d_2 = \frac{-a_2 - \sqrt{a_2^2 - 4a_1 a_3}}{2a_1} \end{aligned} \quad (18)$$

So that the Laplace inversion of each term of the RHS (18) gives

$$\begin{aligned} & g_i(t) = P_k M_a \left[\frac{1}{d_1} \int_0^t e^{d_1(t-u)} \sin \alpha_1 u \, du - \right. \\ & \quad \left. \frac{1}{d_2} \int_0^t e^{d_2(t-u)} \sin \alpha_1 u \, du \right] \\ & + P_k M_b \cos \beta \left[\frac{1}{d_1} \int_0^t e^{d_1(t-u)} \sin \alpha_1 u \, du - \right. \\ & \quad \left. \frac{1}{d_2} \int_0^t e^{d_2(t-u)} \sin \alpha_1 u \, du \right] \end{aligned}$$

$$\begin{aligned} & + P_k M_b \sin \beta \left(\frac{1}{d_1} \int_0^t e^{d_1(t-u)} \cos \alpha_1 u \, du - \right. \\ & \quad \left. \frac{1}{d_2} \int_0^t e^{d_2(t-u)} \cos \alpha_1 u \, du \right) \end{aligned} \quad (19)$$

Further simplification of equation (19) gives

$$\begin{aligned} & g_i(t) = P_k \left\{ \frac{(M_a + M_b \cos \beta)}{d_1} e^{d_1 t} I_1 - \frac{(M_a + M_b \cos \beta)}{d_2} e^{d_2 t} I_3 \right\} \\ & + P_k \left\{ \frac{M_b \sin \beta}{d_1} e^{d_1 t} I_2 - \frac{M_b \sin \beta}{d_2} e^{d_2 t} I_4 \right\} \end{aligned} \quad (20)$$

Where

$$\begin{aligned} & I_1 = \frac{-\alpha_1 e^{-d_1 t} \cos \alpha_1 t + \alpha_1 - d_1 e^{-d_1 t} \sin \alpha_1 t}{d_1^2 + \alpha_1^2}, I_2 = \\ & \frac{-\alpha_1 e^{-d_1 t} \sin \alpha_1 t + d_1 - d_1 e^{-d_1 t} \cos \alpha_1 t}{d_1^2 + \alpha_1^2} \\ & I_3 = \frac{-\alpha_1 e^{-d_2 t} \cos \alpha_1 t + \alpha_1 - d_1 e^{-d_2 t} \sin \alpha_1 t}{d_2^2 + \alpha_1^2}, I_4 = \\ & \frac{-\alpha_1 e^{-d_2 t} \sin \alpha_1 t + d_2 - d_2 e^{-d_2 t} \cos \alpha_1 t}{d_2^2 + \alpha_1^2} \end{aligned} \quad (21)$$

Subjecting (20) to some rearrangements, we obtain

$$\begin{aligned} & g_i(t) = P_k (M_a + M_b \cos \beta) \frac{e^{d_1 t}}{d_1 (d_1^2 + \alpha_1^2)} \\ & * \left((\alpha_1 - \alpha_1 e^{-d_1 t} \cos \alpha_1 t - d_1 e^{-d_1 t} \sin \alpha_1 t) \right) \\ & + P_k M_b \sin \beta \frac{e^{d_1 t}}{d_1 (d_1^2 + \alpha_1^2)} (d_1 - \alpha_1 e^{-d_1 t} \sin \alpha_1 t - \\ & d_1 e^{-d_1 t} \cos \alpha_1 t) \\ & - P_k \frac{e^{d_2 t}}{d_2 (d_2^2 + \alpha_1^2)} (M_a + M_b \cos \beta) * (\alpha_1 - \\ & \alpha_1 e^{-d_2 t} \cos \alpha_1 t - d_2 e^{-d_2 t} \sin \alpha_1 t) \\ & - P_k M_b \sin \beta \frac{e^{d_2 t}}{d_2 (d_2^2 + \alpha_1^2)} (d_2 - \alpha_1 e^{-d_2 t} \sin \alpha_1 t - \\ & d_2 e^{-d_2 t} \cos \alpha_1 t) \end{aligned} \quad (22)$$

3.1 Analytical solutions

Substituting (22) into (9) gives the following transverse displacement of the beam

i. Transverse displacement of the elastic structure with damping effects under two Moving loads with different magnitudes

$$\begin{aligned} & h(x, t) = \frac{2}{L} \sum_{j=1}^{\infty} \left\{ P_k (M_a \right. \\ & \quad \left. + M_b \cos \beta) \frac{e^{d_1 t}}{d_1 (d_1^2 + \alpha_1^2)} \right. \\ & \quad * \left((\alpha_1 - \alpha_1 e^{-d_1 t} \cos \alpha_1 t \right. \\ & \quad \left. - d_1 e^{-d_1 t} \sin \alpha_1 t) \right) \\ & \quad + P_k M_b \sin \beta \frac{e^{d_1 t}}{d_1 (d_1^2 + \alpha_1^2)} * (d_1 - \alpha_1 e^{-d_1 t} \sin \alpha_1 t - \\ & d_1 e^{-d_1 t} \cos \alpha_1 t) \\ & \quad - P_k \frac{e^{d_2 t}}{d_2 (d_2^2 + \alpha_1^2)} (M_a + M_b \cos \beta) * (\alpha_1 - \\ & \alpha_1 e^{-d_2 t} \cos \alpha_1 t - d_2 e^{-d_2 t} \sin \alpha_1 t) \\ & \quad - P_k \frac{e^{d_2 t}}{d_2 (d_2^2 + \alpha_1^2)} M_b \sin \beta * (d_2 - \alpha_1 e^{-d_2 t} \sin \alpha_1 t - \\ & d_2 e^{-d_2 t} \cos \alpha_1 t) \left. \right\} \sin \frac{j\pi x}{L} \end{aligned} \quad (23)$$

The expression (23) is the transverse displacement of the elastic structure with damping effects under two concentrated loads

ii. Transverse displacement of the elastic structure with damping effects under two Moving loads when the initial position, x_0 of the second load is zero

Thus, transverse displacement, $h(x, t)$ becomes

$$h(x, t) = \frac{4M_a P_k}{L} \sum_{j=1}^3 \left\{ \frac{e^{d_1 t}}{d_1 (d_1^2 + \alpha_1^2)} * \left((\alpha_1 - \alpha_1 e^{-d_1 t} \cos \alpha_1 t - d_1 e^{-d_1 t} \sin \alpha_1 t) \right) - P_k \frac{e^{d_2 t}}{d_2 (d_2^2 + \alpha_1^2)} * (\alpha_1 - \alpha_1 e^{-d_2 t} \cos \alpha_1 t - d_2 e^{-d_2 t} \sin \alpha_1 t) \right\} \sin \frac{\pi x}{L} \quad (24)$$

iii. Transverse displacement of the elastic structure with damping effects under two Moving loads with the same magnitudes

$$h(x, t) = \frac{2M_a P_k}{L} \sum_{j=1}^3 \left\{ \frac{e^{d_1 t}}{d_1 (d_1^2 + \alpha_1^2)} (1 + \cos \beta) * \left((\alpha_1 - \alpha_1 e^{-d_1 t} \cos \alpha_1 t - d_1 e^{-d_1 t} \sin \alpha_1 t) \right) + \sin \beta \frac{e^{d_1 t}}{d_1 (d_1^2 + \alpha_1^2)} * (d_1 - \alpha_1 e^{-d_1 t} \sin \alpha_1 t - d_1 e^{-d_1 t} \cos \alpha_1 t) - \frac{e^{d_2 t}}{d_2 (d_2^2 + \alpha_1^2)} (1 + \cos \beta) * (\alpha_1 - \alpha_1 e^{-d_2 t} \cos \alpha_1 t - d_2 e^{-d_2 t} \sin \alpha_1 t) - \frac{e^{d_2 t}}{d_2 (d_2^2 + \alpha_1^2)} \sin \beta * (d_2 - \alpha_1 e^{-d_2 t} \sin \alpha_1 t - d_2 e^{-d_2 t} \cos \alpha_1 t) \right\} \sin \frac{\pi x}{L} \quad (25)$$

The expression (25) is the transverse displacement of the elastic structure with damping effects under two concentrated loads

iv. Transverse displacement of the elastic structure with damping effects under two Moving loads with the same magnitude when the initial position, x_0 of the second load is zero

Thus, transverse displacement, $h(x, t)$ becomes

$$h(x, t) = \frac{4P_k M_a}{L} \sum_{j=1}^3 \left\{ \frac{e^{d_1 t}}{d_1 (d_1^2 + \alpha_1^2)} * \left((\alpha_1 - \alpha_1 e^{-d_1 t} \cos \alpha_1 t - d_1 e^{-d_1 t} \sin \alpha_1 t) \right) - \frac{e^{d_2 t}}{d_2 (d_2^2 + \alpha_1^2)} (\alpha_1 - \alpha_1 e^{-d_2 t} \cos \alpha_1 t - d_2 e^{-d_2 t} \sin \alpha_1 t) \right\} \sin \frac{\pi x}{L} \quad (26)$$

IV. RESULTS AND DISCUSSION

This part uses numerical examples that feature a homogeneous beam with the material and geometric characteristics stated below to demonstrate the analytical formulations previously presented: modulus of elasticity $E=2.9012 \times 10^9 \text{ N/m}^2$, moment of inertia $I=2.87698 \times 10^{-3} \text{ kg}$ beam span $L=50 \text{ m}$, and mass per unit length $\mu=2758.291 \text{ kg/m}$. The moving load is assumed to travel at a constant velocity $V=8.128 \text{ m/s}$. Parametric variations are introduced for the foundation modulus F_{st} (ranging from 0 N/m^3 to $5 \times 10^5 \text{ N/m}^3$), and the distance apart x_0 (ranging from 0.8 m to 3.2 m)

Figures 1 & 2 show the displacement response of the elastic structure with damping effects under two Moving loads for varying moving loads M_a and M_b at constant velocity, respectively. It is observed that higher values of M_a and M_b magnify the amplitude of vibration.

Figure 3 illustrates the transverse displacement response of a simply supported uniform beam subjected to two moving forces for various values of the subgrade modulus F_{st} . The results indicate that as the foundation modulus increases, the vibration amplitude decreases, demonstrating the stabilizing effect of the foundation stiffness.

Figure 4 presents the effect of external damping (r_e) on the deflection profile of a simply supported uniform beam under the action of two moving forces at constant velocity. The graphs show that the response amplitude decreases as the value of external damping (r_e) increases.

Figure 5 displays the effect of internal damping (r_i) on the deflection profile of a simply supported uniform beam under the action of two moving forces at constant velocity. The graphs show that the response amplitude decreases as the value of internal damping (r_i) increases.

Finally, Figure 6 illustrates the deflection profile for different distances between the loads, denoted as x_0 . As the distance between the two loads, x_0 , increases, the deflection profile of the beam decreases. This indicates that greater separation between the two moving loads reduces the beam's dynamic vibration response.

Overall, the numerical results confirm that structural parameters such as foundation stiffness, the distance between the two loads, and damping significantly affect the dynamic behavior of beams under moving loads.

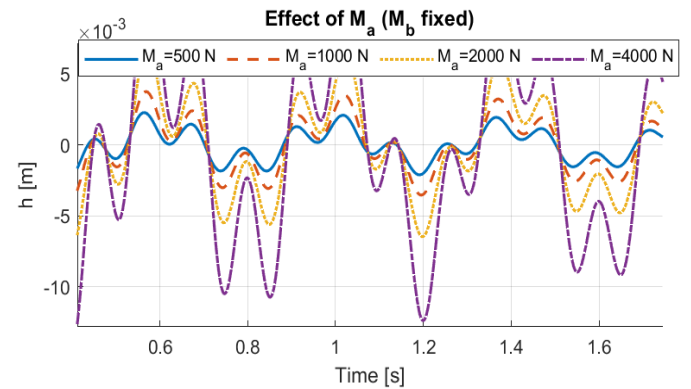


Figure 1: Displacement Profile of elastic Euler-Bernoulli beam with damping effects, for various values of Load M_a and M_b when the beam length $L = 50$ and for fixed values of $F_{st} = 10^5 \text{ N/m}^3$, $M_a = 500 \text{ N}$, $M_b = 100 \text{ N}$, $r_i = 10.5$, $r_e = 413$, $x_0 = 0.8 \text{ m}$ and $c = 8.142 \text{ m/s}$

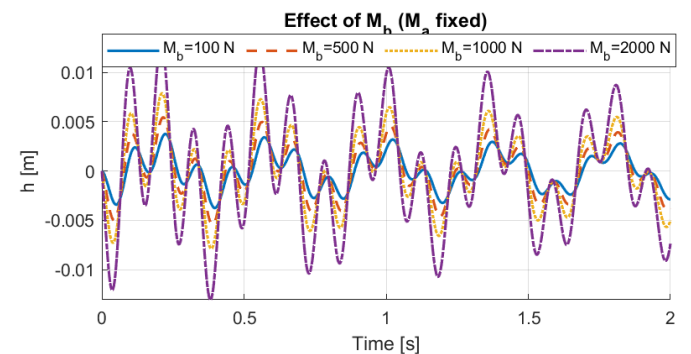


Figure 2: Displacement Profile of elastic Euler-Bernoulli beam with damping effects, for various values of M_b when the beam length $L = 50$ and for fixed values of $F_{st} = 10^5 \text{ N/m}^3$, $M_a = 500 \text{ N}$, $r_i = 10.5$, $r_e = 413$, $x_0 = 0.8 \text{ m}$ and $c = 8.142 \text{ m/s}$

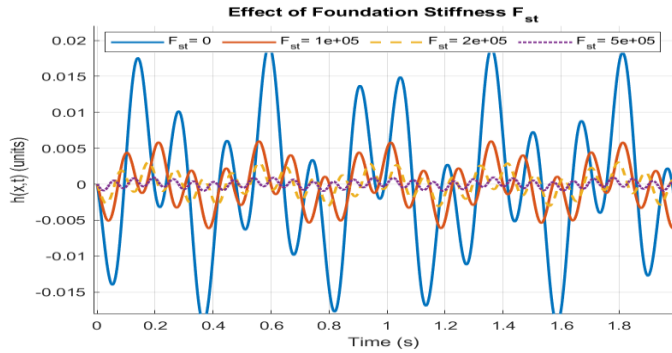


Figure 3: Displacement Profile of elastic Euler-Bernoulli beam with damping effects, for various values of foundation modulus, F_{st} when the beam length $L = 50$ and for fixed values of $M_a = 500N$, $M_b = 100N$, $r_i = 10.5$, $r_e = 413$, $x_0 = 0.8m$ and $c = 8.142 m/s$

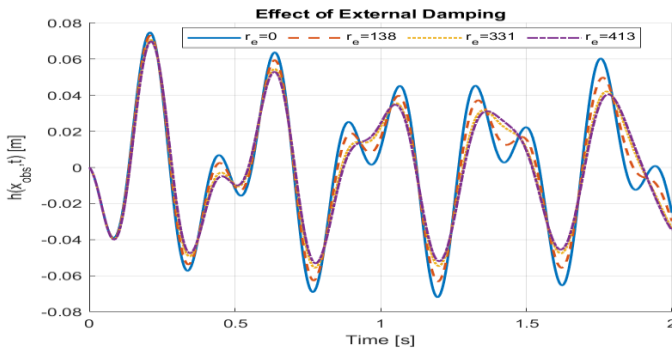


Figure 4: Displacement Profile of elastic Euler-Bernoulli beam with damping effects, for various values of external damping, r_e when the beam length $L = 50$ and for fixed values of $M_a = 500N$, $M_b = 100N$, $r_i = 10.5$, $r_e = 413$, $x_0 = 0.8m$ and $c = 8.142 m/s$

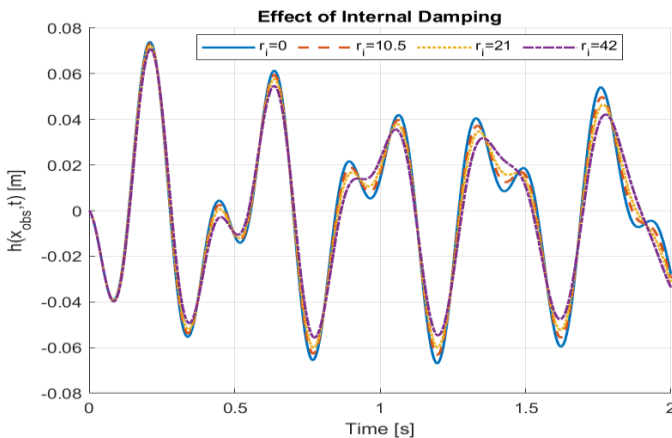


Figure 5: Displacement Profile of elastic Euler-Bernoulli beam with damping effects, for various values of internal damping, r_i when the beam length $L = 50$ and for fixed values of $M_a = 500N$, $M_b = 100N$, $r_i = 10.5$, $r_e = 413$, $x_0 = 0.8m$ and $c = 8.142 m/s$

4.1 System Resonance

The incidence of resonance in a dynamic system is a key issue in design and structural engineering, as it can cause the vibration amplitude of a structural member under moving loads to increase irrepressibly. Therefore, this segment aims to examine and identify the specific conditions under which this unwanted phenomenon may arise. From equation (23), it is evident that the structurally damped system considered in

this study will experience resonance when any of the following conditions are satisfied:

$$\begin{aligned} \text{i.} \quad & d_1^3 - d_2\alpha_1^2 = d_2d_1^2 - d_1\alpha_1^2 \\ \text{ii.} \quad & d_1d_2^2 - d_2\alpha_1^2 = d_2^3 - d_1\alpha_1^2 \end{aligned} \quad (27)$$

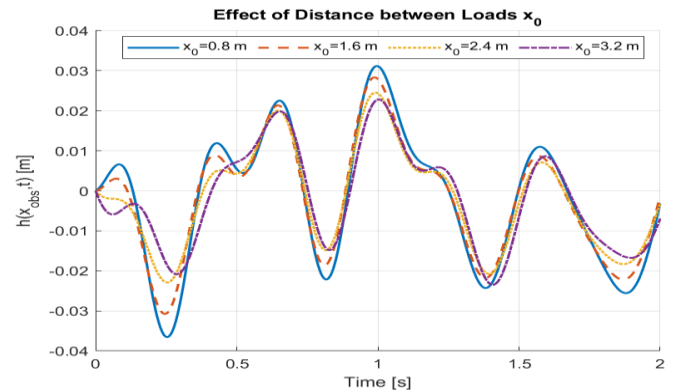


Figure 6: Displacement Profile of elastic Euler-Bernoulli beam with damping effects, for various values of distance, x_0 when the beam length $L = 50$ and for fixed values of $M_a = 500N$, $M_b = 100N$, $r_i = 10.5$, $r_e = 413$ and $c = 8.142 m/s$

and the velocity at which this phenomenon may occur, termed the critical velocity associated with the conditions in (27), is given as

$$c_r = (2a_1)^{-1/2} \frac{L}{k\pi} \sqrt{(a_2 - \sqrt{a_2^2 - 4a_1a_3})} \quad (41)$$

where all the parameters are as previously defined.

V. SUMMARY AND CONCLUSION

This study presents an exact analytical investigation of the dynamic response of a damped beam traversed by two moving concentrated loads travelling at uniform velocity. The structure was modelled as a homogeneous structural member resting on a flexible foundation while incorporating both internal and external damping effects. The main fourth-order partial differential equation was transmuted into a system of second-order ordinary differential equations using the finite Fourier sine transform. Subsequently, the Laplace transform, in conjunction with the convolution theorem, was employed to obtain closed-form analytical solutions for the beam's transverse displacement.

Graphical representations depicted the following:

- An increase in the magnitudes of the first and second moving loads M_a and M_b , respectively, significantly heightened the response amplitude.
- Increasing the Winkler foundation modulus F_{st} significantly diminished the transverse displacement of the Euler-Bernoulli beam by providing greater
- elastic restraint and improving the effective stiffness of the beam-foundation system.
- Increasing the coefficient of external damping r_e gradually reduced the dynamic response through improved energy dissipation, while increasing the coefficient of internal damping r_i suppressed the vibration amplitude by dissipating vibrational energy associated with the beam's flexural deformation.

- e. Furthermore, it was revealed that increasing the distance between the two loads x_0 minimized the dynamic deflection of the beam, showing that an increase in distance between the loads weakens the interaction effects between the moving forces and subsequently reduces the structural response.

The analytical results presented in this work are useful to civil and construction engineers. The results guide the design and assessment of beam-type structures under moving loads, such as railway tracks, highway bridges, and mechanical engineering structures.

The analytical model has certain limitations. The analytical model neglects shear deformation and rotary inertia effects. Investigations may extend the present work by incorporating shear deformation, rotary inertia effects, Timoshenko beam theory, viscoelastic or Pasternak foundation models, nonlinear material behaviors, and moving masses in future studies

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