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A Two-Level Two-Step Picard Method with Coarse-Grid Backtracking Correction for the Stationary Navier-Stokes Equations

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Abstract— *A two-level finite element method is developed for the stationary incompressible Navier-Stokes equations. The method solves the nonlinear problem on a coarse grid, performs two successive Picard linearizations on a fine grid, and returns to the coarse grid for one full linearized residual correction. The correction equation is derived from the nonlinear residual of the second Picard approximation. Under standard approximation, discrete inf-sup, nonsingularity, and adjoint regularity assumptions, stability and a priori error estimates are established. Numerical experiments with the P2-P1 Taylor-Hood element confirm optimal third-order velocity convergence in the L2 norm and second-order convergence in the velocity H1 seminorm and pressure L2 norm. Tests for decreasing viscosity and the lid-driven cavity flow up to Re=2000 show that the proposed method closely reproduces the fully nonlinear fine-grid solution while substantially reducing the computational cost.*

Keywords— *Backtracking correction: finite element method: lid-driven cavity flow: Navier-Stokes equations: Picard iteration: two-level method.*

I. INTRODUCTION

The stationary incompressible Navier Stokes equations form a basic nonlinear model for viscous flow. A direct finite element solution on a fine grid requires the repeated solution of large linearized saddle point systems. Two grid and two level methods reduce this cost by resolving the principal nonlinearity on a relatively small coarse space and transferring only linear problems to the fine space [1], [3]–[7].

Layton and Tobiska [1] proposed a backtracking method consisting of one nonlinear coarse grid solve, one fine grid Oseen solve, and one full linearized correction on the coarse grid. Their analysis demonstrates that the final approximation can recover fine grid asymptotic accuracy without a nonlinear fine grid solve. Shang [2] later studied a three step Oseen correction method in which two linear problems are solved on the fine grid. These results motivate the question of whether a second Picard state can be combined consistently with a coarse grid backtracking correction.

The method considered here contains four computational stages. A nonlinear Navier Stokes system is first solved on the coarse grid. Two Picard problems are then solved successively on the fine grid, with the first coarse velocity and then the first fine velocity used as the frozen convective fields. Finally, the nonlinear residual of the second fine grid state is projected through a full linearization on the coarse grid. In contrast with a simple relabeling of the Oseen step, the second fine grid solve changes the frozen velocity and produces a new residual. The principal contribution of this theoretical draft is the derivation of the matching correction equation and the associated error identity.

The remainder of the paper is organized as follows. Section II introduces the weak formulation, finite element spaces, and analytical assumptions. Section III presents the two-level two-step Picard algorithm and derives the coarse-grid residual correction. Section IV gives estimates for the two Picard states. Section V analyzes the corrected solution in the energy and velocity norms. Section VI discusses mesh scaling and the relation with existing methods. Section VII reports the numerical experiments, and conclusions are stated in Section VIII.

II. MATHEMATICAL PRELIMINARIES

Weak Formulation

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded polygonal or polyhedral domain with boundary $\partial\Omega$. We consider

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in}\Omega, \tag{1}$$

$$\nabla\cdot u = 0 \quad \text{in}\Omega, \qquad u = 0 \quad \text{on}\,\partial\Omega, \tag{2}$$

where u is the velocity, p is the pressure, f is the body force, and $\nu > 0$ is the viscosity. Define

$$X = [H_0^1(\Omega)]^d, \qquad M = L_0^2(\Omega). \tag{3}$$

For $w, u, v \in X$, set

$$a(u,v) = \nu(\nabla u, \nabla v), \qquad b(w,u,v) = \big((w\cdot\nabla)u, v\big). \tag{4}$$

The weak problem is to find $(u,p) \in X \times M$ such that

$$a(u,v) + b(u,u,v) - (p, \nabla\cdot v) + (q, \nabla\cdot u) = (f, v), \tag{5}$$

for every $(v,q) \in X \times M$. The convective form is used in the analysis. The constants below may depend on ν, the domain, and Sobolev norms of the exact solution, but they are independent of h and H.

Finite Element Spaces

Let $\mathcal{T}_H$ and $\mathcal{T}_h$ be shape regular coarse and fine triangulations satisfying $0 < h < H < 1$. The associated conforming mixed finite element spaces are denoted by

$$X^H \times M^H \subset X^h \times M^h \subset X \times M. \tag{6}$$

The nestedness assumption simplifies the residual derivation. It can be replaced by stable transfer operators, but that extension is not pursued here.

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Assumption 1. The pairs (X^μ, M^μ), $\mu = H, h$, satisfy the discrete inf sup condition

$$\inf_{q^\mu \in M^\mu} \sup_{v^\mu \in X^\mu} \frac{(q^\mu, \nabla \cdot v^\mu)}{\| q^\mu \|_0 \| v^\mu \|_1} \geq \beta > 0. \quad (7)$$

Assumption 2. For an integer $k \geq 1$, the spaces have the approximation properties

$$\inf_{v^\mu \in X^\mu} (\| u - v^\mu \|_0 + \mu \| u - v^\mu \|_1) \leq C\mu^{k+1} \| u \|_{k+1},$$
$$\inf_{q^\mu \in M^\mu} \| p - q^\mu \|_0 \leq C\mu^k \| p \|_k. \quad (8)$$

The standard nonlinear finite element solutions on the two grids are assumed to satisfy

$$\| u - u^\mu \|_1 + \| p - p^\mu \|_0 \leq C\mu^k,$$
$$\| u - u^\mu \|_0 \leq C\mu^{k+1}, \qquad \mu = H, h. \quad (9)$$

These estimates follow from the usual regularity, uniqueness, and sufficiently small mesh assumptions for mixed finite element approximations [8], [10].

Linearized Stability and Projection

For $(v, q) \in X \times M$, use the product norm

$$\| (v, q) \|_Y = (\nu \| v \|_1^2 + \| q \|_0^2)^{1/2}. \quad (10)$$

The full linearization at the coarse velocity is

$$\mathcal{B}_H[(w, r); (v, q)] = a(w, v) + b(u^H, w, v)$$
$$+ b(w, u^H, v) - (r, \nabla \cdot v) + (q, \nabla \cdot w). \quad (11)$$

Assumption 3. The exact solution is nonsingular and H is sufficiently small so that the Oseen forms used below and $\mathcal{B}_H$ satisfy uniform mixed inf sup conditions on the relevant continuous and discrete spaces. Thus, for a constant $\gamma > 0$ independent of h and H,

$$\inf_{(w,r)} \sup_{(v,q)} \frac{\mathcal{B}_H[(w,r);(v,q)]}{\| (w,r) \|_Y \| (v,q) \|_Y} \geq \gamma. \quad (12)$$

This is the standard nonsingularity requirement for the linearized Navier Stokes operator [1], [8], [9].

Define the coarse grid Galerkin projection $(Q_H(v, q), R_H(v, q)) \in X^H \times M^H$ by

$$\mathcal{B}_H[(w^H, r^H); (v - Q_H(v, q), q - R_H(v, q))] = 0. \quad (13)$$

for every $(w^H, r^H) \in X^H \times M^H$.

Assumption 4. The linearized adjoint problem is H^2 regular. Consequently,

$$\| (Q_H(v, q), R_H(v, q)) \|_Y \leq C \| (v, q) \|_Y,$$
$$\| v - Q_H(v, q) \|_0 \leq CH \| (v, q) \|_Y. \quad (14)$$

The exact velocity is also assumed to belong to $[H^{k+1}(\Omega) \cap W^{1,\infty}(\Omega)]^d$.

III. Two Level Two Step Picard Method

Coarse Nonlinear Problem

Step I. Find $(u^H, p^H) \in X^H \times M^H$ such that

$$a(u^H, v^H) + b(u^H, u^H, v^H) - (p^H, \nabla \cdot v^H)$$
$$+ (q^H, \nabla \cdot u^H) = (f, v^H), \quad (15)$$

for all $(v^H, q^H) \in X^H \times M^H$.

First Fine Grid Picard Update

Step II. Find $(u_h^{(1)}, p_h^{(1)}) \in X^h \times M^h$ such that

$$a(u_h^{(1)}, v^h) + b(u^H, u_h^{(1)}, v^h) - (p_h^{(1)}, \nabla \cdot v^h)$$
$$+ (q^h, \nabla \cdot u_h^{(1)}) = (f, v^h), \quad (16)$$

for all $(v^h, q^h) \in X^h \times M^h$. This is the fine grid Oseen problem in the classical backtracking method [1].

Second Fine Grid Picard Update

Step III. Find $(u_h^{(2)}, p_h^{(2)}) \in X^h \times M^h$ such that

$$a(u_h^{(2)}, v^h) + b(u_h^{(1)}, u_h^{(2)}, v^h) - (p_h^{(2)}, \nabla \cdot v^h)$$
$$+ (q^h, \nabla \cdot u_h^{(2)}) = (f, v^h), \quad (17)$$

for all $(v^h, q^h) \in X^h \times M^h$. Unlike Step II, the convective field is now the first fine grid velocity.

Residual Derivation and Backtracking Correction

For a coarse test pair, define the nonlinear residual of the second Picard state by

$$\mathcal{R}_h^{(2)}(v^H, q^H) = (f, v^H) - a(u_h^{(2)}, v^H)$$
$$- b(u_h^{(2)}, u_h^{(2)}, v^H) + (p_h^{(2)}, \nabla \cdot v^H)$$
$$- (q^H, \nabla \cdot u_h^{(2)}). \quad (18)$$

Since the spaces are nested, (17) is valid for coarse test functions. Substitution into (18) yields the exact identity

$$\mathcal{R}_h^{(2)}(v^H, q^H) = b(u_h^{(1)} - u_h^{(2)}, u_h^{(2)}, v^H). \quad (19)$$

Thus the right hand side of the correction equation is determined by the nonlinear residual and is not selected heuristically.

Step IV. Find $(e^H, E^H) \in X^H \times M^H$ such that

$$a(e^H, v^H) + b(u^H, e^H, v^H) + b(e^H, u^H, v^H)$$
$$- (E^H, \nabla \cdot v^H) + (q^H, \nabla \cdot e^H)$$
$$= b(u_h^{(1)} - u_h^{(2)}, u_h^{(2)}, v^H). \quad (20)$$

for all $(v^H, q^H) \in X^H \times M^H$. The final approximation is

$$u^* = u_h^{(2)} + e^H, \qquad p^* = p_h^{(2)} + E^H. \quad (21)$$

Theorem 1. Under Assumptions 1–3, Steps II–IV have unique solutions.

Proof. Each step is a finite dimensional mixed linear problem. The discrete pressure inf sup condition controls the saddle point coupling, while Assumption 3 supplies the uniform inf sup stability of the corresponding Oseen or full linearized velocity operator. The Babuška–Brezzi theory therefore gives existence, uniqueness, and stability. ▫

IV. Estimates for the Picard Updates

First Picard State

Subtracting (16) from the continuous problem gives

$$a(u - u_h^{(1)}, v^h) + b(u^H, u - u_h^{(1)}, v^h)$$
$$- (p - p_h^{(1)}, \nabla \cdot v^h) + (q^h, \nabla \cdot (u - u_h^{(1)}))$$
$$= b(u^H - u, u, v^h). \quad (22)$$

The right hand side is controlled by the coarse grid velocity error in L^2:

$$|b(u^H - u, u, v^h)| \leq$$
$$C \| u - u^H \|_0 \| u \|_{1,\infty} \| v^h \|_1. \quad (23)$$

Using the discrete Oseen stability, interpolation, and (9), we obtain

$$\| u - u_h^{(1)} \|_1 + \| p - p_h^{(1)} \|_0 \leq C(h^k + H^{k+1}). \quad (24)$$

An Aubin–Nitsche argument for the adjoint Oseen problem gives

$$\| u - u_h^{(1)} \|_0 \leq C(h^{k+1} + H^{k+1}). \quad (25)$$

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Second Picard State

Subtracting (17) from the continuous problem yields

$$a\left(u-u_h^{(2)},v^h\right)+b\left(u_h^{(1)},u-u_h^{(2)},v^h\right) -\left(p-p_h^{(2)},\nabla\cdot v^h\right)+\left(q^h,\nabla\cdot\left(u-u_h^{(2)}\right)\right) =b\left(u_h^{(1)}-u,u,v^h\right). \tag{26}$$

The right hand side depends on the L^2 error of the first Picard velocity. Combining (25) with the stability of the second Oseen problem gives

$$\| u-u_h^{(2)} \|_1 + \| p-p_h^{(2)} \|_0 \leq C(h^k+H^{k+1}), \tag{27}$$

and the corresponding duality estimate is

$$\| u-u_h^{(2)} \|_0 \leq C(h^{k+1}+H^{k+1}). \tag{28}$$

Equations (24) and (27) have the same asymptotic orders. The second Picard solve should therefore be interpreted as a reduction of the nonlinear residual and error constants rather than an automatic increase of the coarse grid convergence order.

Difference of the Picard States

Subtracting (17) from (16) and rearranging gives

$$a\left(u_h^{(1)}-u_h^{(2)},v^h\right)+b\left(u_h^{(1)},u_h^{(1)}-u_h^{(2)},v^h\right) -\left(p_h^{(1)}-p_h^{(2)},\nabla\cdot v^h\right)+\left(q^h,\nabla\cdot\left(u_h^{(1)}-u_h^{(2)}\right)\right) =b\left(u_h^{(1)}-u^H,u_h^{(1)},v^h\right). \tag{29}$$

Consequently,

$$\| u_h^{(1)}-u_h^{(2)} \|_1 + \| p_h^{(1)}-p_h^{(2)} \|_0 \leq C(h^{k+1}+H^{k+1}), \tag{30}$$

and, in particular,

$$\| u_h^{(1)}-u_h^{(2)} \|_0 \leq C(h^{k+1}+H^{k+1}). \tag{31}$$

The correction residual in (19) is therefore generated by a higher order difference.

V. Error Analysis of the Corrected Solution

Fundamental Error Identity

The second Picard equation and the continuous problem imply

$$\mathcal{B}_H\left[\left(u-u_h^{(2)},p-p_h^{(2)}\right);(v^h,q^h)\right]-\mathcal{R}_h^{(2)}(v^h,q^h) =b\left(u^H-u,u-u_h^{(2)},v^h\right)+b\left(u-u_h^{(2)},u^H-u,v^h\right) +b\left(u-u_h^{(2)},u-u_h^{(2)},v^h\right). \tag{32}$$

Because (e^H,E^H) belongs to the coarse space, the projection property (13) and the correction equation give, with $z^h=v^h-Q_H(v^h,q^h)$,

$$\mathcal{B}_H\left[\left(u-u^*,p-p^*\right);(v^h,q^h)\right] =b\left(u^H-u,u-u_h^{(2)},v^h\right)+b\left(u-u_h^{(2)},u^H-u,v^h\right) +b\left(u-u_h^{(2)},u-u_h^{(2)},v^h\right) +b\left(u_h^{(1)}-u_h^{(2)},u_h^{(2)},z^h\right). \tag{33}$$

This identity is the central analytical relation for the proposed method. The first three terms are products of coarse and second Picard errors. The last term measures the part of the second Picard residual not represented on the coarse grid.

Estimate of the Residual Projection Term

To avoid an inverse estimate for $u_h^{(2)}$, split the second argument as

$$b\left(u_h^{(1)}-u_h^{(2)},u_h^{(2)},z^h\right) =b\left(u_h^{(1)}-u_h^{(2)},u,z^h\right) +b\left(u_h^{(1)}-u_h^{(2)},u_h^{(2)}-u,z^h\right). \tag{34}$$

By (14), (27), (30), and (31),

$$\left|b\left(u_h^{(1)}-u_h^{(2)},u_h^{(2)},z^h\right)\right| \leq CH(h^{k+1}+H^{k+1}) \| (v^h,q^h) \|_Y +C(h^k+H^{k+1})^2 \| (v^h,q^h) \|_Y. \tag{35}$$

Energy and Pressure Estimate

Applying the discrete stability of $\mathcal{B}_H$ to (33), using the trilinear form bounds, and adding the fine grid approximation error yield

$$\| u-u^* \|_1 + \| p-p^* \|_0 \leq Ch^k + CH^k(h^k+H^{k+1}) +C(h^k+H^{k+1})^2 + CH(h^{k+1}+H^{k+1}). \tag{36}$$

Since $0<h<H<1$, the mixed higher order terms can be absorbed into the leading contributions. We obtain the principal energy estimate.

Theorem 2. Under Assumptions 1–4, there exists $H_0>0$ such that, for $0<h<H\leq H_0$,

$$\| u-u^* \|_1 + \| p-p^* \|_0 \leq C(h^k+H^{k+2}+H^{2k+1}). \tag{37}$$

Velocity L2 Estimate

Let (ϕ,ψ) solve the adjoint problem

$$\mathcal{B}_H[(w,r);(\phi,\psi)]=\left(u-u^*,w\right), \tag{38}$$

for every $(w,r)\in X\times M$, with

$$\| \phi \|_2 + \| \psi \|_1 \leq C \| u-u^* \|_0. \tag{39}$$

Testing (33) with a fine grid approximation of (ϕ,ψ), using (37), and estimating the nonlinear terms as in the energy analysis gives the following result.

Theorem 3. Under the assumptions of Theorem 2 and the adjoint regularity (39),

$$\| u-u^* \|_0 \leq C(h^{k+1}+H^{k+2}+H^{2k+1}). \tag{40}$$

Thus the backtracking correction removes the leading H^{k+1} contribution appearing in the uncorrected Picard states and replaces it by higher order coarse grid terms.

VI. Mesh Scaling and Algorithmic Remarks

Representative Mesh Relations

For optimal energy accuracy, it is sufficient to balance

$$h^k \asymp H^{\min\{k+2,2k+1\}}. \tag{41}$$

Representative choices are

$$h\asymp H^3 \quad (k=1), \qquad h\asymp H^{1+2/k} \quad (k\geq 2). \tag{42}$$

For optimal L^2 velocity accuracy, it is sufficient to balance

$$h^{k+1} \asymp H^{\min\{k+2,2k+1\}}, \tag{43}$$

which gives

$$h\asymp H^{3/2} \quad (k=1), \qquad h\asymp H^{(k+2)/(k+1)} \quad (k\geq 2). \tag{44}$$

The relations in (44) are also sufficient for the energy estimate to be of order h^k. They represent a less aggressive fine grid refinement than the balance in (42).

Computational Interpretation

The proposed method requires one nonlinear coarse grid solve, two fine grid Oseen solves, and one linear coarse grid correction. It does not iterate the fine grid Picard process to convergence. Compared with the classical backtracking method [1], one additional fine grid linear system is solved. In exchange,

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the correction is driven by the higher order difference $u_h^{(1)} - u_h^{(2)}$. Compared with the three step Oseen correction method [2], the second Picard problem freezes the convective velocity at the first fine grid state and is followed by an explicit return to the coarse space.

The estimates do not claim a higher asymptotic order than every existing backtracking method. Their significance is that the residual-based correction remains stable and asymptotically optimal after a genuine second Picard update. The numerical experiments below therefore examine the error constants, robustness with respect to viscosity and coarse-grid size, and the additional cost of the second fine-grid solve.

VII. Numerical Experiments

Numerical Setting and Manufactured Solution

All computations were carried out with FreeFEM++ [13]. The velocity and pressure were approximated by the inf-sup stable $P_2 - P_1$ Taylor-Hood pair. The nonlinear coarse- and fine-grid reference problems were solved by Newton iteration with relative increment tolerance 10^{-10}, and a pressure penalty 10^{-10} was used to remove the constant pressure mode. Unless stated otherwise, the proposed approximation is compared with the fully nonlinear fine-grid solution (u_h^F, p_h^F).

For the convergence tests, $\Omega = (0,1)^2$ and the divergence-free exact velocity is generated from the stream function

$$\psi(x,y) = u_c x^2(1-x)^2 \cdot y^2(1-y)^2, \tag{45}$$

$$u_1 = 2u_c x^2(1-x)^2 \cdot y(1-y)(1-2y),$$
$$u_2 = -2u_c x(1-x)(1-2x) \cdot y^2(1-y)^2, \tag{46}$$

$$p(x,y) = x^3 + y^3 - \frac{1}{2},$$
$$\int_\Omega p\, dx = 0, \tag{47}$$

$$f = -\nu\Delta u + (u\cdot\nabla)u + \nabla p. \tag{48}$$

We set $u_c = 10$ and first take $\nu = 0.1$. The nested meshes satisfy $N_h = 4N_H$. The reported errors are

$$E_{L^2}^u = \| u - u_h \|_0,$$
$$E_{H^1}^u = \| \nabla(u - u_h) \|_0,$$
$$E_{L^2}^p = \| p - p_h \|_0. \tag{49}$$

Table I confirms the expected rates $O(h^3)$, $O(h^2)$, and $O(h^2)$ for the velocity L^2 error, velocity H^1 seminorm error, and pressure L^2 error, respectively. On the finest mesh, the proposed solution differs from the fully nonlinear fine-grid solution by only 3.22×10^{-14} in the velocity L^2 norm, while its CPU time is approximately 43% of that required by the fine-grid Newton solve.

TABLE I. Convergence results of the proposed method for ν=0.1.

(N_H, N_h)	Velocity L2 error (rate)	Velocity H1 seminorm (rate)	Pressure L2 error (rate)	CPU (s)
(8,32)	2.927e-6 (-)	8.214e-4 (-)	2.521e-4 (-)	0.490
(12,48)	8.669e-7 (3.001)	3.656e-4 (1.997)	1.121e-4 (2.000)	1.252
(16,64)	3.657e-7 (3.000)	2.057e-4 (1.998)	6.304e-5 (2.000)	2.297
(24,96)	1.084e-7 (3.000)	9.147e-5 (1.999)	2.802e-5 (2.000)	6.176
(32,128)	4.572e-8 (3.000)	5.146e-5 (2.000)	1.576e-5 (2.000)	11.597

Influence of the Viscosity

The coarse and fine grids were fixed at $N_H = 16$ and $N_h = 64$, and the viscosity was varied over $\nu = 1, 0.1, 0.05, 0.01$. As ν decreases, the first Picard approximation becomes less accurate, whereas the second Picard update and the proposed correction remain effective. In Table II, the reduction factors are the ratios of the second-Picard differences to the corresponding proposed-method differences, and the CPU ratio is T_*/T_F.

TABLE II. Effect of the viscosity coefficient (N_H=16, N_h=64).

ν	P2 L2 diff.	Prop. L2 diff.	L2 factor	H1 factor	CPU ratio
1	1.602e-12	1.701e-14	94.2	8.43	0.576
0.1	1.601e-10	1.692e-12	94.6	8.43	0.443
0.05	6.396e-10	6.767e-12	94.5	8.42	0.457
0.01	1.541e-8	1.690e-10	91.2	8.20	0.450

For $\nu = 0.01$, the proposed solution differs from the nonlinear fine-grid solution by 1.69×10^{-10} in the velocity L^2 norm while using about 45% of the reference CPU time.

Influence of the Coarse Grid

To isolate the effect of the coarse mesh, the fine grid was fixed at $N_h = 128$ and $\nu = 0.01$, while N_H was increased from 4 to 32. The results in Table III demonstrate rapid convergence to the fully nonlinear fine-grid solution.

TABLE III. Fixed fine-grid study (N_h=128, ν=0.01).

N_H	H	Prop. L2 diff.	Prop. H1 diff.	P2/Prop. L2	CPU (s)
4	0.2500	7.883e-7	2.069e-5	3.92	11.592
8	0.1250	1.199e-8	6.595e-7	20.73	11.428
16	0.0625	1.615e-10	1.976e-8	95.80	11.478
32	0.03125	3.057e-12	1.100e-9	315.15	12.964

With $N_H = 16$, the velocity L^2 difference is already 1.61×10^{-10}, and the final correction improves the uncorrected second Picard state by a factor of about 96. The observed high-order behavior is relative to the fixed fine-grid reference and is not asserted as an additional theoretical convergence theorem.

Lid-Driven Cavity Flow

The classical lid-driven cavity problem was solved on $\Omega = (0,1)^2$ with

$$u = (1,0) \quad \text{on} y = 1,$$
$$u = (0,0) \quad \text{on the remaining walls}, \qquad \nu = Re^{-1}. \tag{50}$$

The computations used and for. Continuation in the Reynolds number was used to initialize the nonlinear solves. The centerline velocities were compared with the benchmark data in [14].

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TABLE IV. Lid-driven cavity results (N_H=48, N_h=192).

Re	u L2 diff.	u H1 diff.	p L2 diff.	CPU Prop./Ref. (s)
100	8.374e-6	2.841e-4	5.860e-6	32.284/98.155
400	9.802e-5	4.447e-3	3.124e-5	31.917/98.152
1000	2.410e-4	2.086e-2	8.109e-5	31.901/129.781
2000	5.295e-4	5.786e-2	1.563e-4	32.044/114.134

The streamline patterns produced by the proposed method are shown in Fig. 1. As the Reynolds number increases, the primary vortex moves toward the cavity center and the lower corner vortices become progressively stronger.

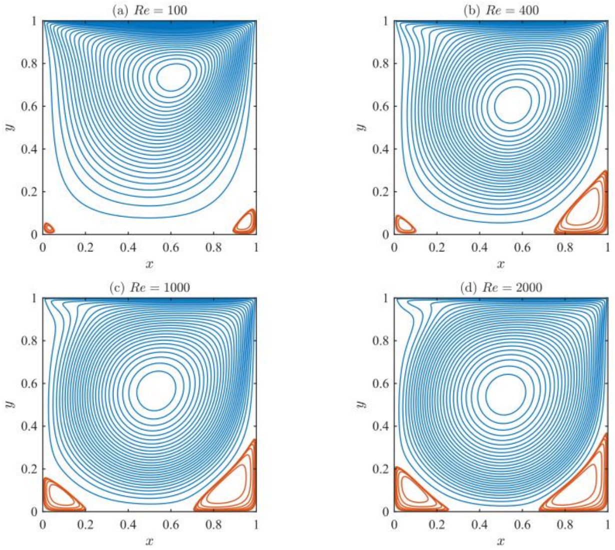

Fig. 1. Streamline patterns obtained by the proposed method: (a) Re=100; (b) Re=400; (c) Re=1000; and (d) Re=2000.

For, the centerline velocity profiles are compared in Fig. 2 with the fully nonlinear fine-grid solution and the benchmark data in [14]. The three sets of results are nearly indistinguishable over most of the cavity.

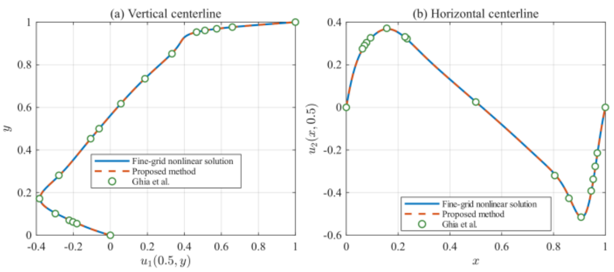

Fig. 2. Centerline velocity profiles at Re=1000 compared with the fully nonlinear fine-grid solution and the benchmark data in [14].

Fig. 3 compares all numerical approximations at $Re = 2000$. Although the second Picard approximation alone does not uniformly outperform the original one-step backtracking solution for this boundary-driven flow, the proposed combination gives the closest approximation to the nonlinear fine-grid solution.

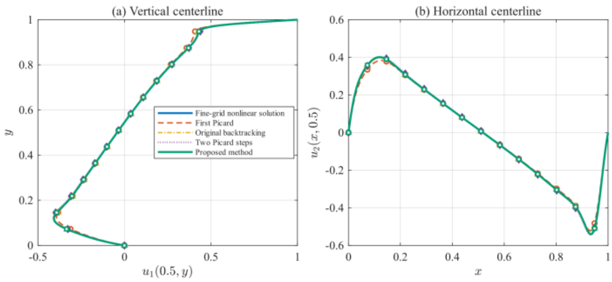

Fig. 3. Comparison of the centerline velocity profiles produced by the different methods at Re=2000.

At $Re = 2000$, the proposed method uses about 28% of the CPU time of the reference Newton method while retaining close agreement in the centerline velocities and the main vortex structure.

VIII. CONCLUSION

A two-level two-step Picard method with coarse-grid backtracking correction has been formulated and analyzed for the stationary incompressible Navier-Stokes equations. The nonlinear residual of the second Picard approximation was derived explicitly and used as the right-hand side of a full linearized coarse-grid correction. Under standard finite element stability, approximation, nonsingularity, and adjoint regularity assumptions, the corrected velocity and pressure satisfy optimal-order estimates for suitable coarse- and fine-grid relations. The numerical experiments confirm the expected Taylor-Hood convergence rates and demonstrate that the second Picard update substantially reduces the nonlinear iteration error. The matching coarse-grid correction further improves the approximation over a broad range of viscosities and coarse mesh sizes. For the lid-driven cavity flow, the proposed method closely agrees with the Ghia benchmark data and the fully nonlinear fine-grid solution up to $Re = 2000$, while requiring only about 25%--33% of the reference CPU time. These results indicate that the method provides a favorable compromise between fine-grid nonlinear accuracy and computational cost.

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