International Journal of Scientific Engineering and Science
Volume 10, Issue 7, pp. 71-79, July 2026. ISSN (Online): 2456-7361

A Parallel Partition-of-Unity Variational Multiscale Method Based on Two Local Gauss Integration Rules for the Steady Natural Convection Equations

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Abstract—*We present a parallel partition-of-unity variational multiscale (PU-VMS) method based on two local Gauss integration rules for the steady natural convection equations. Independent VMS problems are solved on composite grids that are fine in overlapping neighborhoods and coarse elsewhere. Velocity and temperature stabilizations are formed by quadratic-exact and one-point quadrature differences and admit equivalent elementwise* $\boldsymbol{L^2}$*-projection forms. The independently computed local approximations are then combined by a partition of unity to obtain a globally continuous solution. Stability and global and local a priori error estimates are established for the VMS discretization and the final PU approximation. Numerical experiments with* $\boldsymbol{P_2-P_1-P_2}$ *elements confirm the predicted convergence rates and show good agreement with benchmark cavity-flow results.*

Keywords— *Steady natural convection equations; variational multiscale method; two local Gauss integration rules; partition of unity; parallel finite element method.*

I. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$ .We consider the following steady-state incompressible natural convection problem:

$$-Pr\Delta u + (u\cdot\nabla)u + \nabla p = PrRae_jT + f, in\Omega, \quad (1.1)$$

$$\nabla\cdot u = 0, in\Omega, \quad (1.2)$$

$$-\nabla\cdot(\kappa\nabla T) + (u\cdot\nabla)T = g, in\Omega, \quad (1.3)$$

$$u = 0, on\partial\Omega, \quad T = 0, on\Gamma_1, \quad and \quad \frac{\partial T}{\partial n} = 0, on\Gamma_2. \quad (1.4)$$

Here, $\partial\Omega = \Gamma_1 \cup \Gamma_2$with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and meas$(\Gamma_1) > 0$ .The parameters $Pr > 0$ and $Ra > 0$ are the Prandtl and Rayleigh numbers,respectively, $e_j = [0,1]^T$ is the gravitational acceleration vector, f and g are the forcing functions, and n is the outward unit normal on Γ_2.

Natural convection couples nonlinear momentum and heat transport and becomes difficult to resolve as Ra increases. Mixed finite element analysis began with[1]; projection, quadrature-based VMS, and residual VMS stabilizations were subsequently developed in [2-5].VMS methods separate resolved and unresolved scales [6,7]. Here we use the difference between consistent and under-integrated element matrices [8], which is locally assembled and has an elementwise L^2-projection representation.

Parallel composite-grid methods reduce the cost of globally fine discretizations [9,10]. Their combination with VMS stabilization has been studied for incompressible flows [12,13], including partition-of-unity reconstruction[11] , and for natural convection[14,15] . Motivated by these works, we solve independent VMS problems on grids refined only near overlapping target regions and assemble their solutions by subordinate partition-of-unity functions.

The contributions are a two-local-Gauss PU--VMS formulation for both velocity and temperature, global energy- and lower-order-norm estimates, local estimates retaining α_u and α_T, error bounds for the final continuous reconstruction, and $P_2-P_1-P_2$ experiments confirming the theory. Sections II-V present the discretization and analysis; Section VI gives numerical results.

II. MATHEMATICAL PRELIMINARIES

2.1 Functional setting of the steady natural convection problem

For an integer $k \geq 1$, we adopt standard definitions for the Hilbert space $H^k(\Omega)$ or $[H^k(\Omega)]^2$ equipped with the norm $\|\cdot\|_{k,\Omega}$, and denote by $(\cdot,\cdot)$ and $\|\cdot\|_{0,\Omega}$ the $L^2(\Omega)$ or $[L^2(\Omega)]^2$ inner product and norm, respectively. Set

$$X(\Omega) = [H_0^1(\Omega)]^2 = \{w \in [H^1(\Omega)]^2 : w|_{\partial\Omega} = 0\},$$

$$Q(\Omega) = L_0^2(\Omega) = \{q \in L^2(\Omega) : (q,1) = 0\},$$

$$M(\Omega) = \{\psi \in H^1(\Omega) : \psi|_{\Gamma_1} = 0\},$$

$$\tilde{X}(\Omega) = \{u \in X(\Omega) : \nabla\cdot u = 0\}.$$

Defining the trilinear terms $b_1(\cdot,\cdot,\cdot) and b_2(\cdot,\cdot,\cdot)$ as

$$b_1(u,w,\xi) = ((u\cdot\nabla)w,\xi) + \frac{1}{2}((\nabla\cdot u)w,\xi)$$

$$= \frac{1}{2}((u\cdot\nabla)w,\xi) - \frac{1}{2}((u\cdot\nabla)\xi,w), \quad \forall\, u,w,\xi \in X(\Omega),$$

$$b_2(u,T,\psi) = ((u\cdot\nabla)T,\psi) + \frac{1}{2}((\nabla\cdot u)T,\psi)$$

$$= \frac{1}{2}((u\cdot\nabla)T,\psi) - \frac{1}{2}((u\cdot\nabla)\psi,T), \forall\, u \in X(\Omega), T,\psi \in M(\Omega)$$

As in[21-22], the trilinear terms have the following important properties:

$$b_1(u,w,\xi) = -\, b_1(u,\xi,w),$$
$$|b_1(u,w,\xi)| \leq N_1\|\nabla u\|_{0,\Omega}\|\nabla w\|_{0,\Omega}\|\nabla\xi\|_{0,\Omega}, \quad (2.1)$$

$$b_2(u,T,\psi) = -\, b_2(u,\psi,T),$$
$$|b_2(u,T,\psi)| \leq N_2\|\nabla u\|_{0,\Omega}\|\nabla T\|_{0,\Omega}\|\nabla\psi\|_{0,\Omega}, \quad (2.2)$$

where

$$N_1 = \sup_{u,w,\xi\in X(\Omega)} \frac{|b_1(u,w,\xi)|}{\|\nabla u\|_{0,\Omega}\|\nabla w\|_{0,\Omega}\|\nabla\xi\|_{0,\Omega}},$$

International Journal of Scientific Engineering and Science
Volume 10, Issue 7, pp. 71-79, July 2026. ISSN (Online): 2456-7361

$$N_2 = \sup_{u\in X(\Omega),T,\psi\in M(\Omega)} \frac{|b_2(u,T,\psi)|}{\|\nabla u\|_{0,\Omega}\|\nabla T\|_{0,\Omega}\|\nabla\psi\|_{0,\Omega}}.$$

In addition, for any $w \in X(\Omega)$, $\psi \in M(\Omega)$, the following bounds shall be used in later analysis:

$$\|w\|_{0,\Omega} \le c_m\|\nabla w\|_{0,\Omega}, \quad \|\psi\|_{0,\Omega} \le c_n\|\nabla\psi\|_{0,\Omega}. \tag{2.3}$$

$$\|w\|_{L^4(\Omega)} \le 2^{1/4}\|w\|_{0,\Omega}^{1/2}\|\nabla w\|_{0,\Omega}^{1/2}. \tag{2.4}$$

where $c_m > 0$ and $c_n > 0$ are two constants.

We write $x \lesssim y$ if $x \le cy$, where $c > 0$ is independent of the mesh sizes and stabilization parameters unless stated otherwise.

The weak form of problem (1.1)-(1.4) reads as follows: find $(u,p,T) \in X(\Omega)\times Q(\Omega)\times M(\Omega)$ satisfying

$$\begin{cases} Pr(\nabla u,\nabla w) + b_1(u,u,w) - (\nabla\cdot w,p) + (\nabla\cdot u,q) \\ = PrRa(e_jT,w) + (f,w), \forall\,(w,q)\in X(\Omega)\times Q(\Omega), \\ \kappa(\nabla T,\nabla\psi) + b_2(u,T,\psi) = (g,\psi), \quad \forall\,\psi\in M(\Omega). \end{cases} \tag{2.5}$$

The following theorem is available [1,3,21] .

Theorem 2.1 Suppose that Ω is a convex polygon or that $\partial\Omega$ is of class C^2, $(f,g)\in[L^2(\Omega)]^2\times L^2(\Omega)$, and $\widetilde{C} = c_mc_n$.Then problem (2.5) admits at least one solution $(u,p,T)\in[H^2(\Omega)]^2\times H^1(\Omega)\times H^2(\Omega)$satisfying

$$\|\nabla T\|_{0,\Omega} \le \kappa^{-1}\|g\|_{-1,\Omega}, \quad \|\nabla u\|_{0,\Omega} \le \widetilde{A}. \tag{2.6}$$

where $\widetilde{A} = \widetilde{C}\kappa^{-1}Ra\|g\|_{-1,\Omega} + Pr^{-1}\|f\|_{-1,\Omega}$.

2.2 Mixed finite element method

For the sake of introducing the mixed finite element approximations of problem (2.5),let$X_h(\Omega)\subset[H^1(\Omega)]^2$, $Q_h(\Omega)\subset L^2(\Omega)$ and $M_h(\Omega)\subset H^1(\Omega)$ be the three finite element spaces associated with a triangulation$\mathcal{T}^h(\Omega)=\{K\}$ of Ω with a mesh size function $h(x)$. Here, the value of $h(x)$ is the diameter h_K of K for $x\in K$. We assume that the mesh is not exceedingly over-refined locally [9], namely, there exists a constant $l\ge 1$ such that $h^l\le h(x),\forall x\in\Omega$, where $h = h_\Omega = \max_{x\in\Omega} h\,(x)$ is the largest mesh size of $\mathcal{T}^h(\Omega)$. Set

$$X_h^0(\Omega) = X_h(\Omega)\cap X(\Omega), \quad Q_h^0(\Omega) = Q_h(\Omega)\cap Q(\Omega),$$
$$M_h^0(\Omega) = M_h(\Omega)\cap M(\Omega).$$

For a given $D\subset\Omega$, we assume that D is the union of elements of $\mathcal{T}^h(\Omega)$ and set

$$X_h(D) = X_h(\Omega)|_D, \quad Q_h(D) = Q_h(\Omega)|_D,$$
$$M_h(D) = M_h(\Omega)|_D, \quad \mathcal{T}^h(D) = \mathcal{T}^h(\Omega)|_D,$$

in which $\cdot\,|_D$represents the restriction of $\cdot$to D.Besides,let

$$X_0^h(D) = \{w\in X_h(\Omega): \mathrm{supp}w \subset\subset D\},$$
$$Q_0^h(D) = \{q\in Q_h(\Omega): \mathrm{supp}q \subset\subset D\},$$
$$M_0^h(D) = \{\psi\in M_h(\Omega): \mathrm{supp}\psi \subset\subset D\},$$

Given two subdomains G and Ω_0 of Ω fulfilling $G\subset\Omega_0\subset\Omega$, we view $[H_0^1(\Omega_0)]^2$ as a subspace of $[H_0^1(\Omega)]^2$ by extending any function $w\in[H_0^1(\Omega_0)]^2$ to be a function in $[H_0^1(\Omega)]^2$ with zero outside of Ω_0, and utilize the notation $G\subset\subset\Omega_0$ to indicate $\mathrm{dist}(\partial G\setminus\partial\Omega,\partial\Omega_0\setminus\partial\Omega)>0$.

We now state the basic assumptions on the mixed finite element spaces.

A1. Approximation. For each $(u,p,T)\in[H^{s+1}(D)]^2\times H^s(D)\times H^{s+1}(D)$, there exists an approximation $(I_hu,J_hp,\lambda_hT)\in X_h(D)\times Q_h(D)\times M_h(D)$ such that,for s=1,2,

$$h^{-1}\|u-I_hu\|_{0,D} + \|u-I_hu\|_{1,D} \lesssim h^s\|u\|_{s+1,D}, \tag{2.7}$$

$$h^{-1}\|p-J_hp\|_{-1,D} + \|p-J_hp\|_{0,D} \lesssim h^s\|p\|_{s,D}, \tag{2.8}$$

$$h^{-1}\|T-\lambda_hT\|_{0,D} + \|T-\lambda_hT\|_{1,D} \lesssim h^s\|T\|_{s+1,D}. \tag{2.9}$$

A2. Inverse estimate. The inverse inequalities used in the local analysis read as follows: for any $(w,q,\psi)\in X_h(D)\times Q_h(D)\times M_h(D)$,

$$\|w\|_{1,D} \lesssim h^{-1}\|w\|_{0,D}, \quad \|q\|_{0,D} \lesssim h^{-1}\|q\|_{-1,D},$$
$$\|\psi\|_{1,D} \lesssim h^{-1}\|\psi\|_{0,D}. \tag{2.10}$$

A3. Superapproximation. For $D\subset\Omega$, let $\varpi\in C_0^\infty(\Omega)$ with $\mathrm{supp}\varpi\subset\subset D$. Then, for any $(u,p,T)\in X_h(D)\times Q_h(D)\times M_h(D)$, there exists $(w,q,\psi)\in X_0^h(D)\times Q_0^h(D)\times M_0^h(D)$such that

$$\|\varpi u - w\|_{1,D} \lesssim h\|u\|_{1,D}, \|\varpi p - q\|_{0,D} \lesssim h\|p\|_{0,D},$$
$$\|\varpi T - \psi\|_{1,D} \lesssim h\|T\|_{1,D}. \tag{2.11}$$

A4. Stability. The local mixed finite element pair satisfies the inf-sup condition

$$\inf_{q\in Q_0^h(D)}\sup_{w\in X_0^h(D)} \frac{(\nabla\cdot w,q)}{\|\nabla w\|_{0,D}\|q\|_{0,D}} \ge \beta > 0. \tag{2.12}$$

We introduce the discrete divergence-free subspace as follows:

$$X_{h,0}(\Omega) = \{w_h\in X_h^0(\Omega): (\nabla\cdot w_h,q_h) = 0, \forall q_h\in Q_h^0(\Omega)\}.$$

Moreover, the following estimates hold [21,22]

$$\|\nabla\cdot w\|_{0,\Omega} \le \sqrt{2}\|\nabla w\|_{0,\Omega}, \forall w\in X(\Omega),$$

$$\inf_{\omega_h\in X_{h,0}(\Omega)}\|\nabla(w-\omega_h)\|_{0,\Omega} \le (1+\frac{1}{\beta})\inf_{z_h\in X_h^0(\Omega)}\|\nabla(w-z_h)\|_{0,\Omega}, \quad \forall w\in\widetilde{X}(\Omega).$$

A sequence of spaces $\{X_{h,0}(\Omega)\}$ associated with the triangulation$\mathcal{T}^h(\Omega)$ is said to have the optimal approximation property with respect to $\widetilde{X}(\Omega)$ if, for any $(\chi,\tau)\in(\widetilde{X}(\Omega)\cap[H^{s+1}(\Omega)]^2)\times(M(\Omega)\cap H^{s+1}(\Omega))$,there exist $\chi_h\in X_{h,0}(\Omega)$ and $\tau_h\in M_h^0(\Omega)$ such that [21,22], for $s=1,2$,

$$\|\nabla(\chi-\chi_h)\|_{0,\Omega} \lesssim h^s\|\chi\|_{s+1,\Omega}, \tag{2.13}$$

$$\|\nabla(\tau-\tau_h)\|_{0,\Omega} \lesssim h^s\|\tau\|_{s+1,\Omega}. \tag{2.14}$$

With the above assumptions, the finite element discretization is defined as follows: seek $(u_h,p_h,T_h)\in X_h^0(\Omega)\times Q_h^0(\Omega)\times M_h^0(\Omega)$ such that, for all $(w_h,q_h,\psi_h)\in X_h^0(\Omega)\times Q_h^0(\Omega)\times M_h^0(\Omega)$,

$$\begin{cases} Pr(\nabla u_h,\nabla w_h) + b_1(u_h,u_h,w_h) - (\nabla\cdot w_h,p_h) \\ \quad + (\nabla\cdot u_h,q_h) = PrRa(e_jT_h,w_h) + (f,w_h) \\ \quad \kappa(\nabla T_h,\nabla\psi_h) + b_2(u_h,T_h,\psi_h) = (g,\psi_h). \end{cases} \quad (2.15),$$

The following result for problem (2.15)follows from arguments similar to those in[1,3,21] .

Theorem 2.2. Assume that A1-A2,A4 and the conditions of Theorem2.1 hold. Then problem (2.15) admits at least one solution $(u_h,p_h,T_h)\in X_h^0(\Omega)\times Q_h^0(\Omega)\times M_h^0(\Omega)$ satisfying

$$\|\nabla T_h\|_{0,\Omega} \le \kappa^{-1}\|g\|_{-1,\Omega}, \quad \|\nabla u_h\|_{0,\Omega} \le \widetilde{A}. \tag{2.16}$$

III. VMS STABILIZED METHOD BASED ON TWO LOCAL GAUSS INTEGRATION RULES

To stabilize both the velocity and temperature equations, we define the VMS terms based on two local Gauss integration rules by

$$G_u(u_h,w_h) = \alpha_u\sum_{K\in\mathcal{T}^h(\Omega)}\left(\int_{K,k}\nabla u_h:\nabla w_h\,dx - \int_{K,1}\nabla u_h:\nabla w_h\,dx\right) \tag{3.1}$$

$G_T(T_h,\psi_h) =$

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Volume 10, Issue 7, pp. 71-79, July 2026. ISSN (Online): 2456-7361

$$\alpha_T \sum_{K\in T^h(\Omega)} \left(\int_{K,k} \nabla T_h \cdot \nabla\psi_h\,dx - \int_{K,1} \nabla T_h \cdot \nabla\psi_h\,dx\right) \quad (3.2)$$

Here, $\alpha_u > 0$ and $\alpha_T > 0$ are user-selected stabilization parameters, typically chosen as $\alpha_u = \alpha_T = O(h^s)$, $s = 1,2$. The symbols $\int_{K,k}$ and $\int_{K,1}$ denote the quadratic-exact and one-point centroid Gauss rules on K, respectively. Define the piecewise constant space $R_0(\Omega) = \{\eta \in L^2(\Omega): \eta|_K \in P_0(K), \forall K \in T^h(\Omega)\}$, and let $\Pi_h^u: [L^2(\Omega)]^{2\times 2} \longrightarrow [R_0(\Omega)]^{2\times 2}$, $\Pi_h^T: [L^2(\Omega)]^2 \longrightarrow [R_0(\Omega)]^2$ be the elementwise L^2-orthogonal projections satisfying

$$(\Pi_h^u\tau, \chi_h) = (\tau, \chi_h), \forall\tau \in [L^2(\Omega)]^{2\times 2}, \forall\chi_h \in [R_0(\Omega)]^{2\times 2},$$
$$(\Pi_h^T\zeta, \eta_h) = (\zeta, \eta_h), \forall\zeta \in [L^2(\Omega)]^2, \forall\eta_h \in [R_0(\Omega)]^2. \quad (3.3)$$

The projection operators defined by (3.3)satisfy

$$\|(I-\Pi_h^u)\nabla v\|_{0,\Omega} \lesssim \|\nabla v\|_0, \|(I-\Pi_h^T)\nabla\psi\|_{0,\Omega} \lesssim \|\nabla\psi\|_0 \quad (3.4)$$
$$\|(I-\Pi_h^u)\nabla v\|_{0,\Omega} \lesssim h\|v\|_2, \|(I-\Pi_h^T)\nabla\psi\|_{0,\Omega} \lesssim h\|\psi\|_2. \quad (3.5)$$

Here,(3.4) holds for $v \in X(\Omega)$ and $\psi \in M(\Omega)$, whereas (3.5) holds for $v \in [H^2(\Omega)]^2 \cap X(\Omega)$ and $\psi \in H^2(\Omega) \cap M(\Omega)$.

For the $P_2 - P_1 - P_2$ finite element combination,the discrete gradients are affine on each element. Hence, the exactness of the two quadrature rules and the L^2 orthogonality imply[4,8]

$$G_u(u_h, w_h) = \alpha_u((I-\Pi_h^u)\nabla u_h, (I-\Pi_h^u)\nabla w_h)$$
$$= \alpha_u((I-\Pi_h^u)\nabla u_h, \nabla w_h), \quad (3.6)$$
$$G_T(T_h, \psi_h) = \alpha_T((I-\Pi_h^T)\nabla T_h, (I-\Pi_h^T)\nabla\psi_h)$$
$$= \alpha_T((I-\Pi_h^T)\nabla T_h, \nabla\psi_h). \quad (3.7)$$

The quadrature forms (3.1)-(3.2) are used in the implementation, whereas the equivalent projection forms (3.6)-(3.7) are employed in the analysis.

The VMS stabilized finite element method for the steady natural convection equations reads: find $(u_h, p_h, T_h) \in X_h^0(\Omega) \times Q_h^0(\Omega) \times M_h^0(\Omega)$ such that for all $(w_h, q_h, \psi_h) \in X_h^0(\Omega) \times Q_h^0(\Omega) \times M_h^0(\Omega)$,

$$\begin{cases} Pr(\nabla u_h, \nabla w_h) + b_1(u_h, u_h, w_h) - (\nabla\cdot w_h, p_h) + \\ (\nabla\cdot u_h, q_h) + G_u(u_h, w_h) = PrRa(e_jT_h, w_h) + (f, w_h), \\ \kappa(\nabla T_h, \nabla\psi_h) + b_2(u_h, T_h, \psi_h) + G_T(T_h, \psi_h) = (g, \psi_h). \end{cases} \quad (3.8)$$

Theorem 3.1. Problem (3.8) admits at least one solution $(u_h, p_h, T_h) \in X_h^0(\Omega) \times Q_h^0(\Omega) \times M_h^0(\Omega)$ satisfying

$$\|\nabla T_h\|_{0,\Omega} \le \kappa^{-1}\|g\|_{-1,\Omega}, \|\nabla u_h\|_{0,\Omega} \le \tilde{A}. \quad (3.9)$$

Moreover, the discrete solution is unique provided that

$$Pr + \alpha_u - N_1\tilde{A} - \tilde{C}PrRaN_2\kappa\|g\|_{-1,\Omega} > 0. \quad (3.10)$$

IV. Error analysis

4.1 Energy-norm error estimates

Theorem 4.1. *Suppose that* (u, p, T) is a nonsingular solution of the natural convection problem and *, for* $s = 1,2 (u, p, T) \in [H^{s+1}(\Omega)]^2 \times H^s(\Omega) \times H^{s+1}(\Omega)$. Assume also that $\alpha_u \to 0$ and $\alpha_T \to 0$ as h→0 and that the uniqueness condition holds. Then the stabilized solution (u_h, p_h, T_h) of (3.8) satisfies

$$\|\nabla(u-u_h)\|_{0,\Omega} + \|p-p_h\|_{0,\Omega} + \|\nabla(T-T_h)\|_{0,\Omega}$$
$$\lesssim h^s + \alpha_u + \alpha_T, \qquad s = 1,2. \quad (4.1)$$

Proof. Subtract (3.8) from (2.5). The approximation, trilinear, inf--sup, and uniqueness estimates treat the standard terms as in [4]; only the VMS terms are recorded.Taking $w_h = u_h - \chi_h$, where $\chi_h \in X_{h,0}(\Omega)$ is the approximation of u, it follows from (3.4)-(3.7) that $G_u(u_h, w_h) = \alpha_u\|(I-\Pi_h^u)\nabla w_h\|_{0,\Omega}^2 + G_u(\chi_h, w_h)$, where $|G_u(\chi_h, w_h)| \lesssim \alpha_u\big(\|\nabla(u-\chi_h)\|_{0,\Omega} + \|(I-\Pi_h^u)\nabla u\|_{0,\Omega}\big)\|\nabla w_h\|_{0,\Omega} \lesssim \alpha_u\big(h^s\|u\|_{s+1,\Omega} + h\|u\|_{2,\Omega}\big)\|\nabla w_h\|_{0,\Omega}$. Similarly, taking $\psi_h = T_h - \tau_h$, where $\tau_h \in M_h^0(\Omega)$ is the approximation of T , we obtain $G_T(T_h, \psi_h) = \alpha_T\|(I-\Pi_h^T)\nabla\psi_h\|_{0,\Omega}^2 + G_T(\tau_h, \psi_h), and |G_T(\tau_h, \psi_h)| \lesssim \alpha_T\big(h^s\|T\|_{s+1,\Omega} + h\|T\|_{2,\Omega}\big)\|\nabla\psi_h\|_{0,\Omega}$. Young's inequality absorbs the last factors into the diffusion terms, and the standard argument yields (4.1).We finish the proof.

For the velocity and temperature L^2 errors and the pressure H^{-1} error, given $(\varphi_1, \varphi_2, \varphi_3) \in ([L^2(\Omega)]^2, Q(\Omega) \cap H^1(\Omega), L^2(\Omega))$, find $(\Psi_1, \Psi_2, \Psi_3) \in X(\Omega) \times Q(\Omega) \times M(\Omega)$ satisfying

$$\begin{cases} Pr(\nabla w, \nabla\Psi_1) + b_1(u_h, w, \Psi_1) + b_1(w, u_h, \Psi_1) + \\ (\nabla\cdot w, \Psi_2) - (\nabla\cdot\Psi_1, q) - PrRa(e_j\psi, \Psi_1) \\ = (\varphi_1, w) + (\varphi_2, q), \\ \kappa(\nabla\psi, \nabla\Psi_3) + b_2(u_h, \psi, \Psi_3) + b_2(w, T_h, \Psi_3) \\ = (\varphi_3, \psi) \end{cases} \quad (4.2)$$

Assume that (4.2) is H^2 -regular, so that $(\Psi_1, \Psi_2, \Psi_3) \in (X(\Omega) \cap [H^2(\Omega)]^2, Q(\Omega) \cap H^1(\Omega), M(\Omega) \cap H^2(\Omega))$ fulfill[20]

$$\|\Psi_1\|_{2,\Omega} + \|\Psi_2\|_{1,\Omega} + \|\Psi_3\|_{2,\Omega}$$
$$\lesssim \|\varphi_1\|_{0,\Omega} + \|\varphi_2\|_{1,\Omega} + \|\varphi_3\|_{0,\Omega}. \quad (4.3)$$

Its finite element solution $(\Psi_{1h}, \Psi_{2h}, \Psi_{3h}) \in (X_h^0(\Omega), Q_h^0(\Omega), M_h^0(\Omega))$ satisfies, for all (w, q, ψ) in the same discrete product space,

$$\begin{cases} Pr(\nabla w, \nabla\Psi_{1h}) + b_1(u_h, w, \Psi_{1h}) + b_1(w, u_h, \Psi_{1h}) \\ +(\nabla\cdot w, \Psi_{2h}) - (\nabla\cdot\Psi_{1h}, q) - PrRa(e_j\psi, \Psi_{1h}) \\ = (\varphi_1, w) + (\varphi_2, q) \\ \kappa(\nabla\psi, \nabla\Psi_{3h}) + b_2(u_h, \psi, \Psi_{3h}) + b_2(w, T_h, \Psi_{3h}) = (\varphi_3, \psi), \end{cases}$$

and obeys

$$\|\nabla(\Psi_1 - \Psi_{1h})\|_{0,\Omega} + \|\Psi_2 - \Psi_{2h}\|_{0,\Omega} + \|\nabla(\Psi_3 - \Psi_{3h})\|_{0,\Omega}$$
$$\lesssim h(\|\varphi_1\|_{0,\Omega} + \|\varphi_2\|_{1,\Omega} + \|\varphi_3\|_{0,\Omega}), \quad (4.4)$$
$$\|\nabla\Psi_{1h}\|_{0,\Omega} + \|\Psi_{2h}\|_{0,\Omega} + \|\nabla\Psi_{3h}\|_{0,\Omega}$$
$$\lesssim \|\varphi_1\|_{0,\Omega} + \|\varphi_2\|_{1,\Omega} + \|\varphi_3\|_{0,\Omega}. \quad (4.5)$$

4.2 Lower-order norm estimates

Theorem 4.2. Assume that the hypotheses of Theorem4.1 hold and that the linearized dual problem (4.2) is H^2-regular. Then,

$$\|u-u_h\|_{0,\Omega} + \|p-p_h\|_{-1,\Omega} + \|T-T_h\|_{0,\Omega}$$
$$\lesssim h^{s+1} + \alpha_u h + \alpha_T h + \alpha_u^2 + \alpha_T^2, \qquad s = 1,2. \quad (4.6)$$

Proof. We subtract (3.8) from (2.5) and take $(w, q, \psi) = (\Psi_{1h}, \Psi_{2h}, \Psi_{3h})$ to get

$$\begin{cases} Pr(\nabla(u-u_h), \nabla\Psi_{1h}) + b_1(u_h, u-u_h, \Psi_{1h}) + \\ b_1(u-u_h, u_h, \Psi_{1h}) + b_1(u-u_h, u-u_h, \Psi_{1h}) - \\ (\nabla\cdot\Psi_{1h}, p-p_h) + (\nabla\cdot(u-u_h), \Psi_{2h}) \\ - G_u(u_h, \Psi_{1h}) - PrRa(e_j(T-T_h), \Psi_{1h}) = 0, \\ \kappa(\nabla(T-T_h), \nabla\Psi_{3h}) + b_2(u_h, T-T_h, \Psi_{3h}) \\ + b_2(u-u_h, T_h, \Psi_{3h}) + b_2(u-u_h, T-T_h, \Psi_{3h}) - \\ G_T(T_h, \Psi_{3h}) = 0. \end{cases}$$

This and (4.2) yield

$$(\varphi_1, u-u_h) + (\varphi_2, p-p_h) + (\varphi_3, T-T_h)$$
$$= Pr(\nabla(u-u_h), \nabla\Psi_1) + b_1(u_h, u-u_h, \Psi_1)$$
$$+ b_1(u-u_h, u_h, \Psi_1) + (\nabla\cdot(u-u_h), \Psi_2) - (\nabla\cdot\Psi_1, p-p_h)$$

$$
\begin{aligned}
&-PrRa(e_j(T-T_h),\Psi_1)+\kappa(\nabla(T-T_h),\nabla\Psi_3)+b_2(u_h,T\\
&\qquad -T_h,\Psi_3)+b_2(u-u_h,T_h,\Psi_3)\\
&=\underbrace{Pr(\nabla(u-u_h),\nabla(\Psi_1-\Psi_{1h}))}_{I_1}+\underbrace{b_1(u_h,u-u_h,\Psi_1-\Psi_{1h})}_{I_2}\\
&+\underbrace{b_1(u-u_h,u_h,\Psi_1-\Psi_{1h})}_{I_2}\,\underbrace{-b_1(u-u_h,u-u_h,\Psi_{1h})}_{I_3}\\
&-\underbrace{(\nabla\cdot(\Psi_1-\Psi_{1h}),p-p_h)}_{I_4}+\underbrace{(\nabla\cdot(u-u_h),\Psi_2-\Psi_{2h})}_{I_5}\\
&+\underbrace{G_u(u_h,\Psi_{1h})}_{I_6}\,\underbrace{-PrRa(e_j(T-T_h),\Psi_1-\Psi_{1h})}_{I_7}\\
&+\underbrace{\kappa(\nabla(T-T_h),\nabla(\Psi_3-\Psi_{3h}))}_{I_8}+\underbrace{b_2(u_h,T-T_h,\Psi_3-\Psi_{3h})}_{I_9}\\
&+\underbrace{b_2(u-u_h,T_h,\Psi_3-\Psi_{3h}}_{I_9}\,\underbrace{-b_2(u-u_h,T-T_h,\Psi_{3h})}_{I_{10}}\\
&+\underbrace{G_T(T_h,\Psi_{3h})}_{I_{11}}
\end{aligned}
\tag{4.7}
$$

Equations(2.1)-(2.4), (3.1)-(3.2),(3.4)-(3.7),and(4.3)-(4.5)give

$$
\begin{aligned}
&|\boldsymbol{I_1}|\le Pr\|\nabla(u-u_h)\|_{0,\Omega}\|\nabla(\Psi_1-\Psi_{1h})\|_{0,\Omega}\\
&\lesssim h\|\nabla(u-u_h)\|_{0,\Omega}\|\Psi_1\|_{2,\Omega}\\
&\lesssim h\|\nabla(u-u_h)\|_{0,\Omega}(\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}),\\
&|\boldsymbol{I_2}|\le 2N_1\|\nabla u_h\|_{0,\Omega}\|\nabla(u-u_h)\|_{0,\Omega}\|\nabla(\Psi_1-\Psi_{1h})\|_{0,\Omega}\\
&\lesssim h\|\nabla(u-u_h)\|_{0,\Omega}\|\Psi_1\|_{2,\Omega}\\
&\lesssim h\|\nabla(u-u_h)\|_{0,\Omega}(\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}),\\
&|\boldsymbol{I_3}|\le N_1\|\nabla(u-u_h)\|_{0,\Omega}^2\|\nabla\Psi_{1h}\|_{0,\Omega}\\
&\lesssim \|\nabla(u-u_h)\|_{0,\Omega}^2(\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}),\\
&|\boldsymbol{I_4}|\le \|\nabla(\Psi_1-\Psi_{1h})\|_{0,\Omega}\|p-p_h\|_{0,\Omega}\\
&\lesssim h\|p-p_h\|_{0,\Omega}\|\Psi_1\|_{2,\Omega}\\
&\lesssim h\|p-p_h\|_{0,\Omega}(\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}),\\
&|\boldsymbol{I_5}|\le \|\nabla(u-u_h)\|_{0,\Omega}\|\Psi_2-\Psi_{2h}\|_{0,\Omega}\\
&\lesssim h\|\nabla(u-u_h)\|_{0,\Omega}\|\Psi_2\|_{1,\Omega}\\
&\lesssim h\|\nabla(u-u_h)\|_{0,\Omega}(\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}).\\
&|\boldsymbol{I_6}|=|G_u(u_h-u,\Psi_{1h})+G_u(u,\Psi_{1h})|\\
&\le \alpha_u(\|(I-\Pi_h^u)\nabla(u-u_h)\|_{0,\Omega}+\\
&\|(I-\Pi_h^u)\nabla u\|_{0,\Omega})\|(I-\Pi_h^u)\nabla\Psi_{1h}\|_{0,\Omega}\\
&\lesssim \alpha_u(\|\nabla(u-u_h)\|_{0,\Omega}+h\|u\|_{2,\Omega})\|\nabla\Psi_{1h}\|_{0,\Omega}\\
&\lesssim \alpha_u(\|\nabla(u-u_h)\|_{0,\Omega}+h\|u\|_{2,\Omega})(\|\varphi_1\|_0+\|\varphi_2\|_1+\|\varphi_3\|_0),\\
&|\boldsymbol{I_7}|\le PrRa\|T-T_h\|_{0,\Omega}\|\Psi_1-\Psi_{1h}\|_{0,\Omega}\\
&\lesssim \|\nabla(T-T_h)\|_{0,\Omega}\|\nabla(\Psi_1-\Psi_{1h})\|_{0,\Omega}\\
&\lesssim h\|\nabla(T-T_h)\|_{0,\Omega}(\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}),\\
&|\boldsymbol{I_8}|\le \kappa\|\nabla(T-T_h)\|_{0,\Omega}\|\nabla(\Psi_3-\Psi_{3h})\|_{0,\Omega}\\
&\lesssim h\|\nabla(T-T_h)\|_{0,\Omega}\|\Psi_3\|_{2,\Omega}\\
&\lesssim h\|\nabla(T-T_h)\|_{0,\Omega}(\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}),\\
&|\boldsymbol{I_9}|\le N_2(\|\nabla u_h\|_{0,\Omega}\|\nabla(T-T_h)\|_{0,\Omega}\\
&+\|\nabla T_h\|_{0,\Omega}\|\nabla(u-u_h)\|_{0,\Omega})\|\nabla(\Psi_3-\Psi_{3h})\|_{0,\Omega}\\
&\lesssim h(\|\nabla(u-u_h)\|_{0,\Omega}+\|\nabla(T-T_h)\|_{0,\Omega})\|\Psi_3\|_{2,\Omega}\\
&\lesssim h(\|\nabla(u-u_h)\|_{0,\Omega}+\|\nabla(T-T_h)\|_{0,\Omega})\\
&(\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}),\\
&|\boldsymbol{I_{10}}|\le N_2\|\nabla(u-u_h)\|_{0,\Omega}\|\nabla(T-T_h)\|_{0,\Omega}\|\nabla\Psi_{3h}\|_{0,\Omega}\\
&\lesssim \|\nabla(u-u_h)\|_{0,\Omega}\|\nabla(T-T_h)\|_{0,\Omega}\\
&(\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}),\\
&|\boldsymbol{I_{11}}|=|G_T(T_h-T,\Psi_{3h})+G_T(T,\Psi_{3h})|\\
&\le \alpha_T(\|(I-\Pi_h^T)\nabla(T-T_h)\|_{0,\Omega}+\|(I-\Pi_h^T)\nabla T\|_{0,\Omega})\\
&\|(I-\Pi_h^T)\nabla\Psi_{3h}\|_{0,\Omega}\\
&\lesssim \alpha_T(\|\nabla(T-T_h)\|_{0,\Omega}+h\|T\|_{2,\Omega})\|\nabla\Psi_{3h}\|_{0,\Omega}\\
&\lesssim \alpha_T(\|\nabla(T-T_h)\|_0+h\|T\|_{2,\Omega})(\|\varphi_1\|_0+\|\varphi_2\|_1+\|\varphi_3\|_0).
\end{aligned}
$$

Substitution into (4.7), followed by (4.1) and Young's inequality, gives

$$
\begin{aligned}
&\|u-u_h\|_{0,\Omega}+\|p-p_h\|_{-1,\Omega}+\|T-T_h\|_{0,\Omega}\\
&=\sup_{\varphi_1\in[L^2(\Omega)]^2}\frac{|(\varphi_1,u-u_h)|}{\|\varphi_1\|_{0,\Omega}}+\sup_{\varphi_2\in Q(\Omega)\cap H^1(\Omega)}\frac{|(\varphi_2,p-p_h)|}{\|\varphi_2\|_{1,\Omega}}\\
&+\sup_{\varphi_3\in L^2(\Omega)}\frac{|(\varphi_3,T-T_h)|}{\|\varphi_3\|_{0,\Omega}}\lesssim\sup_{(\varphi_1,\varphi_2,\varphi_3)\in([L^2(\Omega)]^2,Q(\Omega)\cap H^1(\Omega),L^2(\Omega))}\\
&\frac{|(\varphi_1,u-u_h)+(\varphi_2,p-p_h)+(\varphi_3,T-T_h)|}{\|\varphi_1\|_{0,\Omega}+\|\varphi_2\|_{1,\Omega}+\|\varphi_3\|_{0,\Omega}}\\
&\lesssim h(\|\nabla(u-u_h)\|_{0,\Omega}+\|p-p_h\|_{0,\Omega}+\|\nabla(T-T_h)\|_{0,\Omega})\\
&+\|\nabla(u-u_h)\|_{0,\Omega}^2+\|\nabla(u-u_h)\|_{0,\Omega}\|\nabla(T-T_h)\|_{0,\Omega}\\
&+\alpha_u(\|\nabla(u-u_h)\|_0+h\|u\|_2)+\alpha_T(\|\nabla(T-T_h)\|_0+h\|T\|_2)\\
&\lesssim h^{s+1}+\alpha_u h+\alpha_T h+\alpha_u^2+\alpha_T^2,\qquad s=1,2.
\end{aligned}
$$

We finish the proof.

4.3 Local a priori estimates

We next record the local estimates for the composite-grid analysis.

Lemma 4.1[9,12,15]. Let $\varpi\in C_0^\infty(\Omega)$ satisfy $supp\varpi\subset\subset\Omega_0$. Then

$$
\begin{aligned}
&Pr\|\varpi\xi\|_{1,\Omega}^2\le Pr(\nabla\xi,\nabla(\varpi^2\xi))+cPr\|\xi\|_{0,\Omega_0}^2,\quad\forall\xi\in X(\Omega),\\
&\kappa\|\varpi\psi\|_{1,\Omega}^2\le\kappa(\nabla\psi,\nabla(\varpi^2\psi))+c\kappa\|\psi\|_{0,\Omega_0}^2,\forall\psi\in M(\Omega).
\end{aligned}
\tag{4.8}
$$

Lemma 4.2. Assume that $D\subset\subset\Omega_0\subset\Omega$, $F\in[H^{-1}(\Omega)]^2$, $G\in H^{-1}(\Omega)$, and $0<h\le h_0$. If the local stabilized solution $(\mu,r,z)\in X_h(\Omega)\times Q_h(\Omega)\times M_h(\Omega)$ is determined by

$$
\begin{cases}
Pr(\nabla\mu,\nabla w)+b_1(u_h,\mu,w)+b_1(\mu,u_h,w)\\
-(\nabla\cdot w,r)+(\nabla\cdot\mu,q)+G_u(\mu,w)=(F,w),\\
\qquad\forall(w,q)\in X_0^h(\Omega_0)\times Q_0^h(\Omega_0),\\
\kappa(\nabla z,\nabla\psi)+b_2(u_h,z,\psi)+b_2(\mu,T_h,\psi)+\\
\qquad G_T(z,\psi)=(G,\psi),\forall\psi\in M_0^h(\Omega_0).
\end{cases}
\tag{4.9}
$$

then

$$
\|\mu\|_{1,D}\lesssim(1+\alpha_u)\|\mu\|_{0,\Omega_0}+\|r\|_{-1,\Omega_0}+\|F\|_{-1,\Omega_0},\tag{4.10}
$$

$$
\|r\|_{0,D}\lesssim(1+\alpha_u)[(1+\alpha_u)\|\mu\|_{0,\Omega_0}+\|r\|_{-1,\Omega_0}+\|F\|_{-1,\Omega_0}],\tag{4.11}
$$

$$
\begin{aligned}
\|z\|_{1,D}\lesssim&(1+\alpha_T)[(1+\alpha_T)\|z\|_{0,\Omega_0}+(1+\alpha_u)\|\mu\|_{0,\Omega_0}\\
&+\|r\|_{-1,\Omega_0}+\|F\|_{-1,\Omega_0}+\|G\|_{-1,\Omega_0}].
\end{aligned}
\tag{4.12}
$$

Proof. The localization argument of [12,15] using (4.8) and (2.10)-(2.12), applies once the VMS terms are bounded. By (3.4)-(3.7), for $w\in X_0^h(\Omega_0)$ and $\psi\in M_0^h(\Omega_0)$, we have $|G_u(\mu,w)|\lesssim\alpha_u\|\nabla\mu\|_{0,\Omega_0}\|\nabla w\|_{0,\Omega_0}$, and $|G_T(z,\psi)|\lesssim\alpha_T\|\nabla z\|_{0,\Omega_0}\|\nabla\psi\|_{0,\Omega_0}$. Young's inequality and the nested subdomains $D\subset\subset D_1\subset\subset D_2\subset\subset\Omega_0$ yield(4.10)-(4.12). We finish the proof.

V. Local and Parallel Algorithms

On every composite grid, G_u and G_T are defined by (3.1)-(3.2) over its triangulation.

5.1 A local VMS stabilized method

Let $D\subset\subset\Omega_0\subset\Omega$ and $0<h\ll H<1$. The composite grid $T_H^h(\Omega)$ refines a coarse grid $T^H(\Omega)$ near D and coincides with the local fine grid $T^h(\Omega_0)$ on Ω_0. Denote its velocity, pressure, and temperature spaces by $X^0_{H,h}$, $Q^0_{H,h}$, and $M^0_{H,h}$.

The local VMS stabilized method seeks $(u^h_H, p^h_H, T^h_H) \in X^0_{H,h}(\Omega) \times Q^0_{H,h}(\Omega) \times M^0_{H,h}(\Omega)$ such that for $(w,q,\psi) \in X^0_{H,h}(\Omega) \times Q^0_{H,h}(\Omega) \times M^0_{H,h}(\Omega)$,

$$\begin{cases} Pr(\nabla u^h_H, \nabla w) + b_1(u^h_H, u^h_H, w) - (\nabla \cdot w, p^h_H) + (\nabla \cdot u^h_H, q) \\ \qquad + G_u(u^h_H, w) = PrRa(e_j T^h_H, w) + (f, w), \\ \kappa(\nabla T^h_H, \nabla \psi) + b_2(u^h_H, T^h_H, \psi) + G_T(T^h_H, \psi) = (g, \psi). \end{cases} \tag{5.1}$$

Thus (5.1) is globally posed but fine only near D.

Lemma 5.1. Under the conclusions of Theorems 4.1 and 4.2, $0 < H \le h_0$, and $D \subset\subset \Omega_0 \subset \Omega$, the solution $(u^h_H, p^h_H, T^h_H) \in X^0_{H,h}(\Omega) \times Q^0_{H,h}(\Omega) \times M^0_{H,h}(\Omega)$ obtained by the local VMS stabilized method fulfills the following estimates for $s = 1,2$:

$$\|u - u^h_H\|_{1,D} \lesssim h^s + H^{s+1} + \alpha_u + \alpha_T + \alpha_u^2 + \alpha_T^2 + \alpha_u^3 + \alpha_u \alpha_T^2, \tag{5.2}$$

$$\|p - p^h_H\|_{0,D} \lesssim h^s + H^{s+1} + \alpha_u + \alpha_T + \alpha_u^2 + \alpha_T^2 + \alpha_u^3 + \alpha_u \alpha_T^2 + \alpha_u^4 + \alpha_u^2 \alpha_T^2, \tag{5.3}$$

$$\|T - T^h_H\|_{1,D} \lesssim h^s + H^{s+1} + \alpha_u + \alpha_T + \alpha_u^2 + \alpha_T^2 + \alpha_u^3 + \alpha_T^3 + \alpha_u^4 + \alpha_T^4 + \alpha_T \alpha_u^3 + \alpha_u \alpha_T^3. \tag{5.4}$$

Proof. Since the auxiliary fine grid $T^h(\Omega)$ coincides with $T^h_H(\Omega)$ on Ω_0, subtracting (5.1) from (3.8)and simplifying the trilinear terms give,for$(w,q,\psi) \in X^h_0(\Omega_0) \times Q^h_0(\Omega_0) \times M^h_0(\Omega_0)$,

$$\begin{cases} Pr(\nabla(u_h - u^h_H), \nabla w) + b_1(u_h, u_h - u^h_H, w) + b_1(u_h - u^h_H, \\ \quad u_h, w) - (\nabla \cdot w, p_h - p^h_H) + (\nabla \cdot (u_h - u^h_H), q) + \\ \quad G_u(u_h - u^h_H, w) = (F, w), \\ \kappa(\nabla(T_h - T^h_H), \nabla \psi) + b_2(u_h, T_h - T^h_H, \psi) + b_2(u_h \\ \quad - u^h_H, T_h, \psi) + G_T(T_h - T^h_H, \psi) = (G, \psi), \end{cases}$$

where

$$(F, w) = PrRa(e_j(T_h - T^h_H), w) + b_1(u_h - u^h_H, u_h - u^h_H, w),$$
$$(G, \psi) = b_2(u_h - u^h_H, T_h - T^h_H, \psi).$$

According to the definitions of the negative norms and(2.1)-(2.2), we have

$$\|F\|_{-1,\Omega_0} \le N_1 \|\nabla(u_h - u^h_H)\|^2_{0,\Omega_0} + C\, Pr\, Ra \|T_h - T^h_H\|_{0,\Omega_0},$$
$$\|G\|_{-1,\Omega_0} \le N_2 \|\nabla(u_h - u^h_H)\|_{0,\Omega_0} \|\nabla(T_h - T^h_H)\|_{0,\Omega_0}.$$

By the triangle inequality,

$$\|u - u^h_H\|_{1,D} \le \|u - u_h\|_{1,D} + \|u_h - u^h_H\|_{1,D}.$$

Bound the first term by(4.1)and the second by Lemma4.2. The composite-grid analogue of Theorem4.2 follows by the same duality argument [15]. Together with(4.6),these estimates give (5.2)-(5.4).We finish the proof.

5.2 A parallel VMS stabilized method

Partition Ω into disjoint D_i and enlarge them to overlapping Ω_i, $D_i \subset\subset \Omega_i \subset \Omega$. The grid$T_i^{H,h}$ has size h on Ω_i and H elsewhere, with spaces $X^{0,i}_{H,h}$, $Q^{0,i}_{H,h}$, and$M^{0,i}_{H,h}$.

1. Seek $(u^h_{H,i}, p^h_{H,i}, T^h_{H,i}) \in X^{0,i}_{H,h}(\Omega) \times Q^{0,i}_{H,h}(\Omega) \times M^{0,i}_{H,h}(\Omega)$, $i = 1, \dots, I$, satisfying for $(w,q,\psi) \in X^{0,i}_{H,h}(\Omega) \times Q^{0,i}_{H,h}(\Omega) \times M^{0,i}_{H,h}(\Omega)$,

$$\begin{cases} Pr(\nabla u^h_{H,i}, \nabla w) + b_1(u^h_{H,i}, u^h_{H,i}, w) - (\nabla \cdot w, p^h_{H,i}) \\ +(\nabla \cdot u^h_{H,i}, q) + G_u(u^h_{H,i}, w) = PrRa(e_j T^h_{H,i}, w) + (f, w), \\ \kappa(\nabla T^h_{H,i}, \nabla \psi) + b_2(u^h_{H,i}, T^h_{H,i}, \psi) + G_T(T^h_{H,i}, \psi) = (g, \psi), \end{cases} \tag{5.5}$$

2.$Set(u^h, p^h, T^h) = (u^h_{H,i}, p^h_{H,i}, T^h_{H,i}) in D_i (i = 1,2, \dots, I)$.

Define the piecewise norms

$$||| \cdot |||_{1,\Omega} = \left(\sum_{i=1}^I \| \cdot \|^2_{1,D_i}\right)^{1/2},$$
$$||| \cdot |||_{0,\Omega} = \left(\sum_{i=1}^I \| \cdot \|^2_{0,D_i}\right)^{1/2}.$$

Theorem 5.1. Assume that the hypotheses of Lemma 5.1 hold uniformly for $i = 1, \dots, I$ and that I is fixed independently of h and H. Then

$$\begin{aligned} &|||u - u^h|||_{1,\Omega} + |||p - p^h|||_{0,\Omega} + |||T - T^h|||_{1,\Omega} \\ &\lesssim h^s + H^{s+1} + \alpha_u + \alpha_T + \alpha_u^2 + \alpha_T^2 + \alpha_u^3 + \alpha_T^3 \\ &+ \alpha_u^4 + \alpha_T^4 + \alpha_u \alpha_T^2 + \alpha_u^2 \alpha_T^2 + \alpha_T \alpha_u^3 + \alpha_u \alpha_T^3, \quad s = 1,2. \end{aligned} \tag{5.6}$$

Proof. Apply Lemma 5.1 on every target subdomain D_i, square the local bounds, sum over i, and use that the number of subdomains is independent of the mesh sizes.We finish the proof.

5.3 Parallel stabilized partition of unity method

Because (u^h, p^h, T^h) is generally discontinuous across target interfaces, we use a partition-of-unity reconstruction [11,16].

For each $i = 1, \dots, I$, choose an open set B_i such that$D_i \subset B_i \subset\subset \Omega_i$ and $\{B_i\}_{i=1}^I$ is an open cover of Ω. Let $\{\chi_i\}_{i=1}^I$ be a partition of unity subordinate to this cover satisfying

$$supp \chi_i \subset \overline{B_i}, \quad \sum_{i=1}^I \chi_i = 1 \quad in \Omega,$$
$$\|\chi_i\|_{\infty,\Omega} \le 1, \qquad \|\nabla \chi_i\|_{\infty,\Omega} \lesssim \frac{1}{diam(B_i)}, i = 1, \dots, I.$$

The layer$\Omega_i \setminus B_i$provides the oversampling required by Lemma 5.1.

For collections $\{w_i\}_{i=1}^I \subset [H^1(\Omega)]^2, \{\psi_i\}_{i=1}^I \subset H^1(\Omega)$, and $\{q_i\}_{i=1}^I \subset L^2(\Omega)$,

$$\begin{aligned} \left\| \sum_{i=1}^I \chi_i w_i \right\|^2_{1,\Omega} &\lesssim \sum_{i=1}^I \| \chi_i w_i \|^2_{1,B_i} \\ \left\| \sum_{i=1}^I \chi_i \psi_i \right\|^2_{1,\Omega} &\lesssim \sum_{i=1}^I \| \chi_i \psi_i \|^2_{1,B_i}, \\ \left\| \sum_{i=1}^I \chi_i q_i \right\|^2_{0,\Omega} &\lesssim \sum_{i=1}^I \| \chi_i q_i \|^2_{0,B_i}. \end{aligned} \tag{5.7}$$

Moreover,$\|\chi_i w_i\|_{1,B_i} \lesssim \|w_i\|_{1,B_i}$, $\|\chi_i \psi_i\|_{1,B_i} \lesssim \|\psi_i\|_{1,B_i}$, $\|\chi_i q_i\|_{0,B_i} \le \|q_i\|_{0,B_i}$.

Numerically, choose χ_i as continuous piecewise linear functions on an auxiliary grid $K^{H_P}(\Omega)$, $where\ 0 < h < H \le H_P < 1$, satisfying

$$\chi_i(x_k) = \delta_{ik} = \begin{cases} 1, i = k, \\ 0, i \ne k, \end{cases} \quad x_k \in K^{H_P}(\Omega).$$

The PU--VMS algorithm is:

1.For each $i = 1, \dots, I$, compute$(u^h_{H,i}, p^h_{H,i}, T^h_{H,i})$ from the

International Journal of Scientific Engineering and Science
Volume 10, Issue 7, pp. 71-79, July 2026. ISSN (Online): 2456-7361

independent composite-grid problem (5.5).

2.Update$(\tilde{u}_i^h, \tilde{p}_i^h, \tilde{T}_i^h) = (u_{H,i}^h, p_{H,i}^h, T_{H,i}^h) in B_i (i = 1, ..., I)$.

3.Construct the globally continuous approximation by

$$(u_{\mathrm{PU}}^h, p_{\mathrm{PU}}^h, T_{\mathrm{PU}}^h) = (\sum_{i=1}^{I} \chi_i \tilde{u}_i^h, p_{\mathrm{PU}}^h - \frac{1}{|\Omega|}\int_\Omega \tilde{p}_{\mathrm{PU}}^h\, dx, \sum_{i=1}^{I} \chi_i \tilde{T}_i^h), \tilde{p}_{\mathrm{PU}}^h = \sum_{i=1}^{I} \chi_i \tilde{p}_i^h. \quad (5.8)$$

The pressure correction in (5.8) ensures that $p_{\mathrm{PU}}^h \in Q(\Omega)$.

The composite-grid problems are independent; communication is needed only for the final reconstruction.

Theorem 5.2. Assume that Lemma5.1 holds with $D = B_i$ and $\Omega_0 = \Omega_i$ for $i = 1, ..., I$ and that the above partition-of-unity estimates hold uniformly.Then the approximation defined by (5.8) satisfies

$$\|u - u_{\mathrm{PU}}^h\|_{1,\Omega} + \|p - p_{\mathrm{PU}}^h\|_{0,\Omega} + \|T - T_{\mathrm{PU}}^h\|_{1,\Omega}$$
$$\lesssim h^s + H^{s+1} + \alpha_u + \alpha_T + \alpha_u^2 + \alpha_T^2 + \alpha_u^3 + \alpha_T^3 + \alpha_u^4 + \alpha_T^4 + \alpha_u \alpha_T^2 + \alpha_u^2 \alpha_T^2 + \alpha_T \alpha_u^3 + \alpha_u \alpha_T^3, \quad s = 1,2. \quad (5.9)$$

Proof. By the definition of the partition of unity,

$$u = \sum_{i=1}^{I} \chi_i u, \quad p = \sum_{i=1}^{I} \chi_i p, \quad T = \sum_{i=1}^{I} \chi_i T.$$

Therefore, using (5.7),

$$\|u - u_{\mathrm{PU}}^h\|_{1,\Omega} + \|p - \tilde{p}_{\mathrm{PU}}^h\|_{0,\Omega} + \|T - T_{\mathrm{PU}}^h\|_{1,\Omega}$$
$$= \left\|\sum_{i=1}^{I} \chi_i (u - \tilde{u}_i^h)\right\|_{1,\Omega} + \left\|\sum_{i=1}^{I} \chi_i (p - \tilde{p}_i^h)\right\|_{0,\Omega} + \left\|\sum_{i=1}^{I} \chi_i (T - \tilde{T}_i^h)\right\|_{1,\Omega}$$
$$\lesssim \left(\sum_{i=1}^{I} \| u - u_{H,i}^h\|_{1,B_i}^2\right)^{1/2} + \left(\sum_{i=1}^{I} \| p - p_{H,i}^h\|_{0,B_i}^2\right)^{1/2} + \left(\sum_{i=1}^{I} \| T - T_{H,i}^h\|_{1,B_i}^2\right)^{1/2}.$$

Since the zero-mean map is the L^2-orthogonal projection onto $Q(\Omega)$, $\|p - p_{\mathrm{PU}}^h\|_{0,\Omega} \le \|p - \tilde{p}_{\mathrm{PU}}^h\|_{0,\Omega}$. Applying the three estimates (5.2)-(5.4) on every $B_i$gives (5.9). This completes the proof.

We solve (5.5) by independent Oseen iterations. Given the previous iterate, find $(u_{H,i}^{h,n}, p_{H,i}^{h,n}, T_{H,i}^{h,n})$ such that, for all $(w, q, \psi) \in X_{H,h}^{0,i} \times Q_{H,h}^{0,i} \times M_{H,h}^{0,i}$,

$$\begin{cases} Pr(\nabla u_{H,i}^{h,n}, \nabla w) + b_1(u_{H,i}^{h,n-1}, u_{H,i}^{h,n}, w) - (\nabla \cdot w, p_{H,i}^{h,n}) \\ +(\nabla \cdot u_{H,i}^{h,n}, q) + G_u(u_{H,i}^{h,n}, w) = PrRa(e_j T_{H,i}^{h,n}, w) + (f, w) \\ \kappa(\nabla T_{H,i}^{h,n}, \nabla \psi) + b_2(u_{H,i}^{h,n-1}, T_{H,i}^{h,n}, \psi) + G_T(T_{H,i}^{h,n}, \psi) = (g, \psi), \end{cases} \quad (5.10)$$

The initial iterate solves the VMS Stokes--temperature problem:

$$\begin{cases} Pr(\nabla u_{H,i}^{h,0}, \nabla w) - (\nabla \cdot w, p_{H,i}^{h,0}) + (\nabla \cdot u_{H,i}^{h,0}, q) + \\ G_u(u_{H,i}^{h,0}, w) = PrRa(e_j T_{H,i}^{h,0}, w) + (f, w), \\ \kappa(\nabla T_{H,i}^{h,0}, \nabla \psi) + G_T(T_{H,i}^{h,0}, \psi) = (g, \psi). \end{cases} \quad (5.11)$$

The VMS terms use the composite-grid quadrature rules:

$$G_u(u_{H,i}^{h,n}, w) = \alpha_u \sum_{K \in T_i^{H,h}(\Omega)} (\int_{K,k} \nabla u_{H,i}^{h,n} : \nabla w\, dx - \int_{K,1} \nabla u_{H,i}^{h,n} : \nabla w\, dx), \quad (5.12)$$

$$G_T(T_{H,i}^{h,n}, \psi) = \alpha_T \sum_{K \in T_i^{H,h}(\Omega)} (\int_{K,k} \nabla T_{H,i}^{h,n} \cdot \nabla \psi\, dx - \int_{K,1} \nabla T_{H,i}^{h,n} \cdot \nabla \psi\, dx). \quad (5.13)$$

The iteration is stopped when

$$E_{\mathrm{iter}} = \max_{1 \le i \le I} \frac{\|u_{H,i}^{h,n} - u_{H,i}^{h,n-1}\|_{0,\Omega}}{\|u_{H,i}^{h,n}\|_{0,\Omega}} < 10^{-6}. \quad (5.14)$$

Equations (5.10)-(5.14) define the implementation; after convergence, (5.8) assembles the global solution.

VI. NUMERICAL EXPERIMENTS

6.1 Convergence verification and comparison

On $\Omega = (0,1)^2$, consider the exact solution

$$u(x, y) = (u_1(x, y), u_2(x, y)) = (2x^2(x-1)^2 y(y-1)(2y-1), -2y^2(y-1)^2 x(x-1)(2x-1)),$$
$$p(x, y) = x^3 + y^3 - 0.5, \quad T(x, y) = u_1(x, y) + u_2(x, y),$$

with f and$g$ obtained from (1.1) and (1.3). We use $P_2 - P_1 - P_2$ elements, $\alpha_u = C_1 h^2$, and$\alpha_T = C_2 h^2$. For the four-subdomain test, $Pr = 0.1, Ra = 100, C_1 = C_2 = 0.1, H = h^{2/3}$, and $h = n^{-3}, n = 2,3,4,5$. In Table 1, Solve time is the maximum local equation-solve time and excludes error evaluation. The rates agree with (5.6).

Table 2 repeats the test with $C_1 = C_2 = 0$; all other settings and the timing definition are unchanged.

Table 3 gives the standard global VMS results on globally fine meshes with $C_1 = C_2 = 0.1$.

Table 4 reports global errors for four overlapping subdomains of width 0.125 with the preceding parameters. Solve time remains the maximum local equation-solve time; the additional reconstruction/error-evaluation times are $0.101, 1.109, 6.384, and\ 25.004$ seconds and are excluded from cross-table timing comparisons. All three rates are approximately second order.

6.2 Buoyancy-driven square cavity flow

On $\Omega=(0,1)^2$, impose no slip, $T=1$ and $T=0$ on the left and right walls, and adiabatic horizontal walls (Figure 1). We set $f=g=0$ and apply the homogeneous formulation to the lifted variable $T=(1-x)+\theta$.

For $Ra=10^4, 10^5$, take $Pr=0.71$, $\kappa=1$,h=1/128,$H=1/64$, overlap 0.125, and $\alpha_u=\alpha_T=10^5 h^2$ to stabilize this convection-dominated test.

Figure 2 shows stronger circulation and thinner vertical-wall thermal layers at $Ra=10^5$; pressure and temperature remain continuous across PU interfaces.

The centerline profiles in Figure 3 confirm larger velocity extrema and steeper wall-temperature gradients at $Ra=10^5$.

On the hot wall,$Nu(y)=-\frac{\partial T}{\partial x}\Big|_{x=0}$ and$Nu_{\mathrm{ave}}=\int_0^1 Nu(y)\, dy$; Nu_{max} and Nu_{min} are its extrema. The cold-wall definition is analogous.

Figure 4 shows the increased wall heat transfer at $Ra=10^5$.

Table 5 compares centerline velocity maxima and hot-wall Nusselt quantities with the reference results in [14,17-19]

All five quantities differ from the de Vahl Davis benchmarks [17] by less than about 0.22%.

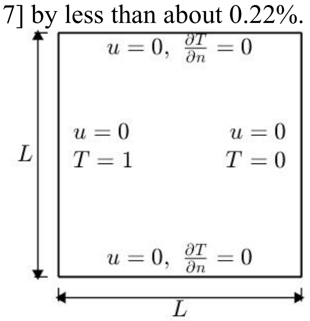

Fig. 1. Physical model and boundary conditions for the differentially heated natural convection cavity

VII. CONCLUSION

The proposed two-local-Gauss PU-VMS method admits stable, concurrent composite-grid solves and a continuous reconstruction with global and local a priori bounds. The $P_2 - P_1 - P_2$ experiments confirm the predicted rates and accurate cavity-flow quantities.

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Table 1: Results of the four-subdomain parallel composite-grid VMS finite element method with C1 = C2 = 0.1, Pr = 0.1, Ra = 100, and κ = 1.

h	H	m	$\|\|\|u - u^h\|\|\|_{1,\Omega}$	u_{H^1}-rate	$\|\|\|p - p^h\|\|\|_{0,\Omega}$	p_{L^2}-rate	$\|\|\|T - T^h\|\|\|_{1,\Omega}$	T_{H^1}-rate	Solve time
1/8	1/4	4	2.31534e−02	-	4.90522e−03	-	3.36428e−03	-	0.060
1/27	1/9	3	1.36113e−03	2.32969	2.58708e−04	2.41891	1.81707e−04	2.39937	0.572
1/64	1/16	3	2.06933e−04	2.18259	4.29048e−05	2.08183	3.05372e−05	2.06646	3.796
1/125	1/25	2	5.50185e−05	1.97888	1.10293e−05	2.02923	7.86625e−06	2.02615	12.783

Table 2: Results of the four-subdomain parallel composite-grid finite element method without VMS stabilization with Pr = 0.1, Ra = 100, and κ = 1.

h	H	m	$\|\|\|u - u^h\|\|\|_{1,\Omega}$	u_{H^1}-rate	$\|\|\|p - p^h\|\|\|_{0,\Omega}$	p_{L^2}-rate	$\|\|\|T - T^h\|\|\|_{1,\Omega}$	T_{H^1}-rate	Solve time
1/8	1/4	3	2.33305e−02	-	4.90517e−03	-	3.36398e−03	-	0.043
1/27	1/9	2	1.36188e−03	2.33551	2.58707e−04	2.41891	1.81709e−04	2.39928	0.445
1/64	1/16	2	2.06952e−04	2.18312	4.29048e−05	2.08183	3.05375e−05	2.06646	3.616
1/125	1/25	2	5.50200e−05	1.97898	1.10293e−05	2.02923	7.86627e−06	2.02616	13.460

Table 3: Results of the standard global VMS finite element method with C1 = C2 = 0.1, Pr = 0.1, Ra = 100, and κ =1.

International Journal of Scientific Engineering and Science
Volume 10, Issue 7, pp. 71-79, July 2026. ISSN (Online): 2456-7361

h	m	$\|\|u - u_h\|\|_{1,\Omega}$	u_{H^1}-rate	$\|\|p - p_h\|\|_{0,\Omega}$	p_{L^2}-rate	$\|\|T - T_h\|\|_{1,\Omega}$	T_{H^1}-rate	Solve time
1/8	4	3.88371e−03	-	2.86365e−03	-	1.52040e−03	-	0.104
1/27	3	2.44659e−04	2.27285	2.50537e−04	2.00284	1.37797e−04	1.97382	1.417
1/64	3	4.16196e−05	2.05238	4.45768e−05	2.00035	2.45920e−05	1.99683	10.786
1/125	2	1.08249e−05	2.01175	1.16850e−05	2.00007	6.44955e−06	1.99933	37.471

Table 4: Results of the final four-subdomain PU–VMS method with C1 = C2 = 0.1, Pr = 0.1, Ra = 100, and κ = 1.

h	H	m	$\|\|u - u_{PU}^h\|\|_{1,\Omega}$	u_{H^1}-rate	$\|\|p - p_{PU}^h\|\|_{0,\Omega}$	p_{L^2}-rate	$\|\|T - T_{PU}^h\|\|_{1,\Omega}$	T_{H^1}-rate	Solve time
1/8	1/4	2	1.41448e−02	-	2.90499e−03	-	2.11966e−03	-	0.048
1/27	1/9	2	1.05777e−03	2.13186	2.19574e−04	2.12307	1.59580e−04	2.12634	0.615
1/64	1/16	2	1.98269e−04	1.93998	3.98482e−05	1.97743	2.88652e−05	1.98125	4.773
1/125	1/25	2	4.75829e−05	2.13189	9.84116e−06	2.08909	7.21727e−06	2.07065	20.068

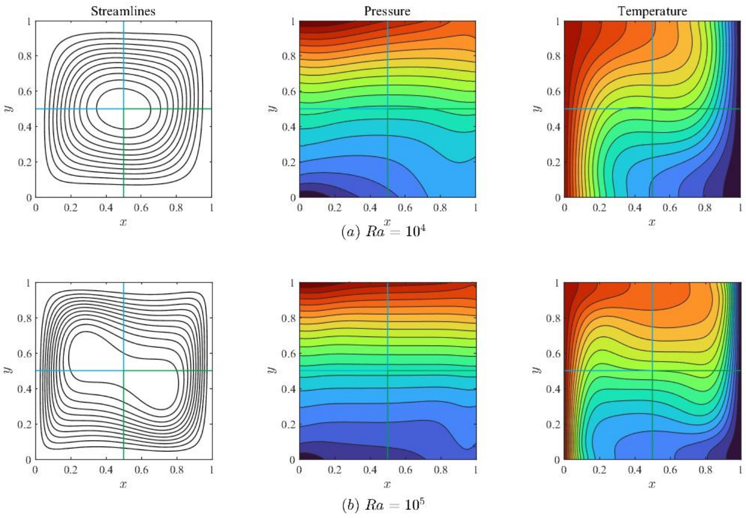

Fig. 2. Streamlines (left), pressure contours (center), and temperature contours (right) of the present PU–VMS approximation for the differentially heated square cavity at (a) Ra = 104 and (b) Ra = 105 . The colored lines indicate the interfaces of the four partition-of-unity subdomains.

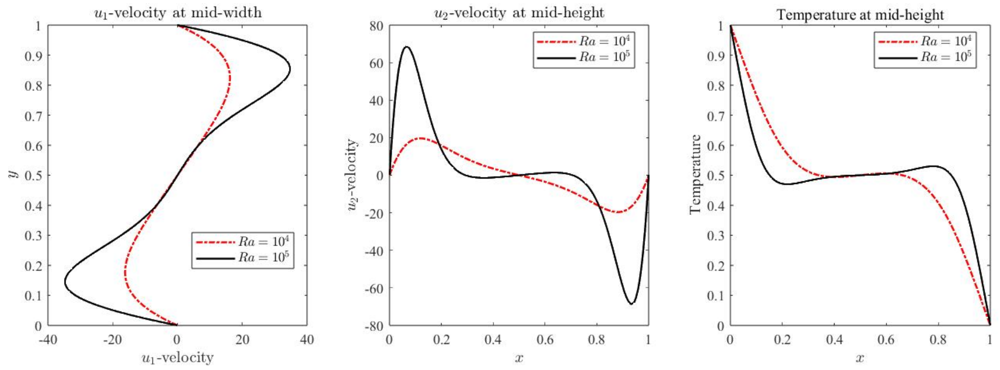

Figure 3: Profiles of the u1-velocity along the vertical centerline x = 0.5 (left), the u2-velocity along the horizontal centerline y = 0.5 (center), and the temperature along y = 0.5 (right) for the differentially heated square cavity at different Rayleigh numbers.

International Journal of Scientific Engineering and Science
Volume 10, Issue 7, pp. 71-79, July 2026. ISSN (Online): 2456-7361

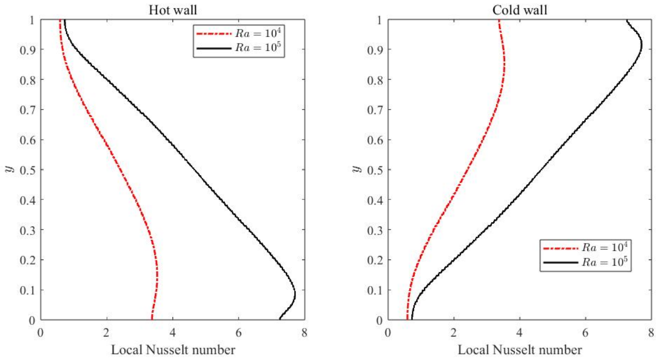

Figure 4: Local Nusselt-number distributions along the hot wall (left) and cold wall (right) of the differentially heated square cavity at different Rayleigh numbers.

Table 5: Comparison of the calculated u1,max, u2,max, Numax, Numin, and Nuave for buoyancy-driven square cavity flow at different Rayleigh numbers.

Method	Ra	$u_{1,\max}$	$u_{2,\max}$	$Nu_{\max}$	$Nu_{\min}$	Nu_{ave}
Present	10^4	16.183	19.6293	3.52863	0.584829	2.24285
	10^5	34.722	68.4802	7.70436	0.727469	4.51263
Shang [14]	10^4	16.1804	19.6257	3.5309	0.585061	-
	10^5	34.7276	68.5124	7.71946	0.728235	-
de Vahl Davis [17]	10^4	16.178	19.617	3.528	0.586	2.243
	10^5	34.73	68.59	7.717	0.729	4.519
Mayne et al. [18]	10^4	16.1798	19.6177	3.5305	0.585	2.2593
	10^5	34.7741	68.692	7.7084	0.7282	4.4832
Wan et al. [19]	10^4	16.122	19.79	3.579	0.577	2.254
	10^5	33.39	70.63	7.945	0.698	4.598