International Journal of Scientific Engineering and Science
Volume 10, Issue 6, pp. 31-39, June 2026. ISSN (Online): 2456-7361

# Two-Grid Method for Fourth-Order Eigenvalue Problems with Periodic Boundary Conditions Based on Ciarlet-Raviart Mixed Finite Elements

Cai Zhou[1], Yuxiang Gao[2], Sainan Huang[3]
[1]School of Mathematical Sciences, Guizhou Normal University, Guiyang, Guizhou, China-550025
[2]School of Mathematical Sciences, Guizhou Normal University, Guiyang, Guizhou, China-550025
[3]School of Mathematical Sciences, Guizhou Normal University, Guiyang, Guizhou, China-550025

***Abstract*—*This paper proposes a two-grid shifted inverse iteration algorithm based on the Ciarlet-Raviart mixed finite element for fourth-order eigenvalue problems with periodic boundary conditions. By introducing auxiliary variables, the original fourth-order equation is transformed into a mixed variational formulation, and relevant mathematical theories for continuous and discrete problems are established within the framework of periodic Sobolev spaces. The algorithm first solves the mixed finite element eigenvalue problem on a coarse grid to obtain initial approximations of eigenpairs. Afterwards, a single-step shifted inverse iteration is adopted to solve the linear system on a fine grid, and high-precision approximate eigenvalues are obtained via Rayleigh quotient correction. Systematic error analysis indicates that the two-grid method achieves optimal convergence rates under appropriate regularity conditions of eigenfunctions, with optimal convergence for eigenfunctions under the first-order Sobolev norm and high-order convergence for eigenvalue errors. Numerical tests defined on a square periodic computational domain verify the theoretical findings, present the convergence trend of the first three eigenvalues and the approximation performance of eigenfunctions, and demonstrate that the developed algorithm drastically cuts computational expenses while retaining numerical accuracy.*

***Keywords*—** *Ciarlet-Raviart mixed variational formulation; eigenvalue approximation; error estimation; periodic boundary conditions.*

## I. INTRODUCTION

Fourth-order eigenvalue problems have a wide range of applications in elasticity, structural vibration, fluid mechanics and other fields. In particular, eigenvalue problems involving the biharmonic [2] operator play an important role in plate and shell theory. Due to the high-order nature of such problems and complex boundary conditions, analytical solutions are generally difficult to obtain, and numerical methods have become the main solution approaches. The finite element method [9] has become a research focus due to its flexibility and adaptability. Among them, the mixed finite element method reduces the order of equations by introducing auxiliary variables, which facilitates the construction of stable discrete schemes. The Ciarlet-Raviart mixed finite element method [3][10][11] is one of the classical approaches for dealing with biharmonic problems, with favorable numerical stability and convergence properties [14][15][16].

Periodic boundary conditions carry important practical significance in physical problems and are widely encountered in fields such as crystal structures, periodic composite materials, and photonic crystals. Compared with Dirichlet or Neumann boundary conditions, the treatment of periodic boundary conditions requires the framework of periodic Sobolev spaces, and their variational formulations and numerical discretizations exhibit particularities. However, existing studies on mixed finite element methods for fourth-order eigenvalue problems mostly focus on simply supported or clamped boundary conditions, and systematic theoretical analysis under periodic boundary conditions [13] remains insufficient. On the other hand, traditional mixed finite element methods incur high computational costs when directly solving eigenvalue problems on fine meshes, which limits their application to large-scale problems. To improve efficiency, various multigrid-based eigenvalue solution strategies have been developed in recent years, such as the two-grid method [7][8][12] and the multigrid method [5][6]. Their core idea is to transform nonlinear eigenvalue problems into linear system solutions, which significantly reduces computational complexity while maintaining accuracy.

In this paper, under periodic boundary conditions, combining the Ciarlet-Raviart mixed finite element and shifted inverse iteration techniques, a two-grid discretization-based shifted inverse iteration algorithm is systematically studied for solving fourth-order eigenvalue problems with periodic boundary conditions. Compared with existing works, the main contributions of this paper are as follows:

- A complete mathematical analysis framework for fourth-order eigenvalue problems under periodic boundary conditions is established, including the definition of periodic Sobolev spaces, the Ciarlet-Raviart mixed variational formulation and its finite element discretization;
- A two-grid algorithm combining coarse-grid initial approximation and fine-grid one-step correction is proposed, which significantly reduces computational cost while ensuring accuracy;
- A systematic error estimation theory is constructed, and the optimal convergence orders of eigenfunctions and eigenvalues are proved by introducing continuous and discrete solution operators;
- Numerical experiments verify the effectiveness of the algorithm and the theoretical results, showing the

Volume 10, Issue 6, pp. 31-39, June 2026. ISSN (Online): 2456-7361

convergence behavior of the first three eigenvalues and the numerical results of eigenfunctions.

The rest of the paper is organized as follows. Section 2 introduces the mathematical modeling of fourth-order eigenvalue problems under periodic boundary conditions and the Ciarlet-Raviart mixed finite element discretization, gives the definitions and approximation properties of continuous and discrete solution operators, establishes the solvability theory of the source problem, and derives error estimates for eigenvalues and eigenfunctions. Section 3 describes in detail the two-grid discretization scheme based on shifted inverse iteration, presents the algorithmic steps, analyzes the convergence of the algorithm via lemmas and theorems, and proves the optimal convergence orders of eigenvalues and eigenfunctions. Section 4 verifies the effectiveness and convergence orders of the proposed method through numerical experiments, shows the numerical results and errors of the first three eigenvalues under different mesh sizes, and presents visualizations of the eigenfunctions. Section 5 summarizes the work of this paper and gives an outlook on future research directions.

## II. PRELIMINARIES

Consider the fourth-order eigenvalue problem on a domain $\Omega \subset \mathbb{R}^2$:

$$\Delta^2 \omega = \xi\omega \quad \text{in } \Omega \tag{2.1}$$

Where $\xi$ is the eigenvalue and ω is the corresponding eigenfunction. To satisfy physical periodicity, the function ω is required to satisfy the following periodic boundary conditions:

$$\omega(x, y) = \omega(x + L_x, y)$$
$$\omega(x, y) = \omega(x, y + L_y), \quad (x, y) \in \Omega \tag{2.2}$$

Where $L_x > 0$ and $L_y > 0$ denote the periodic lengths of the domain in the x-direction and y-direction, respectively.

### *2.1 Ciarlet-Raviart Mixed Variational Formulation*

To reduce the order of the equation and facilitate numerical discretization, an auxiliary variable $\sigma = -\Delta\omega$ is introduced to transform the fourth-order equation (2.1) into the following mixed system:

$$\begin{cases} -\Delta\omega = \sigma & in\ \Omega \\ -\Delta\sigma = \xi\omega & in\ \Omega \end{cases} \tag{2.3}$$

Accordingly, the auxiliary variable σ is also required to satisfy periodic boundary conditions:

$$\sigma(x, y) = \sigma(x + L_x, y)$$
$$\sigma(x, y) = \sigma(x, y + L_y) \qquad (x, y) \in \Omega \tag{2.4}$$

To derive the variational formulation of this system, we first multiply the first equation by a test function $\psi \in H^1_{per}(\Omega)$ and integrate over Ω:

$$-\int_\Omega \Delta\omega\, \psi\, dx = \int_\Omega \sigma\, \psi\, dx$$

Applying Green's formula (the first Green identity):

$$-\int_\Omega \Delta\omega\, \psi\, dx = \int_\Omega \nabla\omega \cdot \nabla\psi\, dx - \int_{\partial\Omega} \frac{\partial\omega}{\partial n}\, \psi\, ds$$

Under periodic boundary conditions, the boundary terms on ∂Ω cancel each other. Specifically,Ω is a rectangular domain whose boundary consists of two pairs of parallel edges. Since $\omega$ and $\psi$ satisfy the periodic boundary conditions (2.2), their function values on each pair of opposite boundaries are equal respectively, while the outward normal derivatives have opposite directions but equal magnitudes on opposite boundaries. Therefore, the sum of the line integrals along the entire boundary $\partial\Omega$ vanishes. The boundary term thus disappears, yielding

$$\int_\Omega \nabla\omega \cdot \nabla\psi\, dx = \int_\Omega \sigma\, \psi\, dx \tag{2.5a}$$

Similarly, multiplying the second equation by a test function $\varphi \in H^1_{per}(\Omega)$ and applying Green's formula, we obtain:

$$\int_\Omega \nabla\sigma \cdot \nabla\varphi\, dx = \xi \int_\Omega \omega\, \varphi\, dx \tag{2.5b}$$

To write the above equations in a symmetric form, we define the following function spaces and bilinear forms:

$$H^1_{per}(\Omega) = \left\{ v \in H^1(\Omega) : v \text{ satisfies (2.2) and } \int_{\partial\Omega} v\, ds = 0 \right\}$$
$$L^2_{per}(\Omega) = \{ v \in L^2(\Omega) | v \text{ satisfies periodicity in (2.2)} \}$$
$$a(\sigma, \psi) = \int_\Omega \sigma\, \psi\, dx, \quad b(\psi, \omega) = -\int_\Omega \nabla\psi \cdot \nabla\omega\, dx$$

The original problem can then be written in the mixed variational formulation: find $(\xi, \sigma, \omega) \in \mathbb{R} \times H^1_{per}(\Omega) \times H^1_{per}(\Omega)$ such that $(\sigma, \omega) \neq (0,0)$ and

$$a(\sigma, \psi) + b(\psi, \omega) = 0, \quad \forall\psi \in H^1_{per}(\Omega) \tag{2.6}$$
$$b(\sigma, \varphi) = -\xi(\omega, \varphi), \quad \forall\varphi \in H^1_{per}(\Omega) \tag{2.7}$$

Where $(\cdot,\cdot)$ denotes the inner product on $L^2_{per}(\Omega)$.This is the Ciarlet–Raviart mixed variational formulation.

### *2.2 Source Problem and Solution Operators*

To study the eigenvalue problem using tools from functional analysis, we introduce the associated source problem and its solution operators. By reformulating the eigenvalue problem as an operator equation, its spectral properties and discrete approximations can be analyzed more clearly.

For a given $f \in L^2_{per}(\Omega)$,consider the source problem: find $(p, \eta) \in H^1_{per}(\Omega) \times H^1_{per}(\Omega)$ such that:

$$a(p, \psi) + b(\psi, \eta) = 0, \quad \forall\psi \in H^1_{per}(\Omega) \tag{2.8}$$
$$b(p, \varphi) = -(f, \varphi), \quad \forall\varphi \in H^1_{per}(\Omega) \tag{2.9}$$

Define the continuous solution operators $T, S: L^2_{per}(\Omega) \to H^1_{per}(\Omega)$ by

$$Tf = \eta, \quad Sf = p \tag{2.10}$$

Then the eigenvalue problem (2.6)–(2.7) can be rewritten in the equivalent operator form:

$$\xi T\omega = \omega, \quad \sigma = S(\xi\omega) \tag{2.11}$$

Under periodic boundary conditions, the operator $T$ can be shown to be self-adjoint and completely compact with respect to the inner product on $L^2_{per}(\Omega)$.

**Remark 2.1** (Solvability of the Source Problem)

For any $f \in L^2_{per}(\Omega)$, the source problem (2.8)–(2.9) admits a unique solution $(p, \eta) \in H^1_{per}(\Omega) \times H^1_{per}(\Omega)$. In fact, considering its corresponding fourth-order weak formulation, the existence and uniqueness of the solution follow from the Lax–Milgram theorem, and the regularity of the solution can be obtained from elliptic regularity theory.

### *2.3 Finite Element Discretization*

International Journal of Scientific Engineering and Science
Volume 10, Issue 6, pp. 31-39, June 2026. ISSN (Online): 2456-7361

Let $\{\pi_h\}$ be a family of shape-regular meshes. Define the periodic finite element space:

$$V_h = \{v \in C(\bar{\Omega}) \cap H^1_{per}(\Omega): v|_K \in P_m, \forall K \in \pi_h\}$$

Where $P_m$ denotes the space of polynomials of degree at most m with $m \geq 1$.

For given $f \in L^2_{per}(\Omega)$, find $(p_h, \eta_h) \in V_h \times V_h$ such that

$$a(p_h, \psi) + b(\psi, \eta_h) = 0, \quad \forall \psi \in V_h \tag{2.12}$$

$$b(p_h, \varphi) = -(f, \varphi), \quad \forall \varphi \in V_h \tag{2.13}$$

Define the discrete solution operators $T_h, S_h: L^2_{per}(\Omega) \to V_h$ by

$$T_h f = \eta_h, \quad S_h f = p_h \tag{2.14}$$

The corresponding discrete mixed variational formulation reads: find $(\xi_h, \sigma_h, \omega_h) \in \mathbb{R} \times V_h \times V_h$ such that $(\sigma_h, \omega_h) \neq (0,0)$ and

$$a(\sigma_h, \psi) + b(\psi, \omega_h) = 0 \ , \quad \forall \psi \in V_h \tag{2.15}$$

$$b(\sigma_h, \varphi) = -\xi_h(\omega_h, \varphi) \ , \quad \forall \varphi \in V_h \tag{2.16}$$

Using the discrete solution operators $T_h, S_h$, the discrete eigenvalue problem can be rewritten in the equivalent operator form:

$$\xi_h T_h \omega_h = \omega_h, \quad \sigma_h = S_h(\xi_h \omega_h) \tag{2.17}$$

Similar to the continuous case, the discrete operator $T_h$ can be shown to be self-adjoint and completely continuous with respect to the inner product on $L^2_{per}(\Omega)$[4].

*2.4 Eigenvalue Ordering and Eigenspaces*

According to the spectral theory of self-adjoint completely continuous operators, the eigenvalues of the continuous problem (2.6)–(2.7) can be ordered as：

$$0 < \xi_1 \leq \xi_2 \leq \cdots \leq \xi_k \leq \cdots \nearrow +\infty$$

The corresponding eigenfunctions $(\sigma_k, \omega_k)$ satisfy the orthonormality condition $(\omega_i, \omega_j) = \delta_{ij}$ The eigenvalues of the discrete problem (2.15)–(2.16) are similarly ordered as：

$$0 < \xi_{1,h} \leq \xi_{2,h} \leq \cdots \leq \xi_{k,h} \leq \cdots \nearrow +\infty$$

and the corresponding discrete eigenfunctions $(\sigma_{k,h}, \omega_{k,h})$ satisfy $(\omega_{i,h}, \omega_{j,h}) = \delta_{ij}$.

For convenience in subsequent discussions, we introduce the notation：

$$\mu_k = \frac{1}{\xi_k}, \quad \mu_{k,h} = \frac{1}{\xi_{k,h}}$$

which denote the eigenvalues of the continuous and discrete solution operators, respectively.Let the algebraic multiplicity of the target eigenvalue $\xi$ be $q$, and denote its continuous eigenspace by：

$$M(\xi) = span\{\omega_j\}_{j=k}^{k+q-1} \subset H^{m+1}(\Omega)$$

The discrete eigenspace $M_h(\xi)$ is defined analogously, i.e., spanned by all discrete eigenfunctions converging to $\xi$.

*2.5 Approximation Properties of Operators*

To analyze the approximation properties of discrete operators and lay a foundation for subsequent eigenvalue error estimates, this section establishes a priori error estimates for finite element solutions. Let $f \in L^2_{per}(\Omega)$, and assume that the continuous solution $(p, \eta) = (Sf, Tf) \in H^{m+1}(\Omega) \times H^{m+1}(\Omega)$ and the discrete solution $(p_h, \eta_h) = (S_h f, T_h f) \in V_h \times V_h$ are solutions to the continuous source problem (2.8)–(2.9) and the discrete source problem (2.12)–(2.13), respectively. First, for the auxiliary variable $p$ (i.e., the approximation of $\sigma$ ,the following $L^2$ and $H^1$ error estimates follow from the standard theory of mixed finite element methods [1][4]:

$$\|p - p_h\|_0 \leq Ch^{m+1} \tag{2.18}$$

$$\|p - p_h\|_1 \leq Ch^m \tag{2.19}$$

Next, to derive error estimates for the main variable $\eta$(i.e., the approximation of $\omega$), we exploit the relationship between its error and that of the auxiliary variable. From the variational formulations of the continuous problem (2.8)–(2.9) and the discrete problem (2.12)–(2.13), for any test function $\psi_h \in V_h$, it holds that:

$$a(p, \psi_h) + b(\psi_h, \eta) = 0, \quad a(p_h, \psi_h) + b(\psi_h, \eta_h) = 0$$

Subtracting the two equations and substituting the definitions of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we obtain the following key integral identity:

$$\int_\Omega (p - p_h)\psi_h \, dx = \int_\Omega \nabla(\eta - \eta_h) \cdot \nabla\psi_h \, dx \tag{2.20}$$

This identity reveals the intrinsic connection between the errors of the auxiliary variable and the main variable, and serves as the basis for subsequent error analysis. Based on the above identity, we establish the $H^1$ error estimate for $\eta_h$. Let $\Pi_h\eta \in V_h$ be the finite element interpolation of $\eta$. Substituting $\psi_h = \Pi_h\eta - \eta_h$ into (2.20) yields:

$$\int_\Omega (p - p_h)(\Pi_h\eta - \eta_h) \, dx = \int_\Omega \nabla(\eta - \eta_h) \cdot \nabla(\Pi_h\eta - \eta_h) \, dx$$

Note that:

$$\int_\Omega |\nabla(\eta - \eta_h)|^2 dx = \int_\Omega \nabla(\eta - \eta_h) \cdot \nabla(\eta - \Pi_h\eta) dx + \int_\Omega \nabla(\eta - \eta_h) \cdot \nabla(\Pi_h\eta - \eta_h) dx$$

Substituting (2.20) into the second term on the right-hand side and applying the Cauchy–Schwarz inequality, we get:

$$\|\nabla(\eta - \eta_h)\|_0^2 \leq \|\nabla(\eta - \eta_h)\|_0 \|\nabla(\eta - \Pi_h\eta)\|_0 + \|p - p_h\|_0 \|\Pi_h\eta - \eta_h\|_0$$

By the Poincaré inequality,$\|\Pi_h\eta - \eta_h\|_0 \leq C\|\nabla(\Pi_h\eta - \eta_h)\|_0$, and from the triangle inequality:

$$\|\nabla(\Pi_h\eta - \eta_h)\|_0 \leq \|\nabla(\eta - \Pi_h\eta)\|_0 + \|\nabla(\eta - \eta_h)\|_0$$

Substituting and rearranging terms gives:

$$\|\nabla(\eta - \eta_h)\|_0^2 \leq \|\nabla(\eta - \eta_h)\|_0 \|\nabla(\eta - \Pi_h\eta)\|_0 + C\|p - p_h\|_0 (\|\nabla(\eta - \Pi_h\eta)\|_0 + \|\nabla(\eta - \eta_h)\|_0)$$

Applying Young's inequality $ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$ and taking sufficiently small ε, we finally obtain:

$$\|\nabla(\eta - \eta_h)\|_0 \leq C(\|\nabla(\eta - \Pi_h\eta)\|_0 + \|p - p_h\|_0)$$

Combining the Poincaré inequality, the standard interpolation estimate $\|\eta - \Pi_h\eta\|_1 \leq Ch\|\eta\|_2$ from finite element interpolation theory and the previous bound $\|p - p_h\|_0 \leq Ch^{m+1}$, we arrive at:

$$\|\eta - \eta_h\|_1 \leq C(\|\eta - \Pi_h\eta\|_1 + \|p - p_h\|_0) \leq Ch^m \tag{2.21}$$

This estimate shows that $\eta_h$ approximates $\eta$ with m-th order convergence in the $H^1$ norm. This result provides essential theoretical support for the subsequent error analysis of eigenvalue problems.

**Lemma 2.1** (Boundedness of Solution Operators)

For any $f \in L^2_{per}(\Omega)$, the discrete solution operators $T_h$ and $S_h$ satisfy the following boundedness estimates:

$$\|T_h f\|_1 \le C\|f\|_0 \quad (2.22)$$
$$\|S_h f\|_0 \le C\|f\|_0 \quad (2.23)$$

Proof. See Ref. [4].

**Lemma 2.2** (Error Estimates for Eigenvalues and Eigenfunctions){}

Let $\xi$ be the $k$-th eigenvalue of the continuous problem (2.6)–(2.7), with $M(\xi) \subset H^{m+1}(\Omega)$ , and let $(\xi_h, \sigma_h, \omega_h)$ be the $k$-th eigenpair of the discrete problem (2.15)–(2.16) satisfying $\|\omega_h\| = 1$ , where $\|\cdot\|$ denotes the $L^2$ norm. Then, under periodic boundary conditions, there exists an eigenfunction $(\sigma, \omega)$of the continuous problem corresponding to ξ (with $\sigma = \mathcal{S}(\xi\omega)$） such that $\|\omega\| = 1$ and the following estimates hold:

$$|\xi - \xi_h| \le Ch^{2m} \quad (2.24)$$
$$\|\sigma - \sigma_h\|_0 \le Ch^{m+1} \quad (2.25)$$
$$\|\omega - \omega_h\|_1 \le Ch^m \quad (2.26)$$

Proof. Let $\mu = \frac{1}{\xi}, \mu_h = \frac{1}{\xi_h}$ , $T, T_h$ be the continuous and discrete solution operators such that $Tf = \eta, T_h f = \eta_h$, and let $S$、$S_h$ be the continuous and discrete solution operators for the auxiliary variable such that $Sf = p, S_h f = P_h$. From (2.21), for any $f \in L^2_{per}(\Omega)$:

$$\|(T - T_h)f\|_1 = \|\eta - \eta_h\|_1 \le Ch^m\|f\|_0$$

Using the Sobolev embedding inequality $\|v\|_0 \le C\|v\|_1$ for all $v \in H^1_{per}(\Omega)$, we have:

$$\|(T - T_h)f\|_0 \le C\|(T - T_h)f\|_1 \le Ch^m\|f\|_0$$

Thus the operator norm estimates are:

$$\|T - T_h\|_{\mathcal{L}(L^2,L^2)} \le Ch^m,\ \ \|T - T_h\|_{\mathcal{L}(L^2,H^1)} \le Ch^m$$

By the eigenvalue approximation theorem for self-adjoint completely continuous operators:

$$|\mu - \mu_h| \le C\|T - T_h\|^2_{\mathcal{L}(L^2,L^2)}$$

which implies

$$|\mu - \mu_h| \le Ch^{2m}$$

Since the eigenvalues of the biharmonic operator are bounded below by a positive constant, there exists $c > 0$ such that $\xi \ge c, \xi_h \ge c$, so $\mu\mu_h \ge c^2$. Therefore,

$$|\xi - \xi_h| = \frac{|\mu_h - \mu|}{\mu\mu_h} \le \frac{Ch^{2m}}{c^2} = Ch^{2m}$$

As $T, T_h$ are self-adjoint completely continuous operators on $L^2_{per}(\Omega)$, by the eigenfunction convergence theorem for such operators, there exists $\omega \in M(\xi)$ such that :

$$\|\omega - \omega_h\|_1 \le C\|T - T_h\|_{\mathcal{L}(L^2,H^1)}\|\omega_h\|_1$$

By the Poincaré inequality,$\|\omega_h\|_1 \le C\|\omega_h\|_0 = C$.Since $\|T - T_h\|_{\mathcal{L}(L^2,H^1)} \le Ch^m$, it follows that :

$$\|\omega - \omega_h\|_1 \le Ch^m$$

Note that $\sigma = S(\xi\omega), \sigma_h = S_h(\xi_h\omega_h)$ . By the triangle inequality,

$$\|\sigma - \sigma_h\|_0 \le \|S(\xi\omega) - S_h(\xi\omega)\|_0 + \|S_h(\xi\omega) - S_h(\xi_h\omega_h)\|_0$$

Let $f = \xi\omega \in L^2_{per}(\Omega)$. From the source problem error estimate (2.18) :

$$\|S(\xi\omega) - S_h(\xi\omega)\|_0 \le Ch^{m+1}\|\xi\omega\|_0$$

Since $\|\xi\omega\|_0 = |\xi| \cdot \|\omega\|_0 = |\xi| \le C$ (eigenvalues are bounded), we have :

$$\|S(\xi\omega) - S_h(\xi\omega)\|_0 \le Ch^{m+1}$$

From (2.23) in Lemma 2.1:

$$\|S_h(\xi\omega) - S_h(\xi_h\omega_h)\|_0 \le C\|\xi\omega - \xi_h\omega_h\|_0$$

Moreover,

$$\|\xi\omega - \xi_h\omega_h\|_0 \le |\xi|\|\omega - \omega_h\|_0 + |\xi - \xi_h|\|\omega_h\|_0$$

By the Aubin–Nitsche duality argument (combined with $\omega \subset H^{m+1}(\Omega)$ and (2.26)), $\|\omega - \omega_h\|_0 \le Ch^{m+1}$ . Together with (2.24), this gives

$$\|\xi\omega - \xi_h\omega_h\|_0 \le C(h^{m+1} + h^{2m}) \le Ch^{m+1}$$

In summary,

$$\|\sigma - \sigma_h\|_0 \le Ch^{m+1}$$

This completes the proof.

**Lemma 2.3** (Error Expansion of the Rayleigh Quotient)

Let $(\xi, \sigma, \omega)$ be an eigenpair of the continuous problem (2.6)–(2.7). Then for any$(\sigma^*, \omega^*) \in H^1_{per}(\Omega) \times H^1_{per}(\Omega)$ with $\omega^* \neq 0$,the Rayleigh quotient is defined as:

$$\xi^r = \frac{a(\sigma^*, \sigma^*) + 2b(\sigma^*, \omega^*)}{-(\omega^*, \omega^*)} \quad (2.27)$$

satisfies the following error expansion:

$$\xi^r - \xi = \frac{a(\sigma^* - \sigma, \sigma^* - \sigma) + 2b(\sigma^* - \sigma, \omega^* - \omega)}{-(\omega^*, \omega^*)} + \xi\frac{(\omega^* - \omega, \omega^* - \omega)}{-(\omega^*, \omega^*)} \quad (2.28)$$

Proof.

Taking $\psi = \sigma$, $\psi = \sigma^*$ in (2.6), we obtain :

$$a(\sigma, \sigma) + b(\sigma, \omega) = 0 \Rightarrow b(\sigma, \omega) = -a(\sigma, \sigma) \quad (2.29)$$
$$a(\sigma, \sigma^*) + b(\sigma^*, \omega) = 0 \Rightarrow a(\sigma, \sigma^*) = -b(\sigma^*, \omega) \quad (2.30)$$

Taking $\varphi = \omega$ and $\varphi = \omega^*$ in (2.7), we get :

$$b(\sigma, \omega) = -\xi(\omega, \omega) \quad (2.31)$$
$$b(\sigma, \omega^*) = -\xi(\omega, \omega^*) \quad (2.32)$$

Combining (2.27) and (2.29) yields:

$$a(\sigma, \sigma) = \xi(\omega, \omega) \quad (2.33)$$

Combining (2.27) and (2.30) yields:

$$a(\sigma, \sigma^*) = \xi(\omega, \omega^*) \quad (2.34)$$

Substituting (2.29)–(2.34) into the left-hand side, we compute:

$$\begin{aligned} &a(\sigma^* - \sigma, \sigma^* - \sigma) + 2b(\sigma^* - \sigma, \omega^* - \omega) \\ &+\xi(\omega^* - \omega, \omega^* - \omega) = a(\sigma^*, \sigma^*) - a(\sigma^*, \sigma) - a(\sigma, \sigma^*) \\ &+a(\sigma, \sigma) + 2b(\sigma^*, \omega^*) - 2b(\sigma^*, \omega) - 2b(\sigma, \omega^*) \\ &+2b(\sigma, \omega) + \xi(\omega^*, \omega^*) - \xi(\omega^*, \omega) - \xi(\omega, \omega^*) + \xi(\omega, \omega) \\ &= a(\sigma^*, \sigma^*) + 2b(\sigma^*, \omega^*) + \xi(\omega^*, \omega^*) \end{aligned}$$

From the definition, we have: $a(\sigma^*, \sigma^*) + 2b(\sigma^*, \omega^*) = -\xi^r(\omega^*, \omega^*)$.Substituting it into the above formula yields:

$$a(\sigma^* - \sigma, \sigma^* - \sigma) + 2b(\sigma^* - \sigma, \omega^* - \omega) + \xi(\omega^* - \omega, \omega^* - \omega) = -\xi^r(\omega^*, \omega^*) + \xi(\omega^*, \omega^*)$$

Rearranging terms yields (2.28). This completes the proof.

**Remark 2.2** (Solvability of the Discrete Source Problem)

For any $f \in L^2_{per}(\Omega)$, the discrete source problem (2.12)–(2.13) admits a unique solution in $V_h \times V_h$. Consider its homogeneous form (i.e.,$f = 0$). Taking $\psi = p_h$、$\varphi = \eta_h$ gives $a(p_h, p_h) = 0$, so $p_h = 0$. Substituting into the first equation yields $\eta_h = 0$. By the theory of linear systems, the original problem has a unique solution.

International Journal of Scientific Engineering and Science
Volume 10, Issue 6, pp. 31-39, June 2026. ISSN (Online): 2456-7361

## III. Two-Grid Discrete Scheme Based on Shifted-Inverse Iteration

In this section, a two-grid discrete scheme based on shifted-inverse iteration is established for the Ciarlet–Raviart mixed variational formulation under periodic boundary conditions. Let $V_H \subset V_h$ and $V_h \subset H^1_{per}(\Omega)$ with $h < H$.

### *3.1 Two-Grid Discrete Scheme Based on Shifted-Inverse Iteration*

Step 1: Solve the eigenvalue problem (2.15)–(2.16) on the coarse grid $\pi_H$ : find $(\xi_H, \sigma_H, \omega_H) \in \mathbb{R} \times V_H \times V_H$ with $\|\omega_H\|_0 = 1$ such that:

$$a(\sigma_H, \psi) + b(\psi, \omega_H) = 0, \quad \forall \psi \in V_H \tag{3.1}$$

$$b(\sigma_H, \varphi) = -\xi_H(\omega_H, \varphi), \quad \forall \varphi \in V_H \tag{3.2}$$

Step 2: Solve the following linear system on the fine grid $\pi_h$h: find $(\sigma', \omega') \in V_h \times V_h$ such that :

$$a(\sigma', \psi) + b(\psi, \omega') = 0, \quad \forall \psi \in V_h \tag{3.3}$$

$$b(\sigma', \varphi) + \xi_H(\omega', \varphi) = -(\omega_H, \varphi), \quad \forall \varphi \in V_h \tag{3.4}$$

Define $\omega^h = \frac{\omega'}{\|\omega'\|_0}, \ \ \sigma^h = \frac{\sigma'}{\|\omega'\|_0}$.

Step 3: Compute the Rayleigh quotient:

$$\xi^h = \frac{a(\sigma^h, \sigma^h) + 2b(\sigma^h, \omega^h)}{-(\omega^h, \omega^h)}$$

Let $(\xi_H, \sigma_H, \omega_H)$ be the $k$-th eigenpair of (3.1)–(3.2). Then $(\xi^h, \sigma^h, \omega^h)$ obtained from Scheme 3.1 is an approximation to the $k$-th eigenpair of (2.6)–(2.7).

Next, we analyze the computational efficiency of Scheme 3.1. To this end, we first introduce $dist(\omega, W) = inf_{v\in W} \| \omega - v\|_0$ The validity of the following lemma will provide a foundation for the subsequent work in this paper.

**Lemma 3.1** Let $(\mu_0, w_0)$ be an approximation to the $k$-th eigenpair $(\mu, \omega)$, where $\mu_0$ is not an eigenvalue of $T_h, w_0 \in V_h, \|w_0\|_0 = 1$. Define $\omega_0 = \frac{T_h w_0}{\|T_h w_0\|_0}$.

Assume the following conditions hold:

(C1) $inf_{v\in M_h(\xi)} \| w_0 - v\|_0 \le \frac{1}{2}$;

(C2) $|\mu_0 - \mu| \le \frac{\rho}{4}$, $|\mu_{j,h} - \mu_j| \le \frac{\rho}{4}$, for $j = k-1, k, k+q (j \neq 0)$ ,where $\rho = \min_{j\neq k} |\mu_j - \mu|$ is the separation constant of the $k$-th eigenvalue $\mu$ ;

(C3) $\omega' \in V_h$, $\omega^h \in V_h$ satisfy:

$$(\mu_0 - T_h)\omega' = \omega_0, \quad \omega^h = \frac{\omega'}{\|\omega'\|_0} \tag{3.5}$$

Then the following estimate holds:

$$dist(\omega^h, M_h(\xi)) \le \frac{C}{\rho} \max_{k\le j\le k+q-1} | \mu_0 - \mu_{j,h}| dist(w_0, M_h(\xi)) \tag{3.6}$$

Proof: Since the eigenfunctions $\{\omega_{j,h}\}_{j=1}^d$ of $T_h$ form an orthonormal basis of $V_h$ with respect to $(\cdot,\cdot)$, we have $\omega_0 = \sum_{j=1}^d(\omega_0, \omega_{j,h})\omega_{j,h}$. Since $\mu_0$ is not an eigenvalue of $T_h$, from (3.5) we obtain:

$$(\mu_0 - \mu_h)\omega' = (\mu_0 - \mu_h)(\mu_0 - T_h)^{-1}\omega_0 = \sum_{j=1}^d \frac{\mu_0 - \mu_h}{\mu_0 - \mu_{j,h}}(\omega_0, \omega_{j,h})\omega_{j,h} \tag{3.7}$$

Using the triangle inequality and condition (C2), we derive

$$|\mu_0 - \mu_h| \le |\mu_0 - \mu| + |\mu - \mu_h| \le \frac{\rho}{4} + \frac{\rho}{4} = \frac{\rho}{2},$$

$$|\mu_0 - \mu_{j,h}| \ge |\mu - \mu_j| - |\mu_0 - \mu| - |\mu_j - \mu_{j,h}| \ge \frac{\rho}{2}$$

For $j = k-1, k+q (j \neq 0)$ . Thus for $j \neq k, k+1, \dots, k+q-1$,

$$|\mu_0 - \mu_{j,h}| \ge \frac{\rho}{2} \tag{3.8}$$

Since $T_h$ is self-adjoint with respect to the inner product $(\cdot,\cdot)$ and $T_h\omega_h = \mu_h\omega_h$, we have for all j = 1,2, ..., d:

$$\begin{aligned}(T_h w_0, \omega_{j,h})\omega_{j,h} &= (w_0, T_h\omega_{j,h})\omega_{j,h} = (w_0, \mu_{j,h}\omega_{j,h})\omega_{j,h} \\ &= (w_0, \omega_{j,h})\mu_{j,h}\omega_{j,h} = (w_0, \omega_{j,h})T_h\omega_{j,h}\end{aligned} \tag{3.9}$$

Note that $\{\omega_{j,h}\}_k^{k+q-1}$ forms an orthonormal basis of $M_h(\xi)$. Using $\omega_0 = \frac{T_h w_0}{\|T_h w_0\|_0}$,together with (3.7), (3.9), (2.22), and (3.8), we deduce:

$$\begin{aligned}
&\left\|(\mu_0 - \mu_h)\omega' - \sum_{j=k}^{k+q-1} \frac{\mu_0 - \mu_h}{\mu_0 - \mu_{j,h}}(\omega_0, \omega_{j,h})\omega_{j,h}\right\|_1 \\
&= \left\|\sum_{j\neq k,k+1,\dots,k+q-1} \frac{\mu_0 - \mu_h}{\mu_0 - \mu_{j,h}}(\omega_0, \omega_{j,h})\omega_{j,h}\right\|_1 \\
&= \frac{1}{\|T_h w_0\|_0}\left\|\sum_{j\neq k,k+1,\dots,k+q-1} \frac{\mu_0 - \mu_h}{\mu_0 - \mu_{j,h}}(T_h w_0, \omega_{j,h})\omega_{j,h}\right\|_1 \\
&= \frac{1}{\|T_h w_0\|_0}\left\|\sum_{j\neq k,k+1,\dots,k+q-1} \frac{\mu_0 - \mu_h}{\mu_0 - \mu_{j,h}}(w_0, \omega_{j,h})T_h\omega_{j,h}\right\|_1 \\
&= \frac{1}{\|T_h w_0\|_0}\left\|T_h\left(\sum_{j\neq k,k+1,\dots,k+q-1} \frac{\mu_0 - \mu_h}{\mu_0 - \mu_{j,h}}(w_0, \omega_{j,h})\omega_{j,h}\right)\right\|_1 \\
&\le \frac{C}{\|T_h w_0\|_0}\left\|\sum_{j\neq k,k+1,\dots,k+q-1} \frac{\mu_0 - \mu_h}{\mu_0 - \mu_{j,h}}(w_0, \omega_{j,h})\omega_{j,h}\right\|_0 \\
&\le \frac{2C}{\rho\|T_h w_0\|_0}|\mu_0 - \mu_h|\left(\sum_{j\neq k,k+1,\dots,k+q-1}(w_0, \omega_{j,h})^2\right)^{\frac{1}{2}} \\
&\le \frac{C}{\rho\|T_h w_0\|_0}|\mu_0 - \mu_h|\left\|w_0 - \sum_{j=k}^{k+q-1}(w_0, \omega_{j,h})\omega_{j,h}\right\|_0 \\
&= \frac{C}{\rho\|T_h w_0\|_0}|\mu_0 - \mu_h| \inf_{v\in M_h(\xi)} \| w_0 - v\|_0 \\
&\le \frac{C}{\rho\|T_h w_0\|_0}|\mu_0 - \mu_h| \mathrm{dist}(w_0, M_h(\xi))
\end{aligned} \tag{3.10}$$

Taking the norm on both sides of (3.7), and using $\omega_0 = \frac{T_h w_0}{\|T_h w_0\|_0}$ and (3.9), we obtain:

$$\|(\mu_0-\mu_h)\omega'\|_0 = \frac{1}{\|T_h w_0\|_0}\left\|\sum_{j=1}^{d}\frac{\mu_0-\mu_h}{\mu_0-\mu_{j,h}}(T_h w_0,\omega_{j,h})\omega_{j,h}\right\|_0$$

$$= \frac{1}{\|T_h w_0\|_0}\left(\sum_{j=1}^{d}\left(\frac{\mu_0-\mu_h}{\mu_0-\mu_{j,h}}(w_0,\mu_{j,h}\omega_{j,h})\right)^2\right)^{\frac{1}{2}}$$

$$\geq \frac{1}{\|T_h w_0\|_0}\min_{k\leq j\leq k+q-1}\left|\frac{\mu_0-\mu_h}{\mu_0-\mu_{j,h}}\right|\left(\sum_{j=k}^{k+q-1}(w_0,\mu_{j,h}\omega_{j,h})^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{\|T_h w_0\|_0}\min_{k\leq j\leq k+q-1}\left|\frac{\mu_0-\mu_h}{\mu_0-\mu_{j,h}}\right|.\left\|w_0-\left(w_0-\sum_{j=k}^{k+q-1}(w_0,\mu_{j,h}\omega_{j,h})\omega_{j,h}\right)\right\|_0$$

$$\geq \frac{1}{2\|T_h w_0\|_0}\min_{k\leq j\leq k+q-1}\left|\frac{\mu_0-\mu_h}{\mu_0-\mu_{j,h}}\right| \quad (3.11)$$

From (3.10) and (3.11), we conclude:

$$dist(\omega^h, M_h(\xi)) = dist(sign(\mu_0-\mu_h)\omega^h, M_h(\xi))$$

$$\leq \left\|sign(\mu_0-\mu_h)\omega^h - \frac{1}{\|(\mu_0-\mu_h)\omega'\|_0}\sum_{j=k}^{k+q-1}\frac{\mu_0-\mu_h}{\mu_0-\mu_{j,h}}(\omega_0,\omega_{j,h})\omega_{j,h}\right\|_1$$

$$= \left\|\frac{(\mu_0-\mu_h)\omega'}{\|(\mu_0-\mu_h)\omega'\|_0} - \frac{1}{\|(\mu_0-\mu_h)\omega'\|_0}\sum_{j=k}^{k+q-1}\frac{\mu_0-\mu_h}{\mu_0-\mu_{j,h}}(\omega_0,\omega_{j,h})\omega_{j,h}\right\|_1$$

$$\leq 2\|T_h w_0\|_0\max_{k\leq j\leq k+q-1}\left|\frac{\mu_0-\mu_{j,h}}{\mu_0-\mu_h}\right|\left\|(\mu_0-\mu_h)\omega' - \sum_{j=k}^{k+q-1}\frac{\mu_0-\mu_h}{\mu_0-\mu_{j,h}}(\omega_0,\omega_{j,h})\omega_{j,h}\right\|_1$$

$$\leq \frac{C}{\rho}\max_{k\leq j\leq k+q-1}|\mu_0-\mu_{j,h}|dist(w_0, M_h(\xi))$$

This completes the proof.

**Theorem 3.1**

Assume $M(\xi)\subset H^{m+1}(\Omega)$. Let $(\xi^h,\sigma^h,\omega^h)$ be the $k$-th approximate eigenpair obtained from Scheme 3.1, with H sufficiently small. Then there exist $\omega\in M(\xi)$ and $\sigma = S(\xi\omega)$ such that :

$$\|\omega^h-\omega\|_1 \leq C(H^{3m}+h^m), \quad (3.12)$$
$$\|\sigma^h-\sigma\|_0 \leq C(H^{2m}+h^{m+1}), \quad (3.13)$$
$$|\xi^h-\xi| \leq C(H^{2m}+h^m)^2 \quad (3.14)$$

where the constant $C>0$ is independent of the mesh sizes $h$ and $H$.

Proof : We use Lemma 3.1 to prove (3.12). First, verify all conditions of Lemma 3.1.

Let $(\xi_H,\sigma_H,\omega_H)$ be obtained from Step 1 of Scheme 3.1. Choose $\mu_0 = \frac{1}{\xi_H}, w_0 = \omega_H$ By the triangle inequality and (2.26),

$$dist(\omega_H, M_h(\xi)) \leq \|\omega_H-\omega\|_1 + dist(\omega, M_h(\xi)) \leq CH^m \quad (3.15)$$

so condition (C1) holds. Next, verify (C2). From (2.24),

$$|\mu_0-\mu| = \frac{|\xi_H-\xi|}{\xi_H\xi} \leq CH^{2m} \leq \frac{\rho}{4},$$

$$|\mu_j-\mu_{j,h}| = \frac{|\xi_{j,h}-\xi_j|}{\xi_{j,h}\xi_j} \leq Ch^{2m} \leq \frac{\rho}{4}$$

thus (C2) holds.

Finally, verify (C3). From (3.3)–(3.4),

$$a(\sigma',\psi)+b(\psi,\omega') = 0, \quad \forall\psi\in V_h,$$
$$b(\sigma',\varphi) = -(\omega_H+\xi_H\omega',\varphi), \quad \forall\varphi\in V_h$$

Combined with (2.12), (2.13), and (2.14), we get :

$$\sigma' = S_h(\xi_H\omega'+\omega_H), \quad (3.16)$$
$$\omega' = T_h(\xi_H\omega'+\omega_H) \quad (3.17)$$

From (3.17),

$$(\xi_H^{-1}-T_h)\omega' = \xi_H^{-1}T_h\omega_H, \quad \omega^h = \frac{\omega'}{\|\omega'\|_0} \quad (3.18)$$

Note that $\xi_H^{-1}T_h\omega_H = \|\xi_H^{-1}T_h\omega_H\|_0\omega_0$ differs from $\omega_0$ only by a constant factor. Therefore Step 2 of Scheme 3.1 is equivalent to :

$$(\xi_H^{-1}-T_h)\omega' = \omega_0, \quad \omega^h = \frac{\omega'}{\|\omega'\|_0}$$

From the above arguments, all assumptions of Lemma 3.1 are satisfied.

We now prove (3.12). Since $M_h(\xi)$ is a q-dimensional space, there exists $\omega^*\in M_h(\xi)$ such that

$$\|\omega^h-\omega^*\|_1 = dist(\omega^h, M_h(\xi)) \quad (3.19)$$

Moreover, for $k\leq j\leq k+q-1$,

$$|\mu_0-\mu_{j,h}| = \left|\frac{1}{\xi_H}-\frac{1}{\xi_{j,h}}\right| \leq \frac{|\xi_H-\xi_{j,h}|}{\xi_H\xi_{j,h}}$$
$$\leq C|\xi_H-\xi_{j,h}| \leq C(|\xi_H-\xi|+|\xi-\xi_{j,h}|) \leq CH^{2m} \quad (3.20)$$

Substituting (3.20) and (3.15) into (3.6) yields

$$\|\omega^h-\omega^*\|_1 = dist(\omega^h, M_h(\xi))$$
$$\leq C\max_{k\leq j\leq k+q-1}|\mu_0-\mu_{j,h}|dist(\omega_H, M_h(\xi)) \leq CH^{3m} \quad (3.21)$$

From (2.26), there exists $\omega\in M(\xi)$ such that $\|\omega^*-\omega\|_1 = dist(\omega^*, M(\xi))$ . Hence :

$$\|\omega^h-\omega\|_1 \leq \|\omega^h-\omega^*\|_1 + \|\omega-\omega^*\|_1 \leq C(H^{3m}+h^m) \quad (3.22)$$

which proves (3.12).

We next prove (3.13). From (2.26), there exists $\omega_h\in M_h(\xi)$ such that

$$\|\omega_H-\omega_h\|_1 \leq CH^m + Ch^m \leq CH^m$$

By (2.22) and the definition of the self-adjoint operator norm,

$$\|(\xi_H^{-1}-T_h)^{-1}T_h(\omega_H-\omega_h)\|_1$$
$$= \|T_h(\xi_H^{-1}-T_h)^{-1}(\omega_H-\omega_h)\|_1$$
$$\leq C\|(\xi_H^{-1}-T_h)^{-1}\|_0\|\omega_H-\omega_h\|_0$$
$$\leq C|(\xi_H^{-1}-\xi_h^{-1})^{-1}|\,\|\omega_H-\omega_h\|_0$$
$$\leq C\frac{1}{|\xi_H-\xi_h|}\|\omega_H-\omega_h\|_0 \quad (3.23)$$

From (3.18),

$$\omega' = (\xi_H^{-1}-T_h)^{-1}(\xi_H^{-1}T_h\omega_H) \quad (3.24)$$

Since $\omega_h\in M_h(\xi)$ and $\{\omega_{j,h}\}_k^{k+q-1}$ is an orthonormal basis of $M_h(\xi)$,

$$T_h\omega_h = T_h\sum_{j=k}^{k+q-1}(\omega_h,\omega_{j,h})\omega_{j,h} = \sum_{j=k}^{k+q-1}(\omega_h,\omega_{j,h})T_h\omega_{j,h}$$
$$= \sum_{j=k}^{k+q-1}(\omega_h,\mu_{j,h}\omega_{j,h})\omega_{j,h}$$

Thus:

International Journal of Scientific Engineering and Science
Volume 10, Issue 6, pp. 31-39, June 2026. ISSN (Online): 2456-7361

$$\|(\xi_H^{-1}-T_h)^{-1}T_h\omega_h\|_0 = \left\|(\xi_H^{-1}-T_h)^{-1}\sum_{j=k}^{k+q-1}(\omega_h,\mu_{j,h}\omega_{j,h})\omega_{j,h}\right\|_0$$
$$= \left\|\sum_{j=k}^{k+q-1}(\omega_h,\mu_{j,h}\omega_{j,h})(\xi_H^{-1}-T_h)^{-1}\omega_{j,h}\right\|_0$$
$$= \left\|\sum_{j=k}^{k+q-1}(\omega_h,\mu_{j,h}\omega_{j,h})(\xi_H^{-1}-\xi_{j,h}^{-1})^{-1}\omega_{j,h}\right\|_0$$
$$= \|(\xi_H^{-1}-\xi_h^{-1})^{-1}T_h\omega_h\|_0$$

Combining the above identity with (3.24) and (3.23),

$$\|\omega'\|_0 = \|(\xi_H^{-1}-T_h)^{-1}(\xi_H^{-1}T_h\omega_H)\|_0$$
$$= \|(\xi_H^{-1}-T_h)^{-1}\xi_H^{-1}T_h(\omega_H-\omega_h+\omega_h)\|_0$$
$$\geq \|(\xi_H^{-1}-T_h)^{-1}\xi_H^{-1}T_h\omega_h\|_0 - \|(\xi_H^{-1}-T_h)^{-1}\xi_H^{-1}T_h(\omega_H-\omega_h)\|_0$$
$$\geq C\left(\|(\xi_H^{-1}-T_h)^{-1}T_h\omega_h\|_0 - \frac{1}{|\xi_h-\xi_H|}\|\omega_H-\omega_h\|_0\right)$$
$$\geq C\left\|\frac{1}{\xi_h-\xi_H}\omega_h\right\|_0 \quad (3.25)$$

Choose $\sigma_h = S_h(\xi_h\omega^*)$ and $\sigma = S(\xi\omega)$. From (3.16), $\sigma^h = \frac{\sigma'}{\|\omega'\|_0}$, (2.23), (3.25), (2.24), and (3.21),

$$\|\sigma^h-\sigma_h\|_0 = \left\|S_h\left(\frac{\omega_H}{\|\omega'\|_1}+\xi_H\omega^h-\xi_h\omega^*\right)\right\|_0$$
$$\leq C\left\|\frac{\omega_H}{\|\omega'\|_0}+\xi_H\omega^h-\xi_h\omega^*\right\|_0$$
$$\leq C\left(\left\|\frac{\omega_H}{\|\omega'\|_0}\right\|_0 + \|\xi_H\omega^h-\xi_H\omega^*\|_0 + \|\xi_H\omega^*-\xi_h\omega^*\|_0\right)$$
$$\leq C(|\xi_H-\xi_h| + \|\omega^h-\omega^*\|_0 + |\xi_H-\xi_h|)$$
$$\leq C(H^{2m}+H^{3m}) \leq CH^{2m} \quad (3.26)$$

Combining (2.25) with (3.26) yields (3.13):

$$\|\sigma^h-\sigma\|_0 \leq \|\sigma^h-\sigma_h\|_0 + \|\sigma_h-\sigma\|_0 \leq C(H^{2m}+h^{m+1})$$

Equation (2.28) will be used to estimate the error of $\xi^h$. We fully exploit the structural properties of the C–R mixed variational formulation to avoid estimating $\|\sigma^h-\sigma\|_1$ .Let $I_h: C(\overline{\Omega}) \to V_h$ be the Lagrange interpolation operator. From (2.6) and (3.3),

$$a(\sigma^h-\sigma,\psi)+b(\psi,\omega^h-\omega)=0, \quad \forall\psi\in V_h \quad (3.27)$$

From Step 3 of Scheme 3.1,

$$\xi^h = \frac{a(\sigma^h,\sigma^h)+2b(\sigma^h,\omega^h)}{-(\omega^h,\omega^h)}$$

In (2.28), set $\xi^r = \xi^h, \sigma^* = \sigma^h, \omega^* = \omega^h$ Using (3.27), (3.12), (3.13), and interpolation error estimates,

$$\xi^h-\xi = \frac{a(\sigma^h-\sigma,\sigma^h-\sigma)+2b(\sigma^h-\sigma,\omega^h-\omega)}{-(\omega^h,\omega^h)} + \xi\frac{(\omega^h-\omega,\omega^h-\omega)}{-(\omega^h,\omega^h)}$$
$$= \frac{-a(\sigma^h-\sigma,\sigma^h-\sigma)+2a(\sigma^h-\sigma,\sigma^h-\sigma)+2b(\sigma^h-\sigma,\omega^h-\omega)}{-(\omega^h,\omega^h)} + \xi\frac{(\omega^h-\omega,\omega^h-\omega)}{-(\omega^h,\omega^h)}$$
$$= \frac{-a(\sigma^h-\sigma,\sigma^h-\sigma)+2a(\sigma^h-\sigma,I_h\sigma-\sigma)+2b(I_h\sigma-\sigma,\omega^h-\omega)}{-(\omega^h,\omega^h)} + \xi\frac{(\omega^h-\omega,\omega^h-\omega)}{-(\omega^h,\omega^h)}$$
$$\leq C((H^{2m}+h^{m+1})^2 + h^{m+1}(H^{2m}+h^{m+1}) + h^m(H^{3m}+h^m)+(H^{3m}+h^m)^2)$$
$$\leq C(H^{2m}+h^m)^2 \quad (3.28)$$

Thus (3.14) is proved.

## IV. Numerical Experiments

In this section, numerical experiments are carried out on the square domain $\Omega = [0,1] \times [0,1]$ with uniform triangular meshes to verify the efficiency of the two-grid shifted inverse iteration method proposed in this paper for periodic eigenvalue problems. Let $\xi_t$ denote the reference exact eigenvalue, $\xi_{t,H}$ the approximate eigenvalue on the coarse mesh, and $\xi_{t,h}$ the corrected eigenvalue obtained by the two-grid method. Then $\xi_{t,H}$ - $\xi_t$ stands for the error between the coarse-mesh approximation and the exact value, while $\xi_{t,h}$ -$\xi_t$ denotes the error of the two-grid corrected solution relative to the exact value. The first three reference exact eigenvalues are given as follows:

$$\xi_1 = \pi^4 \approx 97.409091034002;$$
$$\xi_2 = 4\pi^4 \approx 389.63636413601;$$
$$\xi_3 = 16\pi^4 \approx 1558.54545654404.$$

Table 1 lists the numerical results and absolute errors of the first three eigenvalues under different mesh sizes. It can be observed from the table that as the mesh is successively refined, the error of the two-grid corrected solution $\xi_{t,h}$ decreases significantly. Taking the first eigenvalue as an example, when H is refined from $\frac{\sqrt{2}}{32}$ to $\frac{\sqrt{2}}{128}$ , the error drops from 0.575506 to 0.035866, showing an obvious reduction. The third eigenvalue follows a similar trend, with the error decreasing from 37.169985 to 9.208099 and then to 2.296744. This not only demonstrates that the proposed method can effectively approximate the exact eigenvalues, but also verifies the effectiveness of the algorithm.

TABLE I. Numerical results and errors of the first three eigenvalues under different mesh sizes.

| $t$ | $H$ | $h$ | $\xi_{t,H}$ | $\xi_{t,h}$ | $\xi_{t,H}$ -$\xi_t$ | $\xi_{t,h}$ -$\xi_t$ |
|---|---|---|---|---|---|---|
| 1 | $\frac{\sqrt{2}}{32}$ | $\frac{\sqrt{2}}{64}$ | 99.521344 | 97.984597 | 2.112253 | 0.575506 |
| 1 | $\frac{\sqrt{2}}{64}$ | $\frac{\sqrt{2}}{128}$ | 97.932294 | 97.552638 | 0.523203 | 0.143546 |
| 1 | $\frac{\sqrt{2}}{128}$ | $\frac{\sqrt{2}}{256}$ | 97.539589 | 97.444957 | 0.130498 | 0.035866 |
| 2 | $\frac{\sqrt{2}}{32}$ | $\frac{\sqrt{2}}{64}$ | 417.037736 | 396.780369 | 27.401372 | 7.144005 |
| 2 | $\frac{\sqrt{2}}{64}$ | $\frac{\sqrt{2}}{128}$ | 396.356824 | 391.412925 | 6.720460 | 1.776560 |
| 2 | $\frac{\sqrt{2}}{128}$ | $\frac{\sqrt{2}}{256}$ | 391.308210 | 390.079911 | 1.671846 | 0.443547 |
| 3 | $\frac{\sqrt{2}}{32}$ | $\frac{\sqrt{2}}{64}$ | 1698.745042 | 1595.715442 | 140.199586 | 37.169985 |
| 3 | $\frac{\sqrt{2}}{64}$ | $\frac{\sqrt{2}}{128}$ | 1592.341501 | 1567.753556 | 33.796045 | 9.208099 |
| 3 | $\frac{\sqrt{2}}{128}$ | $\frac{\sqrt{2}}{256}$ | 1566.916698 | 1560.842200 | 8.371241 | 2.296744 |

Fig. 1 shows the convergence trend of eigenvalue errors with respect to mesh sizes and the comparison between different methods. As the mesh is gradually refined, the errors of the two-grid solutions decay regularly, and the convergence rate achieves the optimal order of $O(h^2)$ . This is completely

International Journal of Scientific Engineering and Science
Volume 10, Issue 6, pp. 31-39, June 2026. ISSN (Online): 2456-7361

consistent with the conclusion derived from the theoretical analysis that the eigenvalue error satisfies $|\xi^h - \xi| \le C(H^{2m} + h^m)^2$ under the parameter settings in this paper, which verifies the theoretical convergence of the proposed algorithm.

Fig.2 presents the numerical profiles of the first three corrected eigenfunctions by the two-grid method under different mesh sizes. With successive mesh refinement, the smoothness of the eigenfunctions is significantly improved, the numerical oscillations gradually disappear, and their profiles tend to be stable. In particular, the third eigenfunction exhibits obvious distortion on the coarse grid, but agrees well with the theoretical solution after fine-grid correction. This indicates that the proposed method possesses excellent approximation ability for eigenfunctions.

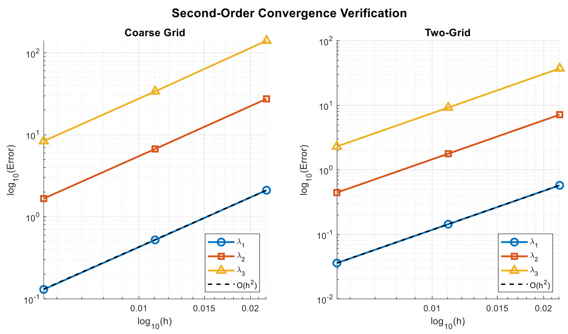


Fig. 1. Second-order convergence analysis of the coarse-grid method and the two-grid method

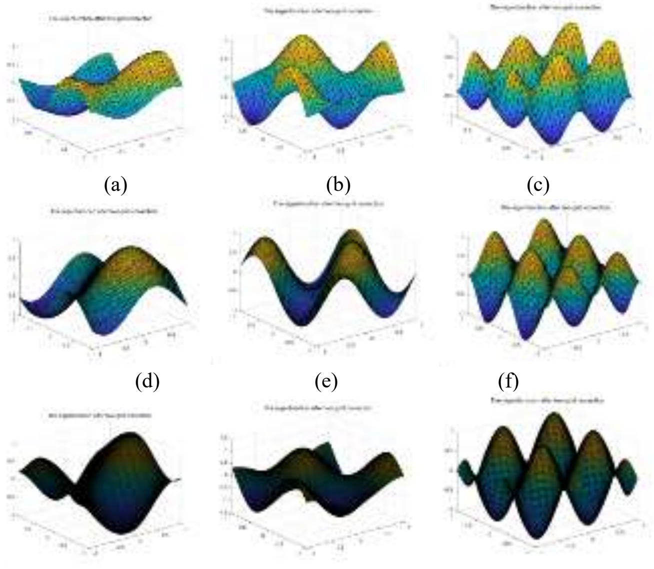


Fig. 2. The first three corrected eigenfunctions by the two-grid method under three successively refined meshes ((a)(b)(c): $H = \frac{\sqrt{2}}{32}, h = \frac{\sqrt{2}}{64}$; (d)(e)(f): $H = \frac{\sqrt{2}}{64}, h = \frac{\sqrt{2}}{128}$; (g)(h)(i): $H = \frac{\sqrt{2}}{128}, h = \frac{\sqrt{2}}{256}$; (a)(d)(g) denote the first eigenfunction; (b)(e)(h) denote the second eigenfunction; (c)(f)(i) denote the third eigenfunction )

## V. Conclusions

This paper focuses on the numerical solution of fourth-order eigenvalue problems under periodic boundary conditions, and establishes a complete theoretical framework and algorithm implementation of the two-grid shifted inverse iteration method based on the Ciarlet-Raviart mixed finite element. Different from existing studies mainly focusing on Dirichlet or Neumann boundary conditions, this paper systematically analyzes the well-posedness of the mixed variational formulation within the framework of periodic Sobolev spaces, presents the approximation properties of continuous and discrete solution operators, and proves the optimal convergence orders of eigenvalues and eigenfunctions, which provides a reference for the theoretical analysis of mixed finite element eigenvalue problems under periodic boundary conditions.

At the level of algorithm design, the two-grid discrete scheme proposed in this paper fully utilizes the complementary advantages of high computational efficiency on coarse grids and good approximation accuracy on fine grids. By solving the original eigenvalue problem on the coarse grid to obtain the initial approximation, and then solving only one linear system and performing Rayleigh quotient correction on the fine grid, the computational bottleneck of directly solving large-scale eigenvalue problems on fine grids is effectively avoided. Theoretical analysis shows that when the coarse grid size H and the fine grid size h satisfy an appropriate relation, the corrected eigenvalue error can reach $O((H^{2m} + h^m)^2)$, and the error of eigenfunctions in the $H^1$-norm is $O(H^{3m} + h^m)$, which verifies that the method can significantly reduce the computational cost while maintaining optimal convergence accuracy.

Numerical experiments are carried out to verify the first three eigenvalues on a square periodic domain, and the results are highly consistent with the theoretical analysis. The eigenvalue errors show a second-order convergence trend with mesh refinement, and the numerical profiles of eigenfunctions gradually become smooth and stable, further confirming the reliability and practicability of the proposed method. In addition, the theoretical analysis framework and two-grid algorithm structure established in this paper have strong universality and can be extended to other types of mixed finite element schemes and more general elliptic eigenvalue problems.

In summary, this paper forms systematic research results in three aspects: theoretical analysis, algorithm design and numerical verification, which provides new ideas and theoretical support for the efficient solution of high-order eigenvalue problems under periodic boundary conditions. Future work can further explore the extended applications of non-uniform mesh partitioning, adaptive refinement strategies and nonlinear eigenvalue problems.

## Acknowledgment

The author sincerely thanks the supervisor for continuous guidance and valuable suggestions throughout the whole research work. Gratitude is also given to the laboratory members and university for providing experimental conditions and academic communication environment to complete this study.

## References

[1] R.S. Falk, J.E. Osborn, "Error estimates for mixed methods", RAIRO. Anal. Numér., vol.14, no.3, pp.249-277, 1980.
[2] Y. Yang, H. Bi, Y. Zhang, "The adaptive Ciarlet-Raviart mixed method for biharmonic problems with simply supported boundary condition", Appl. Math. Comput., vol.339, pp.206-219, 2018.

Volume 10, Issue 6, pp. 31-39, June 2026. ISSN (Online): 2456-7361


[3] Han, Jiayu, Yang, Yidu, Hai, Hao, "Mixed methods for the Helmholtz transmission eigenvalues", SIAM J. Sci. Comput., vol.38, no.3, pp.A1383-A1403, 2016.

[4] Yu, Zhang, Hai, Bi, Yidu, Yang, "The two-grid discretization of Ciarlet-Raviart mixed method for biharmonic eigenvalue problems", Appl. Numer. Math., vol.138, pp.94-113, 2019.

[5] Y. Yang, H. Bi, J. Han, Y. Yu, "The shifted-inverse iteration based on the multigrid discretizations for eigenvalue problems", SIAM J. Sci. Comput., vol.37, no.6, pp.A2583-A2606, 2015.

[6] H. Chen, H. Xie, F. Xu, "A full multigrid method for eigenvalue problems", J. Comput. Phys., vol.322, pp.747-759, 2016.

[7] Y. Liu, Y. Chen, Y. Huang, "Efficient algorithm based on two-grid method for semiconductor device problem", Comput. Math. Appl., vol.144, pp.221-228, 2023.

[8] Chen, Yp, Zhou, Jw, Hou, Tl, "Two-grid Method of Expanded Mixed Finite Element Approximations for Parabolic Integro-differential Optimal Control Problems", Acta Math. Appl. Sin. Engl. Ser., vol.41, pp.1106–1129, 2025.

[9] Hu J, Liu Z, Ma R, Wang R, "A Mixed Finite Element Method for Coupled Plates",Computational Methods in Applied Mathematics, vol.25, pp.601-618, 2025.

[10] Wang Z, Wang Y, Zhu R, Beauwens R, "Stability and convergence analysis of the mixed finite element method for subdiffusive Oseen equations", Math. Comput. Simul., vol.245, pp.483-511, 2026.

[11] Y. Zhou, X. Leng, Y. Li, Q. Deng, Z. Luo, "A novel two-grid Crank-Nicolson mixed finite element method for nonlinear fourth-order sin-Gordon equation", AIMS Math., vol.9, no.11, pp.31470-31494, 2024.

[12] J. Wang, B. Yin, Y. Liu, H. Li, "A fast two-grid mixed element method for a fourth-order fractional diffusion-wave equation", Numer. Sin., vol.44, no.4, pp.496-507, 2022.

[13] D. VandenHeuvel, S. Olver, "A Sparse hp-Finite Element Method for Piecewise-Smooth Differential Equations with Periodic Boundary Conditions", arXiv, Dec. 19, 2025.

[14] Yu, Zhang, Hai, Bi, Yidu, Yang, "The two-grid discretization of Ciarlet–Raviart mixed method for biharmonic eigenvalue problems", Appl. Numer. Math., vol.138, pp.94-113, 2019.

[15] D.Shi, H. Zhang, "Convergence analysis of nonconforming quadrilateral finite element methods for nonlinear coupled Schrödinger-Helmholtz equations", J. Comput. Math., vol.42, no.4, pp.979-998, 2024.

[16] Hughes V, Harris I, Sun J, "The anisotropic interior transmission eigenvalue problem with a conductive boundary", Commun. Anal. Comput., vol.4, pp.60–82, 2025.