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The Positive Integer Solutions of Arithmetic Function Equation $t\left(\varphi_2\left(\frac{n(n+1)}{2}\right)\right)^2 = S\left(SL\left(n^{19}\right)\right)$

WU Yongjun, SHI Fenglin, Guan Yanlin
School of Mathematical Sciences, Guizhou Normal University, Guiyang, Guizhou 550025, China

Abstract— *This paper investigates the solvability of the equation* $t\left(\varphi_2\left(\frac{n(n+1)}{2}\right)\right)^2 = S\left(SL\left(n^{19}\right)\right)$ *where* $t \in Z^+, n \in Z^+$, *by utilizing the definitions and properties of the Smarandache function* $S(n)$, *the Smarandache LCM function* $SL(n)$, *Euler's totient function* $\varphi(n)$, *the generalized Euler function* $\varphi_2(n)$, *and employing elementary number theory methods. It is obtained that the equation admits positive integer solutions only for* $t = 5, 10, 22, 42$ *and all positive integer solutions* $(t,n) = (5,5), (10,4), (22,2), (42,3)$ *are given.*[1]

Keywords— *Smarandache function ; Smarandache LCM function ; Generalized Euler function; Positive integer solutions*.

I. Introduction

The Smarandache function, the Smarandache LCM function, Euler's totient function, and the generalized Euler function are important functions in number theory. The Smarandache function is defined as $S(n) = \min\{m : m \in Z^+, n \mid m!\}$, and the Smarandache LCM function is defined as $SL(n) = \min\{k \in Z^+ : n \mid [1,2,\ldots,k]\}$. Many results have been obtained in the study of these functions. Reference [1] studied the solutions of the number-theoretic functional equation $\varphi_2(n) = S(SL(n^k))$; Reference [2] investigated the solvability of the number-theoretic functional equation $(\varphi_2(n))^2 = S(SL(n^{3,4}))$; Reference [3] studied the solvability of the number-theoretic functional equation $S(SL(n)) = \varphi_2(n)$. Euler's totient function is defined as the number of integers in the sequence that are coprime to $1,2,\ldots,n-1$. Many results have also been obtained in the study of Euler's totient function .Reference [4] studied the positive integer solutions of the composite Euler function equation $\varphi(x - \varphi(\varphi(x))) = 6$ and obtained five positive integer solutions. The generalized Euler function $\varphi_e(n)$ is defined as the number of integers in the sequence $1,2,\ldots,\left[\frac{n}{e}\right]$ that are coprime to n. Similarly, many results on the generalized Euler function $\varphi_e(n)$ have been obtained in recent years. Reference [5] investigated the positive integer solutions of the equation involving two generalized Euler functions $\varphi_3(n) = 2^{\omega(n)}$ and $\varphi_4(n) = 2^{\omega(n)}$. In this paper, using the definitions and properties of the Smarandache function $S(n)$, the Smarandache LCM function $SL(n)$, Euler's totient function $\varphi(n)$, and the generalized Euler function $\varphi_2(n)$, together with elementary number theoretic methods, we study the solvability of the number-theoretic functional equation

$$t\left(\varphi_2\left(\frac{n(n+1)}{2}\right)\right)^2 = S\left(SL\left(n^{19}\right)\right).$$

II. Main Lemma

Lemma 1[1] When $n>2$, we have $2 \mid \varphi(n)$.

Lemma 2[1] When $n \geqslant 3$, we have $\varphi_2(n) = \frac{\varphi(n)}{2}$.

Lemma 3[1] If a positive integer $n = p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}$, where $p_1 < p_2 < \cdots < p_k$ is a prime, then

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right),$$

$$SL(n) = max\left\{p_1^{l_1}, p_2^{l_2} \cdots, p_k^{l_k}\right\},$$

$$S(n) = max\left\{S\left(p_1^{l_1}\right), S\left(p_2^{l_2}\right), \ldots, S\left(p_k^{l_k}\right)\right\}.$$

Lemma 4[5] For a prime p and a positive integer k, we have $S\left(p^k\right) \leqslant kp$; in particular, when $k<p$, we have $S\left(p^k\right) = kp$.

Lemma 5[5] For any positive integer n, we must have $\varphi(n) \geqslant \sqrt{\frac{n}{2}}$.

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[1] Wu Yongjun (1994–), male, master's degree candidate, main research direction: number theory and its applications, E-mail: 1561048376@qq.com;

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Lemma 6[5] For any positive integers m and n, we have $\varphi(mn)=\frac{(m,n)\varphi(m)\varphi(n)}{\varphi(m,n)}=\varphi(nm)$.. In particular, when $\gcd(m,n)=1$, we have $\varphi(mn)=\varphi(m)\varphi(n)$.

III. Theorem and Its Proof

Theorem 1. Let $t,n\in\mathrm{N}^*$, the number-theoretic functional equation

$$t\left(\varphi_2\left(\frac{n(n+1)}{2}\right)\right)^2=S\left(SL\left(n^{19}\right)\right). \quad (1)$$

has positive integer solutions, and the positive integer solutions are

$(t,n)=(5,5),(10,4),(22,2),(42,3)$.

Proof:

(1) When $n=1,\varphi_2(1)=0,S\left(SL\left(1^{19}\right)\right)=S(1)=1$.

Substituting into equation (1) shows that the equation has no positive integer solutions in this case.

(2) When $n=2,\varphi_2(3)=1,S\left(SL\left(2^{19}\right)\right)=S\left(2^{19}\right)=22$.

Substituting into equation (1) yields $t=22$; then the positive integer solution of equation (1) in this case is unique.

(3) When $n\geqslant 3$, Let $n=p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}$, where p is a prime satisfying $p^l=max\left\{p_1^{l_1},p_2^{l_2}\cdots,p_k^{l_k}\right\}$.

From Lemma 3 and Lemma 4, we obtain

$$S\left(SL\left(n^{19}\right)\right)=S\left(p^{19l}\right)\leqslant 19lp. \quad (2)$$

From $n\geqslant 3, \frac{n(n+1)}{2}>3$, by Lemma 2 and Lemma 5 we obtain

$$t\left(\varphi_2\left(\frac{n(n+1)}{2}\right)\right)^2=t\left(\frac{1}{2}\varphi\left(\frac{n(n+1)}{2}\right)\right)^2\geq\frac{t}{4}\left(\sqrt{\frac{n(n+1)}{4}}\right)^2\geq\frac{t}{4}\cdot\frac{n(n+1)}{4}\geq\frac{tn}{4}. \quad (3)$$

Combining (2) and (3) yields

$$\frac{tn}{4}\leqslant 19lp\Rightarrow tn\leqslant 76lp.$$

Therefore $1\leqslant t\leqslant 76lp,\ 3\leqslant n\leqslant 76lp. \quad (4)$

Let $n=p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}=p^l m\geq 3,$ and $\gcd=\left(p^l,m\right)=1$.

① When n is odd, $\gcd=\left(n,\frac{n+1}{2}\right)=1$. combining Lemma 5, Lemma 3, and equation (3)

$$t\left(\frac{1}{2}\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=\frac{t}{4}\left(\varphi(n)\varphi\left(\frac{n+1}{2}\right)\right)^2\geq\frac{1}{4}\left(\varphi(n)\right)^2=\frac{1}{4}\left(\varphi(p^l)\varphi(m)\right)^2\geq\frac{1}{4}p^{2l-2}(p-1)^2.$$

Combining (2)

$$\frac{1}{4}p^{2l-2}(p-1)^2\leqslant 19lp\Rightarrow p^{2l-3}(p-1)^2\leqslant 76l. \quad (5)$$

② When n is even, $\gcd=\left(\frac{n}{2},n+1\right)=1,$ take $p=2,n=2^l m,$, combining Lemma 5, Lemma 3, and equation (3)

$$t\left(\frac{1}{2}\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=\frac{t}{4}\left(\varphi\left(\frac{n}{2}\right)\varphi(n+1)\right)^2$$

$$\geqslant\frac{1}{4}\left(\varphi\left(\frac{n}{2}\right)\right)^2\left(\sqrt{\frac{n+1}{2}}\right)^2\geqslant\frac{1}{2}\left(\varphi\left(2^{l-1}\right)\varphi(m)\right)^2$$

$$\geqslant\frac{1}{2}\left(\varphi\left(2^{l-1}\right)\right)^2=\frac{1}{2}\cdot 2^{2l-4}.$$

Combining (2) $\frac{1}{2}\cdot 2^{2l-4}\leqslant 19l\cdot 2\Rightarrow 2^{2l-4}\leqslant 76l.$ (6)

When $p>2,n=p^l m$, combining Lemma 5, Lemma 3, and equation (3)

$$t\left(\frac{1}{2}\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=\frac{t}{4}\left(\varphi\left(\frac{n}{2}\right)\varphi(n+1)\right)^2$$

$$\geqslant\frac{1}{4}\left(\varphi\left(\frac{n}{2}\right)\right)^2\left(\sqrt{\frac{n+1}{2}}\right)^2\geqslant\frac{1}{4}\left(\varphi\left(p^l\right)\varphi[\frac{m}{2}]\right)^2$$

$$\geqslant\frac{1}{4}\left(\varphi\left(p^l\right)\right)^2=\frac{1}{4}\cdot p^{2l-2}.$$

Combining (2) $\frac{1}{4}\cdot p^{2l-2}\leqslant 19lp\Rightarrow p^{2l-3}\leqslant 76l.$ (7)

When $l\geqslant\frac{3}{2}$, combining (5) (6) (7) $2^{2l-3}\leqslant 76l$

Take the logarithm of both sides of equation (10)

$$(2l-3)\log 2\leqslant\log^{76}+\log l$$

We obtain the solution $l\leq 5$. If $l<\frac{3}{2}$, then $l\leqslant 5$ is satisfied.

Next, we discuss five cases for l.

Case 1 If $l=1$, since p is a prime, we have $p\geqslant 2$.

When $p=2$, $3\leq n\leq 152$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(2^{19}\right)=4\times 22=88$, calculation

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yields the solution $(t,n)=(22,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

When $p=3$, $1\le t\le 228$, $3\le n\le 228$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(3^{19}\right)=4\times 42=168$, calculation yields the solution $(t,n)=(42,3)$. Substituting back into functional equation (1) for verification shows that $(t,n)=(42,3)$ is a solution of the equation.

When $p=5$, $1\le t\le 380$, $3\le n\le 380$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(5^{19}\right)=4\times 80=320$, calculation yields the solution $(t,n)=(5,5),(20,4),(80,3)$. Substituting back into functional equation (1) for verification shows that $(t,n)=(5,5)$ is a solution of the equation.

When $p=7$, $1\le t\le 532$, $3\le n\le 532$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(7^{19}\right)=4\times 119=476$, calculation yields the solution $(t,n)=(119,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

When $p=11$, $1\le t\le 836$, $3\le n\le 836$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(11^{19}\right)=4\times 198=792$, calculation yields the solution $(t,n)=(198,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

When $p=13$, $1\le t\le 988$, $3\le n\le 988$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(13^{19}\right)=4\times 234=936$, calculation yields the solution $(t,n)=(234,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

When $p=17$, $1\le t\le 1292$, $3\le n\le 1292$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(17^{19}\right)=4\times 306=1224$, calculation yields the solution $(t,n)=(306,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

When $p=19$, $1\le t\le 1444$, $3\le n\le 1444$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(19^{19}\right)=4\times 61=1444$, calculation yields the solution $(t,n)=(361,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

When $p>19$, $1\le t\le 76p$, $3\le n\le 76p$, Therefore, by Lemma 4 we know that $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4S\left(p^{19}\right)=76p$. Since $n=p^l m$ and $\gcd(p,m)=1$, we have $(p-1)\mid 76$, thus $(p-1)\le 76$, that is $p\le 77$. However, the primes of $19<p\le 77$ do not satisfy $(p-1)\mid 76$. Hence, when $p>19$, equation (1) has no positive integer solutions.

Case 2 If $l=2$, from equation (7) and the fact that p is a prime, we obtain $p\le 6$.

When $p=2$, $1\le t\le 304$, $3\le n\le 304$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(2^{19\times 2}\right)=4\times 40=160$, calculation yields the solution $(t,n)=(10,4),(40,3)$. Substituting back into functional equation (1) for verification shows that $(t,n)=(10,4)$ is a solution of the equation.

When $p=3$, $1\le t\le 456$, $3\le n\le 456$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(3^{19\times 2}\right)=4\times 81=324$, calculation yields the solution $(t,n)=(81,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

When $p=5$, $1\le t\le 760$, $3\le n\le 760$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(5^{19\times 2}\right)=4\times 155=620$, calculation yields the solution $(t,n)=(155,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

Case 3 If $l=3$, from equation (7) and the fact that p is a prime, we obtain $p\le 3$.

When $p=2$, $1\le t\le 456$, $3\le n\le 456$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(2^{19\times 3}\right)=4\times 62=248$, calculation yields the solution $(t,n)=(62,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

When $p=3$, $1\le t\le 684$, $3\le n\le 684$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2=4\times S\left(3^{19\times 3}\right)=4\times 117=468$, calculation yields the solution $(t,n)=(117,3)$. Substituting

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back into functional equation (1) for verification shows that equation (1) has no solution in this case.

Case 4 If $l=4$, from equation (7) and the fact that p is a prime, we obtain $p \le 2$.

When $p=2$, $1 \le t \le 608$, $3 \le n \le 608$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2 = 4 \times S\left(2^{19\times 4}\right) = 4 \times 80 = 320$, calculation yields the solution $(t,n)=(5,5),(20,4),(80,3)$. Substituting back into functional equation (1) for verification shows that $(t,n)=(5,5)$ is a solution of the equation.

Case 5 If $l=5$, from equation (7) and the fact that p is a prime, we obtain $p \le 2$.

When $p=2$, $1 \le t \le 760$, $3 \le n \le 760$, so from $t\left(\varphi\left(\frac{n(n+1)}{2}\right)\right)^2 = 4 \times S\left(2^{19\times 5}\right) = 4 \times 98 = 392$, calculation yields the solution $(t,n)=(98,3)$. Substituting back into functional equation (1) for verification shows that equation (1) has no solution in this case.

In summary, through analysis, all positive integer solutions of the number-theoretic functional equation $t\left(\varphi_2(\frac{n(n+1)}{2})\right)^2 = S(SL(n^{19})$ involving the Smarandache function, the Smarandache LCM function, Euler's totient function, and the generalized Euler function are $(t,n)=(5,5),(10,4),(22,2),(42,3)$.

IV. Conclusion

This paper uses the definitions and properties of the Smarandache function, the Smarandache LCM function, Euler's totient function, and the generalized Euler function, together with elementary number theory methods, to study the basic situation of positive integer solutions of $t\left(\varphi_2\left(\frac{n(n+1)}{2}\right)\right)^2 = S\left(SL(n^{19})\right)$, where $t \in Z^+, n \in Z^+$_, providing a reference for future research on such equations.

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