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Spectral Post-processing Method for Two-Dimensional Nonlinear Volterra Integral Equations and Its Error Analysis

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Abstract— *For the two-dimensional nonlinear case, this paper proposes an algorithmic framework for the spectral post-processing method. First, the integration domain is uniformly partitioned, and an initial approximate solution is obtained using classical methods. The original domain is then transformed into a standard orthogonal space, and the integral terms are discretized using the Gauss quadrature formula. Spectral post-processing iterative operations are carried out to complete the numerical solution. The core innovation of this algorithm lies in the use of the most recently generated numerical solution in each iteration. Theoretical analysis confirms that the algorithm achieves high-order convergence under two norms; numerical experiments validate the effectiveness and correctness of the algorithm. Both computational efficiency and solution accuracy are significantly improved, providing a more efficient technical approach for solving multidimensional nonlinear integral equations.*

Keywords— *Two-dimensional Volterra integral equation, Spectral Post-processing methods, Error analysis, numerical experiment.*

I. Introduction

Two-dimensional integral equations have important applications in various fields of applied science and engineering. Integral-differential equations serve as models for many practical problems in engineering and mechanics, and their numerical simulation has received considerable attention. Obtaining exact solutions for two-dimensional Volterra integral equations (2D-VIEs) is the most challenging task; therefore, providing their numerical solutions is of significant importance.

In recent years, numerous scholars have studied approximate solution methods for two-dimensional Volterra integral equations. Brunner[1][2] introduced the collocation method and the iterative collocation method to analyze the convergence of approximate polynomial spline functions and estimate the approximation order of two-dimensional Volterra integral equations; in [3], a set of basis functions composed of two-dimensional orthogonal trigonometric functions was used for approximate solution; in [4], based on Haar wavelets, a numerical method was proposed to solve two-dimensional nonlinear Fredholm, Volterra, and Volterra-Fredholm integral equations of the first and second kinds; in [5], the discrete collocation method based on radial basis functions for second-kind two-dimensional nonlinear Volterra-Fredholm integral equations was discussed; in [6][7], a method based on the hybrid function interpolant approach was applied to solve two-dimensional nonlinear Volterra-Fredholm integral equations. In [8][15], the two-dimensional Legendre wavelet method and simple scheme was utilized for the numerical treatment of nonlinear hybrid two-dimensional Volterra-Fredholm integral equations (2D-VFIEs). The Legendre-Gauss quadrature formula is one of the most commonly used quadrature methods for integral equations. Therefore, we provide an efficient and fast numerical scheme by combining Lagrange interpolation functions and the Legendre-Gauss quadrature formula.

Inspired by the spectral post-processing method for solving one-dimensional Volterra integral equations presented in [13], this paper studies the spectral post-processing numerical algorithm and error estimation for two-dimensional nonlinear Volterra integral equations. Firstly, by classical methods, an approximate initial value is estimated, followed by iterative solution of the two-dimensional nonlinear Volterra integral equations using the spectral post-processing method. Numerical experiments demonstrate that this method efficiently converges when solving two-dimensional Volterra integral equations. Moreover, error analysis for this method is established, providing error estimates for the numerical solution under two types of norms. In fact, this paper offers a general structure and framework for numerical algorithms and error analysis of multidimensional Fredholm and Volterra integral equations.

II. Solution Scheme For 2D Second-Order Nonlinear Vies System

This paper studies the following two-dimensional nonlinear Volterra integral equation model:

$$u(x,y)+\int_a^x\int_a^y k(x,y,s,t,u(s,t))\mathrm{d}s\mathrm{d}t=g(x,y),\quad (x,y)\in\Lambda=[a,b]^2, \tag{1}$$

Among them, the source function $f:\Lambda\to\mathbb{R}$ ($\mathbb{R}$ is the set of real numbers) is a known continuous function, and $u(x,y)$ is the unknown function to be determined, the kernel function $k(x,y,s,t)$ is defined on the set $\Xi=D\times\mathbb{R}$ ($D:=\{(x,y,s,t): a\le s\le x\le b,\ a\le t\le y\le b\}$) as a sufficiently smooth continuous function.

Define the space $L^2_\omega(\Omega) := \{u \mid \int_\Omega |u(x,y)|^2 \omega(x,y)\mathrm{d}x\mathrm{d}y < \infty\}$ of weighted integrable functions on $\Omega = [-1,1]^2$, which is a Banach space endowed with the norm $\|u\|_{L^2_\omega(\Omega)} = \left(\int_\Omega |u(x,y)|^2 \omega(x,y)\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}$, and a Hilbert space with respect to the inner product $(u,v)_\omega = \int_\Omega u(x,y)v(x,y)\omega(x,y)\mathrm{d}x\mathrm{d}y$. $L^\infty(\Omega)$ is a Banach space consisting of bounded measurable functions $u:\ \Omega \to R$ outside sets of measure zero, and this space is equipped with the norm (or modulus) $\|u\|_{L^2_\omega(\Omega)} = ess \sup_{(x,y)\in\Omega} |u(x,y)|$. Given $\alpha = (\alpha_1, \alpha_2)$ multi-index a of non-negative integers, let $|\alpha| = \alpha_1 + \alpha_2$, have $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_x^{\alpha_1} \partial_y^{\alpha_2}}$, based on this we define $H^m_\omega(\Omega) := \{u \mid u \in L^2_\omega(\Omega),\ D^\alpha u \in L^2_\omega(\Omega),\ |\alpha| \le m\}$. This is a Hilbert space with respect to the inner product $(u,v)_{\omega,m} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u(x,y) D^\alpha v(x,y)\omega(x,y)\mathrm{d}x\mathrm{d}y$, and the seminorm is $|u|_{H^{m,N}_\omega(\Omega)} = \left(\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2_\omega(\Omega)}\right)^{1/2}$.

A. Post-processing methods for spectra

Obtain an approximate value of (1) using the classical composite trapezoidal method as the initial value for the spectral post-processing method. For an integer $N > 0$, let $\tilde{P}_N = P_N \times P_N$, P_N denote the space consisting of univariate polynomials of maximum degree N. Define $\{\theta_i\}_{i=0}^N$ and $\{\varphi_j\}_{j=0}^N$ as the sets of (N+1)-order Gaussian points, with $\{\omega_i\}_{i=0}^N$ and $\{\upsilon_j\}_{j=0}^N$ representing the corresponding weight function values, respectively. Use a variable transformation to change the integral interval from $(x,y) \in [a,b]^2$ to $(x_i^s, y_j^t) \in \Omega = [-1,1]^2$, that is, let

$$x_i^s = \frac{b-a}{2}\theta_i + \frac{b+a}{2},\ y_j^t = \frac{b-a}{2}\varphi_j + \frac{b+a}{2}.$$

Thus, equation (1) becomes

$$u(x_i^s, y_j^t) + \int_a^{x_i^s}\int_a^{y_j^t} k(x_i^s, y_j^t, s, t, u(s,t))\mathrm{d}s\mathrm{d}t = \mathrm{g}(x_i^s, y_j^t),\quad 0 \le i,j \le N. \tag{2}$$

then perform a linear transformation

$$s_i = s(x_i^s, \theta) = \frac{x_i^s - a}{2}\theta + \frac{x_i^s + a}{2},\ \ t_j = t(y_j^t, \varphi) = \frac{y_j^t - a}{2}\varphi + \frac{y_j^t + a}{2},\quad -1 \le \theta, \varphi \le 1,$$

The integral term of equation (2) can be transformed into

$$\int_a^{x_i^s}\int_a^{y_j^t} k(x_i^s, y_j^t, s, t, u(s,t))\mathrm{d}s\mathrm{d}t = \frac{x_i^s - a}{2}\frac{y_j^t - a}{2}\int_{-1}^1\int_{-1}^1 k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u(s(x_i^s,\theta), t(y_j^t,\varphi)))\mathrm{d}\theta\mathrm{d}\varphi,$$

Therefore

$$u(x_i^s, y_j^t) + \frac{x_i^s - a}{2}\frac{y_j^t - a}{2}\int_{-1}^1\int_{-1}^1 k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u(s(x_i^s,\theta), t(y_j^t,\varphi)))\mathrm{d}\theta\mathrm{d}\varphi = \mathrm{g}(x_i^s, y_j^t). \tag{3}$$

Based on the spectral method outlined in [12] and the study of the spectral post-processing method for one-dimensional Volterra integral equations in [13] , it can be concluded that

$$u_N(x_i^s, y_j^t) = \mathrm{g}(x_i^s, y_j^t) - \frac{x_i^s - a}{2}\frac{y_j^t - a}{2}\sum_{k=0}^N\sum_{l=0}^N k(x_i^s, y_j^t, s(x_i^s,\theta_k), t(y_j^t,\varphi_l), u(s(x_i^s,\theta_k), t(y_j^t,\varphi_l)))\omega_k\upsilon_l. \tag{4}$$

The post-processing spectral method proposed in this paper can accelerate the convergence of standard methods (such as the collocation method) and is a spectral method known for its high accuracy. For achieving high precision, we use the classical composite trapezoidal method to derive a set of approximate values of equation (2) on a uniform grid. This set of approximations serves as the initial values for the spectral post-processing. In equation (2), due to the nonlinearity of the kernel function, handling the kernel function is somewhat challenging. Specifically, we use the interpolation method: first,

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we interpolate the initial values $\left\{u(x_i^{h_1},y_j^{h_2})\right\}_{i,j=0}^{N}$ and $\left\{u(x_i^{s},y_j^{t})\right\}_{i,j=0}^{N}$ obtained from the classical method, and then use $\left\{u(x_i^{s},y_j^{t})\right\}_{i,j=0}^{N}$ to interpolate $u(s(x_i^s,\theta_k),t(y_j^t,\varphi_l))$. The advantage of this method lies in its use of the most recently updated in each iteration, eliminating the need to solve a complex system of equations. By obtaining results through a direct iterative process, it can significantly reduce data storage requirements and computation time, as well as accelerate the convergence rate.

III. Error Analysis

This section conducts a convergence analysis of the numerical scheme, aiming to demonstrate that the proposed method exhibits exponential convergence, thereby establishing the spectral accuracy of the approximate solution provided by the meth od.

First, some basic lemmas are presented, which are important for deriving the main results of the subsequent error analysis. The following analysis examines the errors of the spectral post-processing method in the sense of two different norms, respectively.

Lemma 2.1 [14] Assume $v(x,y)\in H_\omega^m(\Omega)$, $m>1,\ \alpha,\beta>-1$ and $\phi\in L_\omega^2(\Omega)$, then there exists a constant C and an exponent related only to N, such that

$$|(u,\phi)_\omega-(u,\phi)_{\omega,N}|\le CN^{-m}\|\phi\|_{L_\omega^2(\Omega)}|u|_{H_\omega^{m,N}(\Omega)}.$$

Lemma 2.2 [9] Let $\|I_N^c\|_\infty:=\max\limits_{(x,y)\in\Omega}\sum_{i=0}^{N}\sum_{j=0}^{N}F_i(x)F_j(y)$, $\{F_i(x)\}_{i=0}^N$, $\{F_j(y)\}_{i=0}^N$ be the Nth-order Lagrange interpolation polynomials associated with the Jacobi-Gauss points $\{x_m\}_{m=0}^N$ and $\{y_n\}_{n=0}^N$, respectively. Then

$$\|I_N^c\|_\infty=\begin{cases}O((\log N)^2) & , \quad -1<\alpha,\beta\le-\frac{1}{2}\\ O((N^{\max(\alpha,\beta)+\frac{1}{2}})^2) & , \quad -\frac{1}{2}<\alpha,\beta<0\\ O((N^{\alpha+\frac{1}{2}})^2) & , \quad -1<\beta\le-\frac{1}{2},\ -\frac{1}{2}<\alpha<0\\ O((N^{\beta+\frac{1}{2}})^2) & , \quad -1<\alpha\le-\frac{1}{2},\ -\frac{1}{2}<\beta<0\end{cases}$$

Lemma 2.3 [11] For any bounded function $v(x,y)$ there exists a constant C independent of $v(x,y)$, such that

$$\sup_N\|I_N^c v\|_{L_\omega^2(\Omega)}\le C\max_{(x,y)\in\Omega}\|v\|_\infty,$$

where $I_N^c v=\sum_{i=0}^{N}\sum_{j=0}^{N}v(x_i,y_j)F_i(x)F_j(y)$ is the interpolation representation of the function v, $F_i(x),\ F_j(y),\ i,j=0,1,\ldots,N$ are Lagrange interpolation basis functions based on Jacobi Gaussian points of the corresponding weight order.

Lemma 2.4 [16] Assume $v(x,y)\in H_\omega^m(\Omega)$, $m>4$ and $\alpha,\beta>-1$, $(I_N^c u)(x,y)$ represents the interpolation polynomial associated with multidimensional Jacobi-Gauss points, then the following estimate holds:

$$\|u-I_N^c u\|_{L_\omega^2(\Omega)}\le CN^{-m}|u|_{H_\omega^{m,N}(\Omega)},\quad \|u-I_N^c u\|_{L^\infty(\Omega)}\le CN^{4-m}|u|_{H_\omega^{m,N}(\Omega)},$$

here, C is a constant.

Lemma 2.5 [10] Assume $M>0$, a non-negative integrable function $E(x,y)$ satisfies

$$E(x,y)\le M\int_{-1}^{x}\int_{-1}^{y}E(s,t)\mathrm{d}s\mathrm{d}t+G(x,y),\quad (x,y)\in\Omega.$$

If $G(x,y)$ is also an integrable function, then we have

$$\|E\|_{L_\omega^2(\Omega)}\le C\|G\|_{L_\omega^2(\Omega)},\ \|E\|_{L^\infty(\Omega)}\le C\|G\|_{L^\infty(\Omega)}.$$

here, C is a constant.

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Introduce some notation. For $r \geq 0$ and $k \in [0,1]$, $C^{r,k}([0,T])$ denotes the function space of functions whose r-th derivative is Hölder continuous with exponent k, equipped with the norm $\|\cdot\|_{r,k}$, For each non-negative integer r and k, there exists a constant $C_{r,k}$ C such that for any function $v \in C^{r,k}([0,T])$, there exists a polynomial function $\Im_N v \in \tilde{P}_N$, such that

$$\| v - \Im_N v \|_\infty \leq C_{r,k} N^{-(r+k)} \| v \|_{r,k}, \tag{5}$$

Here $\Im_N$ is a linear operator from $C^{r,k}([0,T])$ to $\tilde{P}_N$. Define the integral operator M, M is a bounded linear compact operator from $C(\bar{\Omega})$ to $C^{0,k}(\bar{\Omega})$, that is

$$Mv = \int_{-1}^{x}\int_{-1}^{y} \tilde{k}(x,y,\xi,\eta,u(\xi,\eta))\mathrm{d}\xi\mathrm{d}\eta .$$

Let

$$Mv_1v_2 = \int_{-1}^{x}\int_{-1}^{y} [\tilde{k}(x,y,\xi,\eta,v_1(\xi,\eta)) - \tilde{k}(x,y,\xi,\eta,v_2(\xi,\eta))]\mathrm{d}\xi\mathrm{d}\eta. \tag{6}$$

Theorem 2.1 Suppose that $u(x,y)$ is a sufficiently smooth solution of the two-dimensional nonlinear Volterra integral equation (2), and $\tilde{k}(x,y,\xi,\eta,u)$ with respect to u satisfies an m-th order Lipschitz condition. Let $u_N^d(x,y)$ be the solution obtained by the spectral post-processing method in equation (4), d denotes the number of iterations, and $u \in H_\omega^m(\Omega)$, $m \geq 1$, then we have

$$\| u - u_N^d \|_{L^\infty(\Omega)} = C(N^{4-m} | u |_{H_\omega^{m,N}(\Omega)} + N^{-m} K^* \| I_N^c \|_\infty),$$

here C is a constant, $\| I_N^c \|_\infty$ as shown in Lemma 2.2, $K^* = \max_{(x,y)\in\Omega} \left| \tilde{k}(x,y,\xi,\eta,u) \right|_{H_\omega^{m,N}(\Omega)}$.

Proof: First, let $u_{m,N}^d$ represents the approximate solution obtained after d iterations, $0 \leq m \leq N$, which can be obtained according to the d-iteration solution and (4), can be obtained

$$u_{m,N}^{d+1} = \mathrm{g}(x_i^s, y_j^t) - \frac{x_i^s - a}{2}\frac{y_j^t - a}{2}\int_{-1}^{1}\int_{-1}^{1} k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u_N^d(s(x_i^s,\theta), t(y_j^t,\varphi)))\mathrm{d}\theta\mathrm{d}\varphi + H_1^d, \tag{7}$$

Here u_N^d represents the interpolating polynomial obtained through $\left\{x_i^s\right\}_{i=0}^N$, $\left\{y_j^t\right\}_{j=0}^N$ and $\left\{u_{m,N}^{d-1}\right\}_{m=0}^N$, and

$$H_1^d = \frac{x_i^s - a}{2}\frac{y_j^t - a}{2}\Bigg[\int_{-1}^{1}\int_{-1}^{1} k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u_N^d(s(x_i^s,\theta), t(y_j^t,\varphi)))\mathrm{d}\theta\mathrm{d}\varphi$$
$$- \sum_{k=0}^{N}\sum_{l=0}^{N} k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u_N^d(s(x_i^s,\theta), t(y_j^t,\varphi)))\omega_k\upsilon_l\Bigg].$$

Using Lemma 2.1 on the integral estimation result of Gaussian quadrature polynomials, we have

$$\left|H_1^d\right| \leq CN^{-m} \left| k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u_N^d(s(x_i^s,\theta), t(y_j^t,\varphi))) \right|_{H_\omega^{m,N}(\Omega)}. \tag{8}$$

Based on equation (4) analyzing the kernel function, we have

$$\begin{aligned} &\left| k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u_N^d(s(x_i^s,\theta), t(y_j^t,\varphi))) \right| \\ &\leq \left| k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u(s(x_i^s,\theta), t(y_j^t,\varphi))) \right| \\ &+ \left| k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u_N^d(s(x_i^s,\theta), t(y_j^t,\varphi))) \right. \\ &\left. - k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u(s(x_i^s,\theta), t(y_j^t,\varphi))) \right|, \end{aligned} \tag{9}$$

combining the above two equations, we have

$$\begin{aligned} \left|H_1^d\right| \leq CN^{-m} \Big(K^* + &\left| k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u_N^d(s(x_i^s,\theta), t(y_j^t,\varphi))) \right. \\ &\left. - k(x_i^s, y_j^t, s(x_i^s,\theta), t(y_j^t,\varphi), u(s(x_i^s,\theta), t(y_j^t,\varphi))) \right|_{H_\omega^{m,N}(\Omega)} \Big). \end{aligned} \tag{10}$$

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Continue calculating the right-hand side of the above expression

$$\left|k(x_i^s,y_j^t,s(x_i^s,\theta),t(y_j^t,\varphi),u_N^d(s(x_i^s,\theta),t(y_j^t,\varphi)))-k(x_i^s,y_j^t,s(x_i^s,\theta),t(y_j^t,\varphi),u(s(x_i^s,\theta),t(y_j^t,\varphi)))\right|_{H_\omega^{m,N}(\Omega)}$$

$$=\left(\sum_{\alpha_1+\alpha_2=0}^{m}\left\|\frac{\partial^{\alpha_1+\alpha_2}k(x_i^s,y_j^t,s,t,u_N^d(s,t))}{\partial s^{\alpha_1}\partial t^{\alpha_2}}-\frac{\partial^{\alpha_1+\alpha_2}k(x_i^s,y_j^t,s,t,u(s,t))}{\partial s^{\alpha_1}\partial t^{\alpha_2}}\right\|_{L_\omega^2(\Omega)}^2\right)^{1/2}$$

$$\le\sum_{\alpha_1+\alpha_2=0}^{m}\left\|\frac{\partial^{\alpha_1+\alpha_2}k(x_i^s,y_j^t,s,t,u_N^d(s,t))}{\partial s^{\alpha_1}\partial t^{\alpha_2}}-\frac{\partial^{\alpha_1+\alpha_2}k(x_i^s,y_j^t,s,t,u(s,t))}{\partial s^{\alpha_1}\partial t^{\alpha_2}}\right\|_{L_\omega^2(\Omega)}$$

$$\le\sum_{\alpha_1+\alpha_2=0}^{m}L_{\alpha_1+\alpha_2}\left\|u_N^d(s,t)-u(s,t)\right\|_{L_\omega^2(\Omega)}\le C\left\|u_N^d-u\right\|_{L_\omega^2(\Omega)},$$

Among $L_{\alpha_1+\alpha_2}$ is the differentiable Lipschitz constant. Therefore

$$\left|H_1^d\right|\le CN^{-m}\left(K^*+\left\|u_N^d-u\right\|_{L_\omega^2(\Omega)}\right). \tag{11}$$

Subtract (7) from (2) to get

$$u(x_i^s,y_j^t)-u_{m,N}^{d+1}=\lambda\int_a^{x_i^s}\int_b^{y_j^t}[k(x_i^s,y_j^t,s,t,u(s,t))-k(x_i^s,y_j^t,s,t,u_N^d(s,t))]\mathrm{dsdt}-H_1^d,(x_i^s,y_j^t)\in[-1,1]^2, \tag{12}$$

Multiply both sides of the above equation by $F(x_i)F(y_j)$ and sum from $i,j=0$ to N obtain

$$I_N^c u(x_i^s,y_j^t)-u_{m,N}^{d+1}=I_N^c\left(\lambda\int_a^{x_i^s}\int_a^{y_j^t}[k(x_i^s,y_j^t,s,t,u(s,t))-k(x_i^s,y_j^t,s,t,u_N^d(s,t))]\mathrm{dsdt}\right)-I_N^c H_1^d,$$

let $e^{d+1}(x_i^s,y_j^t)=u_{m,N}^{d+1}(x_i^s,y_j^t)-u(x_i^s,y_j^t)$, Then the above expression can be expressed as

$$e^{d+1}(x_i^s,y_j^t)=I_N^c\left(\int_a^{x_i^s}\int_a^{y_j^t}[k(x_i^s,y_j^t,s,t,u(s,t))-k(x_i^s,y_j^t,s,t,u_N^d(s,t))]\mathrm{dsdt}\right)+J_1(x_i^s,y_j^t)+J_2(x_i^s,y_j^t), \tag{13}$$

here $J_1(x_i^s,y_j^t)=u(x_i^s,y_j^t)-I_N^c u(x_i^s,y_j^t)$, $J_2(x_i^s,y_j^t)=-I_N^c H_1^d(x_i^s,y_j^t)$. Applying the Lagrange mean value theorem for differentials to the above expression, there exists a function $\phi(s,t)=u_N^d(s,t)+ke^d(s,t)(0<k<1)$, such that

$$e^{d+1}(x_i^s,y_j^t)=I_N^c\left(\int_a^{x_i^s}\int_a^{y_j^t}\frac{\partial k(x_i^s,y_j^t,s,t,\phi(s,t))}{\partial u}e^d(s,t)\mathrm{dsdt}\right)+J_1(x_i^s,y_j^t)+J_2(x_i^s,y_j^t),$$

namely

$$e^{d+1}(x_i^s,y_j^t)=\int_a^{x_i^s}\int_a^{y_j^t}\frac{\partial k(x_i^s,y_j^t,s,t,\phi(s,t))}{\partial u}e^d(s,t)\mathrm{dsdt}+J_1(x_i^s,y_j^t)+J_2(x_i^s,y_j^t)+J_3(x_i^s,y_j^t), \tag{14}$$

here $J_3(x_i^s,y_j^t)=I_N^c(Me^d)-Me^d$. Therefore

$$\left|e^{d+1}(x_i^s,y_j^t)\right|=C\int_a^{x_i^s}\int_a^{y_j^t}\left|e^d(s,t)\right|\mathrm{dsdt}+\left|J_1(x_i^s,y_j^t)\right|+\left|J_2(x_i^s,y_j^t)\right|+\left|J_3(x_i^s,y_j^t)\right|.$$

According to Lemma 2.5 , we have

$$\| e^{d+1}\|_{L^\infty(\Omega)}\le C\left(\| J_1\|_{L^\infty(\Omega)}+\| J_2\|_{L^\infty(\Omega)}+\| J_3\|_{L^\infty(\Omega)}\right). \tag{15}$$

Estimate now $\| J_1\|_{L^\infty(\Omega)}$, $\| J_2\|_{L^\infty(\Omega)}$, $\| J_3\|_{L^\infty(\Omega)}$, First, we obtain it through Lemma 2.4

$$\| J_1\|_{L^\infty(\Omega)}=\| u-I_N^c u\|_{L^\infty(\Omega)}\le CN^{4-m}\,|u|_{H_\omega^{m,N}(\Omega)}, \tag{16}$$

Secondly, according to Lemma 2.2 and equation (11), we have

$$\|J_2\|_{L^\infty(\Omega)} = \|I_N^c H_1^d(x,y)\|_{L^\infty(\Omega)} \le C \max_{(x,y)\in\bar{\Omega}} |I_N^c(x,y)| \max_{(x,y)\in\bar{\Omega}} \sum_{i=0}^{N}\sum_{j=0}^{N} F_i(x)F_j(y) \\ \le CN^{-m} \|I_N^c\|_\infty (K^* + \|e^d\|_{L^\infty(\Omega)}), \tag{17}$$

Finally, combining the interpolation relation formula $I_N^c u(x,y) = u(x,y)$, $(I_N^c - I)u(x,y) = 0$, I as the identity operator, we have

$$\|J_3\|_{L^\infty(\Omega)} = \|I_N^c(Me^d) - Me^d\|_{L^\infty(\Omega)} = \|(I_N^c - I)(Me^d - \Im_N(Me^d))\|_{L^\infty(\Omega)} \\ \le (1 + \|I_N^c\|_\infty)\|Me^d - \Im_N(Me^d))\|_{L^\infty(\Omega)} \le \\ CN^{-k}(1 + \|I_N^c\|_\infty)\|Me^d\|_{0,k} \le CN^{-k}\|I_N^c\|_\infty \|e^d\|_{L^\infty(\Omega)}, \tag{18}$$

The estimation results are obtained by combining (15) to (18).

Theorem 2.2 Suppose $u(x,y)$ is a solution satisfying the conditions of Theorem 2.1, and $u_N^d(x,y)$ is the solution obtained by the spectral post-processing method in equation (4), where d denotes the number of iterations. Then we have

$$\|u - u_N^d\|_{L^\infty(\Omega)} = C\left(N^{4-m-k}|u|_{H_\omega^{m,N}(\Omega)} + N^{-m}K^*(1 + N^{-k}\|I_N^c\|_\infty)\right),$$

here C is a constant, $\|I_N^c\|_\infty$ as shown in Lemma 2.2, $K^* = \max_{(x,y)\in\Omega}\left|\tilde{k}(x,y,\xi,\eta,u)\right|_{H_\omega^{m,N}(\Omega)}$.

Proof: From (21) and Lemma 2.5, we get

$$\|e^{d+1}\|_{L_\omega^2(\Omega)} \le C\left(\|J_1\|_{L_\omega^2(\Omega)} + \|J_2\|_{L_\omega^2(\Omega)} + \|J_3\|_{L_\omega^2(\Omega)}\right). \tag{19}$$

According to Lemma 2.4 $\|J_1\|_{L_\omega^2(\Omega)} = \|u - I_N^c u\|_{L_\omega^2(\Omega)} \le CN^{-m}|u|_{H_\omega^{m,N}(\Omega)}$, (20)

By Lemma 2.3 and (11), we have

$$\|J_2\|_{L_\omega^2(\Omega)} = \|I_N^c H_1^d(x,y)\|_{L_\omega^2(\Omega)} \le C\max_{(x,y)\in\bar{\Omega}}|I_N^c(x,y)| \le CN^{-m}(K^* + \|e^d\|_{L_\omega^2(\Omega)}), \tag{21}$$

Combining Lemma 2.3 and (5), have

$$\|J_3\|_{L_\omega^2(\Omega)} = \|I_N^c(Me^d) - Me^d\|_{L_\omega^2(\Omega)} = \|(I_N^c - I)(Me^d - \Im_N(Me^d))\|_{L_\omega^2(\Omega)} \\ \le \|I_N^c(Me^d - \Im_N(Me^d))\|_{L_\omega^2(\Omega)} + \|Me^d - \Im_N(Me^d))\|_{L_\omega^2(\Omega)} \\ \le C\|Me^d - \Im_N(Me^d))\|_{L^\infty(\Omega)} \le CN^{-k}\|Me^d\|_{0,k} \le CN^{-k}\|e^d\|_{L^\infty(\Omega)} \tag{22}$$

Using the estimation result of Theorem 2.1 in combination with (19) to (22), the L^2 norm error estimate is obtained.

IV. NUMERICAL EXPERIMENTS

In this section, the correctness of the theoretical analysis will be verified through the solution of numerical examples. Err ors will be presented in two forms: $\|\cdot\|_{L_\omega^2(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$, and all numerical calculations are implemented using Matlab2024b.

4.1 Example 1

Consider the following two-dimensional nonlinear Volterra integral equation

$$u(x,y) + \int_0^x\int_0^y (s^2 + e^{-2t})u^2(s,t)\mathrm{d}s\mathrm{d}t = x^2\mathrm{e}^y - \frac{x^7}{14} + \frac{x^7 e^{2y}}{14} + \frac{x^5 y}{5},\ (x,y)\in[0,1]^2,$$

Its exact solution is $u(x,y) = \sin(2x+y)$. After four iterations ($d = 4$), the error results between the approximate solution and the exact solution are listed in Table 1. The error results in the $\|\cdot\|_{L_\omega^2(\Omega)}$ norm and the $\|\cdot\|_{L^\infty(\Omega)}$ norm demonstrate the feasibility and effectiveness of the method presented in this paper. To visually display the effectiveness of the algorithm, the corresponding error variation graph of Table 1 is depicted in Figure 1.

Table 1. Variation of the error $\left\| u_N^d - u \right\|$ with N in Example 1

N	4	6	8	10	12
L_ω^2 -误差	1.0629e-04	8.5060e-07	2.8918e-09	7.2161e-12	1.2044e-14
L^∞ -误差	1.2839e-04	1.2638e-06	4.4365e-09	1.1976e-11	2.2482e-14

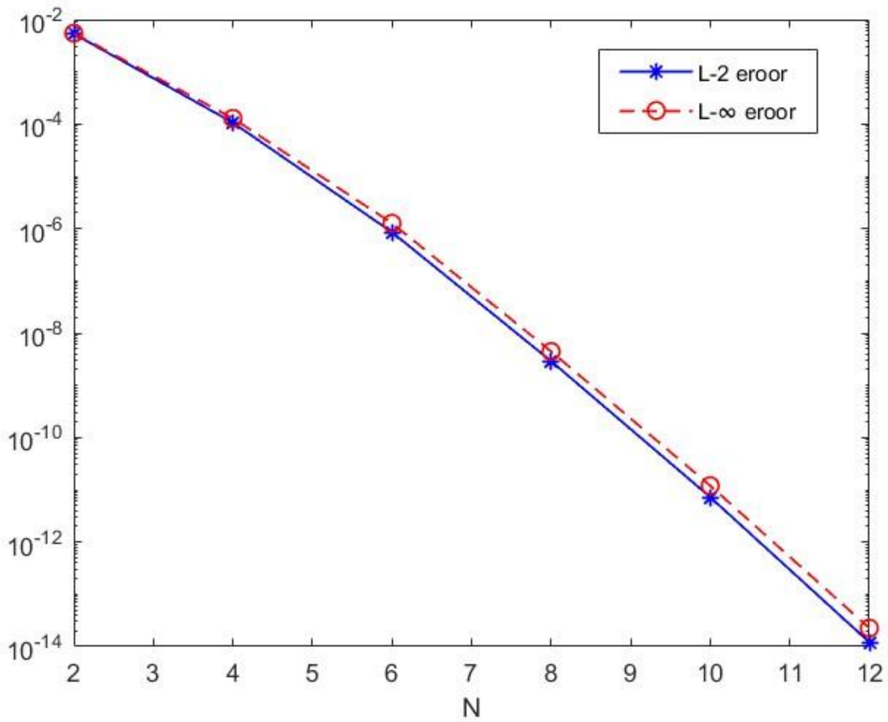

Figure 1. Example 1 Error variation diagram under two types of norms

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