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Efficient Spectral Method for Elliptic Equations Problems in Complex Region

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Abstract**—** *This paper proposes and studies a spectral method for solving complex elliptical problems with Neumann boundary conditions. In theory, we transform the region into polar coordinates through transformation. Subsequently, we established a weak form and a discrete variational form. And constructed a set of appropriate basis functions that satisfy the pole conditions, transformed the problem into a matrix system, and verified its effectiveness through a large number of numerical examples, improving spectral accuracy.*

Keywords**—** *Elliptic problems, complex regions, spectral method, algorithm design, numerical experiment.*

I. INTRODUCTION

Solving elliptic equations remains a central research topic in many scientific and engineering disciplines. Elliptical equations are widely used in fields such as fluid mechanics, electromagnetics, and quantum mechanics, playing an indispensable role in accurately describing and understanding various physical phenomena and processes. These issues are deeply rooted in important physical backgrounds and have wide applications in quantum mechanics, fluid mechanics, modern science and technology, engineering, and other fields [1-5]. So far, many methods have been developed to solve eigenvalue problems [6-11], including traditional methods, finite element method (FEM), and finite difference method (FDM). The construction of direct spectral methods for complex geometric problems presents significant challenges. Generally speaking, there are two types of direct spectral methods: one is to embed complex geometric domains into a larger regular domain, known as the virtual domain method; Another approach is to use mapping to transform complex geometry into regular domains. Gu and Shen [12] applied the full spectrum method in the virtual domain to solve complex geometric elliptic differential equations. In reference [13], Wang et al. considers polar coordinate transformation to transform complex geometric shapes convert to a unit disc. Provided some basic properties of polar coordinate transformation. In applications, we consider elliptic equations in two-dimensional complex geometry, and proved the existence and uniqueness of weak solutions and constructed the Fourier-Legendre spectral Galerkin scheme. In reference [14], Wang et al. proposed and studied a new hybrid spectral method for solving fourth-order problems with Neumann boundary conditions. And extended the algorithm to complex domains and verified its effectiveness. Pan et al.[15] first proposed differential spectral approximation of Allen-Cahn equation in circular domain. J. Zheng and J. An [16] transformed the fourth-order equation into a second-order coupled system and solved the Cahn-Hilliard equation in a two-dimensional complex domain using domain transformation. In the context of finite element methods, achieving high-precision solutions typically requires a significant amount of computational storage. The finite difference method provides greater flexibility in scheme construction and is conceptually simpler, but as geometric shapes become more complex, they become increasingly difficult to implement. Therefore, using spectral methods to solve second-order elliptic equations in complex regions is of great significance.

This paper presents an efficient spectral method for solving second-order elliptic problems in complex domains. The approach transforms Cartesian coordinates into polar coordinates adapted to the complex geometry, and then applies variable transformations to convert functions defined on convex regions in polar coordinates into functions on standard domains. A set of effective basis functions is constructed, and approximate solutions are obtained using these basis functions. The governing equations are discretized by means of the Legendre spectral method, resulting in a linear system that can be solved efficiently. Numerical examples are provided, and the results demonstrate the effectiveness and convergence of the proposed method.

The rest of this article is arranged as follows. In Section 2, We have defined a class of Sobolev spaces and established their variational and corresponding discrete formats. In Section 3, we focus on constructing the basis functions and implementing the algorithm. In Section 4, we present a numerical example to validate the theoretical findings and the effectiveness of the algorithm. Finally, we give in Section 5 a concluding remark.

II. WEAK FORM AND DISCRETE FORMAT

In this paper, we consider the second-order elliptic problem under Riemann boundary conditions as follows:

$$-\Delta u+\alpha u=f, \qquad in\ \Omega, \qquad (2.1)$$

$$\frac{\partial u}{\partial n}=0, \qquad on\ \Omega, \qquad (2.2)$$

where Ω represents a complex region, while α represent positive constants, n is the outward normal vector at boundary $\partial\Omega$.

Introduce the usual Sobolev space:

$$L^2(\Omega):=\{u:\int_\Omega u^2dx<\infty\}, \quad H^1(\Omega):=\{u:u\in L^2(\Omega)\},$$

endowed with the norms:

$$(u,v):=\int_{\Omega,\omega}uv\omega dx, \quad \|u\|_\omega=(u,v)^{\frac{1}{2}}.$$

Define a class of weighted Sobolev space:

$$H_*^1(D) := \{\hat{u} : \int_D (1+\xi)[1+\frac{(\partial_\theta R(\theta))^2}{R^2(\theta)}] |\partial_\xi \hat{u}|^2 + (1+\xi)^{-1} |\partial_\theta \hat{u}|^2$$

$$-\frac{\partial_\theta R(\theta)}{R(\theta)}(\partial_\xi \hat{u} \partial_\theta \bar{\hat{u}} + \partial_\theta \hat{u} \partial_\xi \bar{\hat{u}}) + (1+\xi) R^2(\theta) |\hat{u}|^2 \, d\xi d\theta < \infty,$$

$$\partial_\theta \hat{u}(-1,\theta) = 0, \hat{u}(\xi,\theta) = \hat{u}(\xi,\theta+2\pi)\}.$$

endowed with the norms:

$$(\hat{u},\hat{v})_{1,*} = \int_D (1+\xi)[1+\frac{(\partial_\theta R(\theta))^2}{R^2(\theta)}] \partial_\xi \hat{u} \partial_\xi \bar{\hat{v}} + (1+\xi)^{-1} \partial_\theta \hat{u} \partial_\theta \bar{\hat{v}}$$

$$-\frac{\partial_\theta R(\theta)}{R(\theta)}(\partial_\xi \hat{u} \partial_\theta \bar{\hat{v}} + \partial_\theta \hat{u} \partial_\xi \bar{\hat{v}}) + (1+\xi) R^2(\theta) \hat{u} \bar{\hat{v}} d\xi d\theta,$$

$$\|\hat{u}\|_{1,*} = [(\hat{u},\hat{u})_{1,*}]^{\frac{1}{2}}.$$

Define transformation form as follows:

$$x = rR(\theta)\cos\theta, y = rR(\theta)\sin\theta, (r,\theta) \in (0,1) \times [0,2\pi),$$

where

$$r = \frac{1+\xi}{2}, (\xi,\theta) \in [-1,1] \times [0,2\pi). \tag{2.3}$$

Let $\hat{u}(\xi,\theta) = u(x,y), \hat{f}(\xi,\theta) = f(x,y)$.

By applying Green's formula, we can deduce the variational form of (2.1)-(2.2) :

$$a(\hat{u},\hat{v}) = F(\hat{v}), \forall \hat{v} \in H_*^1(D), \tag{2.4}$$

where

$$a(\hat{u},\hat{v}) = \int_D (1+\xi)(1+\frac{(\partial_\theta R(\theta))^2}{R^2(\theta)}) \hat{u}_\xi \hat{v}_\xi - \frac{\partial_\theta R(\theta)}{R(\theta)}(\hat{u}_\xi \hat{v}_\theta + \hat{u}_\theta \hat{v}_\xi)$$

$$+\frac{1}{1+\xi}\hat{u}_\theta \hat{v}_\theta d\xi d\theta + \alpha \int_D \frac{1+\xi}{4} R(\theta) \hat{u}\hat{v} d\xi d\theta,$$

$$F(\hat{v}) = \int_D (1+\xi) R(\theta) \hat{f} \hat{v} d\xi d\theta.$$

Let P_N be a space of polynomials of degree N, and Define approximation space

$$X_{MN} = span\{p_{mN} e^{im\theta} : p_{mN} \in P_{mN}, |m| \le M\},$$

where

$$p_{mN} = \{p_{mN} : mp_{mN}(-1) = 0, p_{mN} \in P_N\}.$$

Then, a spectral-Galerkin approximation of (2.4) reads: Find $\hat{u} \in H_*^1$, such that

$$a(\hat{u}_{MN}, \hat{v}_{MN}) = F(\hat{v}_{MN}), \forall \hat{v}_{MN} \in X_{MN}. \tag{2.5}$$

III. THE ALGORITHMIC IMPLEMENTION

In this section, we will select appropriate basis functions to help us obtain effective solutions. Firstly, we construct a set of appropriate basis functions. Let

$$\phi_{mk}(\xi) = L_k(\xi) - L_{k+2}(\xi), 0 \le k \le N+2$$

$$\phi_{m,N-1}(\xi) = \frac{\xi+1}{2}, \phi_{m,N}(\xi) = 1.$$

here, L_k is a k-th degree Legendre polynomial. It is obvious that

$$X_{MN} = \{\phi_{mk}(\xi) e^{im\theta}, |m| \le M, 0 \le k \le N-2\}.$$

We have

$$u_{MN} == \sum_{|m|=0}^{M} \sum_{j=0}^{N-\mathrm{sgn}(|m|)} u_{ij} \phi_{mk}(\xi) e^{im\theta}. \tag{3.1}$$

Specifically, u_{ij}represents the expansion coefficient, which is presented as follows

$$U = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} \\ u_{10} & u_{11} & \cdots & u_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N0} & u_{N1} & \cdots & u_{NN} \end{bmatrix}.$$

Then, the equivalent form of variational format (2.4) is as follows

$$A_1 = a_{kj}^{nm} = \int_D (1+\xi)(1+\frac{(\partial_\theta R(\theta))^2}{R^2(\theta)}) \phi'_{mj}(\xi) \phi'_{nk}(\xi) e^{i(m-n)\theta} d\xi d\theta;$$

$$A_2 = b_{kj}^{nm} = (-in) \int_D \frac{\partial_\theta R(\theta)}{R(\theta)} \phi'_{mj}(\xi) \phi_{nk}(\xi) e^{i(m-n)\theta} d\xi d\theta;$$

$$A_3 = c_{kj}^{nm} = (im) \int_D \frac{\partial_\theta R(\theta)}{R(\theta)} \phi_{mj}(\xi) \phi'_{nk}(\xi) e^{i(m-n)\theta} d\xi d\theta;$$

$$A_4 = d_{kj}^{nm} = mn \int_D \frac{1}{1+\xi} \phi_{mj}(\xi) \phi_{nk}(\xi) e^{i(m-n)\theta} d\xi d\theta;$$

$$A_5 = \gamma_{kj}^{nm} = \int_D \frac{1+\xi}{4} \phi_{mj}(\xi) \phi_{nk}(\xi) e^{i(m-n)\theta} d\xi d\theta;$$

Plugging the expressions of (3.1) into (2.4), we can obtain the following linear system:

$$A\overline{U}^m = F \tag{3.2}$$

Where

$$\overline{U}^m = (u^{-M}, u^{-M+1}, \ldots, u^0, \ldots, u^M)^T; u^m = (u_0^m, u_1^m, \ldots, u_{N-\mathrm{sgn}(|m|)}^m),$$

$$F = (f^{-M}, f^{-M+1}, \ldots, f^0, \ldots, f^M)^T; f^m = (f_0^m, f_1^m, \ldots, f_{N-\mathrm{sgn}(|m|)}^m),$$

$$A = A_1 - A_2 - A_3 + A_4 + \alpha A_5; F = \frac{1}{4}\int_D (1+\xi) R(\theta) f \phi_{nk}(\xi) e^{-in\theta} d\xi d\theta$$

IV. NUMERICAL EXPERIMENT

To substantiate the theoretical analysis and the spectral accuracy of the algorithm, we will present numerous numerical examples in this section. We will conduct programmatic calculations on the MATLAB R2021a platform.

Example 1: We take $\Omega = \{(x,y) \in R^2 : x^4 + x^4 < 1\}$, $\alpha = 0.5$, $D = \{(\xi,\theta) : -1 < \xi < 1, 0 < \theta < 2\pi\}$ as our example. We choose the exact solution $u(x,y) = xy(x^4 + y^4 - 1)^4$. Obviously, the exact solution satisfies the boundary conditions (2.2). Then, we substitute the exact solution $u(x,y)$ into equation (2.1), and obtain $f(x,y)$. By calculating the approximate solution, we obtain: for different N=20, M=60, the errors between approximate solution and exact solutions are listed in Tables 1. To further demonstrate the spectral accuracy of our algorithm intuitively, we present in Figures 1 the images of the exact solutions and the approximate solutions. In addition, we also plot the error plots under L^2norm between the exact and approximate solutions in Figure 2.

TABLE 1. Errors between $u(x,y)$ and $u_{N,60}(x,y)$ under L_2-norms for various values of N.

N	5	10	15	20	25	30
$\left\|u(x,y) - u_{N,60}(x,y)\right\|_{L^2}$	6.2067e-03	1.2930e-04	1.2577e-08	6.8380e-14	6.8386e-14	6.8391e-14

TABLE 1. Errors between $u(x,y)$ and $u_{20,M}(x,y)$ under L_2-norms for various values of M.

M	10	20	30	40	50	60
$\left\|u(x,y) - u_{20,M}(x,y)\right\|_{L^2}$	1.9084e-04	4.4301e-06	1.7920e-08	4.7405e-10	2.1109e-12	6.8380e-14

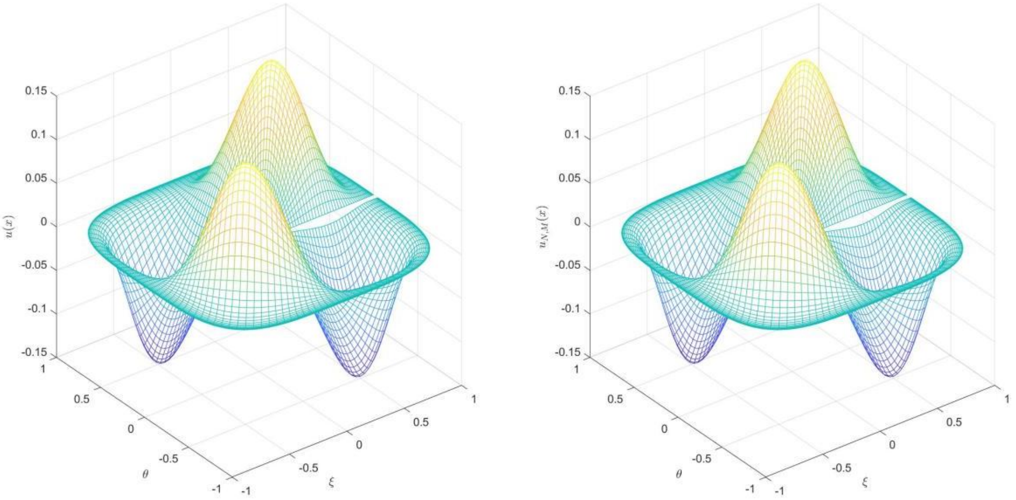

Fig. 1.Images of exact solution u(x) (left) and approximate solution $u_{N,M}$ (x) (right) with N = 20, M = 60

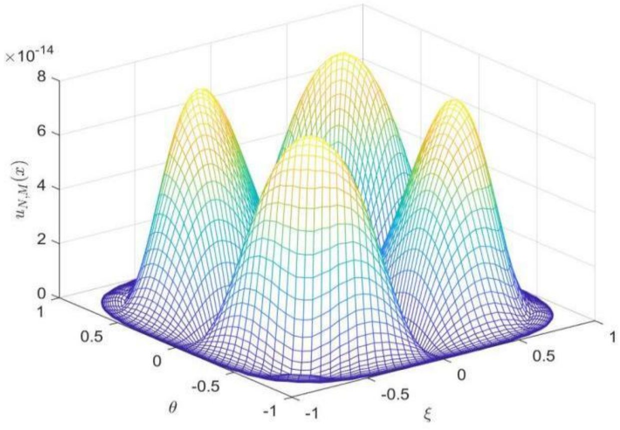

Fig. 2.Error images between u(x, y) and $u_{N,M}$ (x, y) with N = 20, M = 60

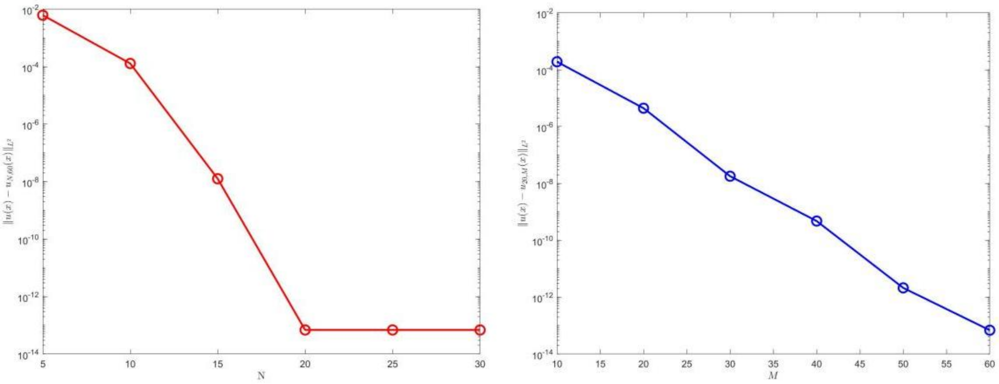

Fig. 3. Error curves between $u_{N,M}$ (x, y) with different N (left) and M (right)

By examining Figures 1-3, it becomes apparent that our algorithm exhibits convergence and spectral accuracy.

V. CONCLUDING REMARKS

This article proposes and studies an effective spectral Galerkin approximation based on complex regions. The region of elliptic problems under Neumann boundary conditions. Theoretically analyzed the variational and discrete forms of the problem. Finally, the effectiveness of the method was verified through numerical examples.

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