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The Integer Solutions of an Indeterminate Equation $x^3 \pm 6859 = 38y^2$

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Abstract**—***Indeterminate equations are important objects of study in number theory, and research on them can further promote the development of number theory. Using properties of congruences, this paper investigates the integer solutions of the indeterminate equations* $x^3 \pm 6859 = 38y^2$ *,and proves that the indeterminate equation* $x^3 + 6859 = 38y^2$ *has only the positive integer solutions* $(x,y) = (19,19),(437,1482)$*,while the indeterminate equation* $x^3 - 6859 = 38y^2$ *has only the integer solution* $(x,y) = (19,0)$*. This enriches the study of integer solutions for indeterminate equations of the form* $x^3 \pm a^3 = Dy^2$.

Keywords**—***indeterminate equation; congruence; integer solution.*

I. Introduction

From the Babylonians' search for Pythagorean triples, to Fermat's famous marginal note that he had "a truly marvelous proof," and then to Wiles's final proof of Fermat's Last Theorem after seven years of silence, the study of indeterminate equations has run through almost the entire history of mathematical development. As an important topic in number theory, it has had a far-reaching influence on the advancement of the field. In recent years, researchers have continued to investigate indeterminate equations.

The indeterminate equation

$$x^3 \pm a^3 = Dy^2 \quad (1)$$

It is an important class of cubic indeterminate equations, and many researchers have studied the integer solutions of this equation for different values of and D. Reference [1] proved that when $a = \pm 1$ and $D > 1$ (where D is square-free and is not divisible by any prime of the form $6k+1$), equation (1) has no integer solutions other than $(x,y) = (1,\pm 1),(23,\pm 78)$. Reference [2] found that when $a = \pm 1, D = 2$, equation (1) has positive integer solutions $(x,y) = (1,1),(23,78)$. References [3] and [4] proved respectively that, in the case $a = -1$, when D=749 and D=1043, equation (1) has integer solutions, and the corresponding integer solutions were obtained. Reference [5] studied the integer solutions of equation (1) when $a = -2,\ D = 13$. Reference [6] investigated the integer solutions of equation (1) when $a = \pm 3,\ D = pq$. References [7] and [8] studied the integer solutions of equation (1) when $a = \pm 5,\ D = 6q$ and when $a = \pm 13, D = 26$ respectively. Reference [9] studied the integer solutions of equation (1) when $a = \pm 17,\ D = 34$. In short, much research has been done on this equation. However, when $a = \pm 4,\ D = 152$, the integer solutions of equation (1) had not yet been studied. Therefore, this paper uses properties of congruences to investigate the integer solutions of the indeterminate equation $x^3 \pm 6859 = 38y^2$. It is found that $x^3 + 6859 = 38y^2$ has only the positive integer solutions $(x,y) = (19,19),(437,1482)$, while $x^3 - 6859 = 38y^2$ has only the integer solution $(x,y) = (19,0)$.

II. Main Lemma

Lemma 1 [9] The indeterminate equation $x^3 + 1 = 2y^2$ has only the positive integer solutions $(x,y) = (1,1),(23,78)$; the indeterminate equation $x^3 - 1 = 2y^2$ has only the integer solution $(x,y) = (1,0)$.

III. Theorem and Its Proof

Theorem 1: The indeterminate equation

$$x^3 + 6859 = 38y^2 \quad (2)$$

has only positive integer solutions $(x,y) = (19,19),(437,1482)$.

Proof: When $x \equiv 0 (\bmod 19)$, we have $x^3 \equiv 0 (\bmod 19^3)$. Then from equation (2), it follows that $38y^2 = x^3 + 6859 \ (\bmod 19^3)$, so $19^2 \mid y^2$, and hence $y \equiv 0 (\bmod 19)$. Let $x = 19x_1$, $y = 19y_1$. Substituting these into equation (2) and simplifying, we obtain $x_1^3 + 1 = 2y_1^2$, By Lemma 1, the indeterminate equation $x^3 + 1 = 2y^2$ has only the positive integer solutions $(x,y) = (1,1),(23,78)$. Therefore, we obtain $x = 19, y = 19$ or $x = 437, y = 1482$. Hence, the indeterminate equation $x^3 + 6859 = 38y^2$ has only the positive integer solutions $(x,y) = (19,19)$, $(437,1482)$.

When x is not divisible by 19, since $x^3 + 6859 = x^3 + 19^3 = (x+19)(x^2 - 19x + 361)$ and $(x+19, x^2 - 19x + 361) = (x+19, (x+19)^2 - 57(x+19) + 19^2 \times 3) = 1$ or 3, the equation $x^3 + 6859 = 38y^2$ has the following eight possible cases when $(a,b) = 1$ and $a \neq 0$:

Case 1: $x + 19 = a^2, x^2 - 19x + 361 = 38b^2, y = ab$;

Case 2: $x + 19 = 2a^2, x^2 - 19x + 361 = 19b^2, y = ab$;

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Case 3: $x+19=3a^2, x^2-19x+361=114b^2, y=3ab$;
Case 4: $x+19=6a^2, x^2-19x+361=57b^2, y=3ab$;
Case 5: $x+19=38a^2, x^2-19x+361=b^2, y=ab$;
Case 6: $x+19=19a^2, x^2-19x+361=2b^2, y=ab$;
Case 7: $x+19=114a^2, x^2-19x+361=3b^2, y=3ab$;
Case 8: $x+19=57a^2, x^2-19x+361=6b^2, y=3ab$.
These eight cases are discussed separately below:

Case 1: $x+19=a^2, x^2-19x+361=38b^2, y=ab$.

Taking both sides of $x^2-19x+361=38b^2$ modulo 19, we obtain $x^2-19x+361\equiv 38b^2 \pmod{19}$, that is $x^2\equiv 0 \pmod{19}$. Hence $x^2\equiv 0 \pmod{19}$, which contradicts the assumption that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3+6859=38y^2$.

Case 2: $x+19=2a^2, x^2-19x+361=19b^2, y=ab$.

As in Case 1, from $x^2-19x+361\equiv 19b^2 \pmod{19}$, it follows that $x^2\equiv 0 \pmod{19}$. This contradicts the fact that x is not divisible by 19.Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3+6859=38y^2$.

Case 3: $x+19=3a^2, x^2-19x+361=114b^2, y=3ab$.

As in Case 1, from $x^2-19x+361\equiv 114b^2 \pmod{19}$, it follows that $x^2\equiv 0 \pmod{19}$. This contradicts the fact that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3+6859=38y^2$.

Case 4: $x+19=6a^2, x^2-19x+361=57b^2, y=3ab$.

As in Case 1, from $x^2-19x+361\equiv 57b^2 \pmod{19}$, it follows that $x^2\equiv 0 \pmod{19}$. This contradicts the fact that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3+6859=38y^2$.

Case 5: $x+19=38a^2, x^2-19x+361=b^2, y=ab$.

From $x+19=38a^2$,we get $x=38a^2-19=19(2a^2-1)$, hence $19\mid x$, which contradicts the assumption that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3+6859=38y^2$.

Case 6: $x+19=19a^2, x^2-19x+361=2b^2, y=ab$.

From $x+19=19a^2$, we get $x=19(a^2-1)$, hence $19\mid x$, which contradicts the assumption that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3+6859=38y^2$.

Case 7: $x+19=114a^2, x^2-19x+361=3b^2, y=3ab$.

From $x+19=114a^2$, we get $x=114a^2-19=19(6a^2-1)$, hence $19\mid x$, which contradicts the assumption that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3+6859=38y^2$.

Case 8: $x+19=57a^2, x^2-19x+361=6b^2, y=3ab$.

From $x+19=57a^2$, we get $x=57a^2-19=19(3a^2-1)$, hence $19\mid x$, which contradicts the assumption that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3+6859=38y^2$.

In conclusion, the indeterminate equation $x^3+6859=38y^2$ has only the positive integer solutions $(x,y)=(19,19)$, $(437,1482)$.

Theorem 2: The indeterminate equation

$$x^3-6859=38y^2 \quad (3)$$

has only positive integer solutions $(x,y)=(19,0)$.

Proof: When $x\equiv 0 \pmod{19}$, we have $x^3\equiv 0 \pmod{19^3}$, Then from equation (3), it follows that $38y^2=x^3-6859\equiv 0 \pmod{19^3}$. Similarly, we obtain $y\equiv 0 \pmod{19}$. Let $x=19x_1$ and $y=19y_1$. Substituting these into equation (3) and simplifying, we get $x_1^3-1=2y_1^2$. By Lemma 1, the indeterminate equation $x^3-1=2y^2$ has only the integer solution $(x,y)=(1,0)$. Therefore, we obtain $x=19, y=0$. Hence, the indeterminate equation $x^3-6859=38y^2$ has only the integer solution $(x,y)=(19,0)$.

When x is not divisible by 19, since $x^3-6859=x^3-19^3=(x-19)(x^2+19x+361)$ and $(x-19, x^2+19x+361)=(x-19,(x-19)^2+57(x-19)+19^2\times 3)=1$ or 3, the equation $x^3-6859=38y^2$ has the following eight possible cases when $(a,b)=1$ and $a\neq 0$:

Case 1: $x-19=a^2, x^2+19x+361=38b^2, y=ab$;
Case 2: $x-19=2a^2, x^2+19x+361=19b^2, y=ab$;
Case 3: $x-19=3a^2, x^2+19x+361=114b^2, y=3ab$;
Case 4: $x-19=6a^2, x^2+19x+361=57b^2, y=3ab$;
Case 5: $x-19=38a^2, x^2+19x+361=b^2, y=ab$;
Case 6: $x-19=19a^2, x^2+19x+361=2b^2, y=ab$;
Case 7: $x-19=114a^2, x^2+19x+361=3b^2, y=3ab$;
Case 8: $x-19=57a^2, x^2+19x+361=6b^2, y=3ab$.
These eight cases are discussed separately below:

Case 1: $x-19=a^2, x^2+19x+361=38b^2, y=ab$.

Taking both sides of $x^2+19x+361=38b^2$ modulo 19, we obtain $x^2+19x+361\equiv 38b^2 \pmod{19}$, and hence $x^2\equiv 0 \pmod{19}$. This contradicts the assumption that x is

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not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3-6859=38y^2$.

Case 2: $x-19=2a^2, x^2+19x+361=19b^2, y=ab$.

As in Case 1, reducing modulo 19 gives $x^2\equiv 0(\bmod 19)$, which contradicts the fact that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3-6859=38y^2$.

Case 3: $x-19=3a^2, x^2+19x+361=114b^2, y=3ab$.

As in Case 1, reducing modulo 19 gives $x^2\equiv 0(\bmod 19)$, which contradicts the fact that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3-6859=38y^2$.

Case 4: $x-19=6a^2, x^2+19x+361=57b^2, y=3ab$.

As in Case 1, reducing modulo 19 gives $x^2\equiv 0(\bmod 19)$, which contradicts the fact that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3-6859=38y^2$.

Case 5: $x-19=38a^2, x^2+19x+361=b^2, y=ab$.

From $x-19=38a^2$, we get $x=38a^2+19=19(2a^2+1)$, hence $19\mid x$, which contradicts the assumption that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3-6859=38y^2$.

Case 6: $x-19=19a^2, x^2+19x+361=2b^2, y=ab$.

From $x-19=19a^2$, we get $x=19(a^2+1)$, hence $19\mid x$, which contradicts the assumption that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3-6859=38y^2$.

Case 7: $x-19=114a^2, x^2+19x+361=3b^2, y=3ab$.

From $x-19=114a^2$, we get $x=114a^2+19=19(6a^2+1)$, hence $19\mid x$, which contradicts the assumption that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3-6859=38y^2$.

Case 8: $x-19=57a^2, x^2+19x+361=6b^2, y=3ab$.

From $x-19=57a^2$, we get $x=57a^2+19=19(3a^2+1)$ hence $19\mid x$, which contradicts the assumption that x is not divisible by 19. Therefore, in this case there is no positive integer solution to the indeterminate equation $x^3-6859=38y^2$.

In conclusion, the indeterminate equation $x^3-6859=38y^2$ has only the integer solution $(x,y)=(19,0)$.

IV. CONCLUSION

Using properties of congruences, this paper proves that the indeterminate equation $x^3+6859=38y^2$ has only the positive integer solutions $(x,y)=(19,19),(437,1482)$, and that the indeterminate equation $x^3-6859=38y^2$ has only the integer solution $(x,y)=(19,0)$. For indeterminate equations of the form $x^3\pm a^3=Dy^2$, a general solution method has not yet been obtained. 今 Future research may continue to investigate the integer solutions for different values of a and D, thereby further enriching the study of integer-solution problems for indeterminate equations of the form $x^3\pm a^3=Dy^2$.

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