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An exploration of the Positive Integer Solutions of the Arithmetic Function Equation $t\varphi(n)+\varphi_2^2(n)=S(SL(n^{31}))$

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Abstract*—In this paper, using the definitions and properties of the Smarandache function S(n), the Smarandache LCM function SL(n), the Euler function φ(n), and the generalized Euler function* $\varphi_2(n)$*, together with elementary number theory methods, we study the solvability of the arithmetic function equation* $t\varphi(n)+\varphi_2^2(n)=S\left(SL\left(n^{31}\right)\right)$*, where* $t\in Z^+,n\in Z^+$*. It is obtained that the equation has positive integer solutions only when* $t=1,2,4,6,10,14,15,20,23,31$*, and all positive integer solutions (t,n) are given* $(t,n)=(1,1),(2,40),(4,48),(6,27),(6,54),(10,24),(14,16),(15,12),(20,9),(20,18),(23,8),(31,2)$.

Keywords*—Smarandache function S(n), Smarandache LCM function SL(n), generalized Euler function* $\varphi_2(n)$*, positive integer solutions*.

I. INTRODUCTION

Euler's totient function $\varphi(n)$ is defined as, for any positive integer n, the number of positive integers in the sequence $1,2,\ldots,n-1$ that are coprime to n. $\varphi_e(n)$ is a generalized Euler totient function, defined as the number of integers in the sequence $1,2,\ldots,\left[\frac{n}{e}\right]$ that are coprime to n. When $e=1$, the generalized Euler totient function $\varphi_e(n)$ is the Euler totient function $\varphi(n)$.There have been many studies on the Euler totient function and the generalized Euler totient function. For instance, Niu Jiaxing[1] et al. studied the solvability of the arithmetic functional equation $\varphi_8(n)=\frac{n}{d}(n>8)$ and obtained two sets of positive integer solutions for this equation. Zhang Mingli[2] et al. studied the solvability of two composite Euler totient functional equations: $\varphi\left(\varphi\left(n-\varphi\left(\varphi(n)\right)\right)\right)=8$ and $\varphi\left(\varphi\left(n-\varphi\left(\varphi(n)\right)\right)\right)=10$. Wang Yu[3] et al. investigated the solvability of the quaternary nonlinear equation related to the Euler totient function $\varphi(n)$ and the generalized Euler totient function $\varphi_e(n):\varphi(abcd)=3\varphi(a)+3\varphi(b)+3\varphi(c)+6\varphi_2(d)-11$ and obtained 84 sets of positive integer solutions for this equation. Smarandache function $S(n)$ is defined as $S(n)=\min\{m:m\in Z^+,n\mid m!\}$, Smarandache LCM function $SL(n)$ is defined as $SL(n)=\min\{k\in Z^+:n\mid[1,2,\ldots,k]\}$. Research on the functions $S(n)$ and $SL(n)$ has also yielded many results. For example, Gao Li[4] studied the solvability of the composite functional equation $k\varphi_2\left(n(n+1)\right)+\varphi(n)=2S\left(SL\left(n^{11}\right)\right)$ and obtained all positive integer solutions $(k,n)=(1,1)$, $(27,2),(26,3),(24,5)$ Li Xinxin[5] et al. discussed the solvability of the arithmetic functional equation $\left(\varphi_2(n)\right)^2=S\left(SL\left(n^k\right)\right)$ for $k=2,5$, and provided all corresponding positive integer solutions; Zhu Shanshan[6] et al.studied $t\varphi(n)+\varphi_2(n)=S\left(SL\left(n^{13}\right)\right)$ and $t\varphi(n)+\varphi_2(n)=S\left(SL\left(n^{18}\right)\right)$.Based on the above research, this paper investigates the solvability of the arithmetic functional equation $t\varphi(n)+\varphi_2^2(n)=S\left(SL\left(n^{31}\right)\right)$, and obtains that the equation has positive integer solutions only for $t=1,2,4,6,10,14,15,20,23,31$. All positive integer solutions of the equation are given as $(t,n)=(1,1)$, $(2,40)$, $(4,48),(6,27),(6,54)$, $(10,24),(14,16)$, $(15,12),(20,9)$, $(20,18),(23,8),(31,2)$.

II. MAIN LEMMA

Lemma 1 [9] When n^3 3, there is $j_2(n)=\frac{j(n)}{2}$.

Lemma 2 [9] If a positive integer $n=p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}$, where $p_1<p_2<\ldots<p_k$ are primes, then:

$$\varphi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right),$$

$$SL(n)=max\left\{p_1^{l_1},p_2^{l_2}\cdots,p_k^{l_k}\right\},$$

$$S(n)=max\left\{S\left(p_1^{l_1}\right),\ S\left(p_2^{l_2}\right),\ldots,\ S\left(p_k^{l_k}\right)\right\}.$$

Lemma 3 *[9] For a prime p and a positive integer k, $S\left(p^k\right)\leqslant kp$; in particular, when $k<p$, $S\left(p^k\right)=kp$.*

Lemma 4 [9] For any positive integer n, $\varphi(n)\geqslant\sqrt{\frac{n}{2}}$.

Lemma 5 [9] For arbitrary positive integers m and n, there is $j(mn)=\frac{\gcd(m,n)j(m)j(n)}{j(\gcd(m,n))}=j(nm)$, especially when $\gcd(m,n)=1$, there is $j(mn)=j(m)j(n)$.

III. THEOREM AND ITS PROOF

Theorem 1 Let $t,n\in\mathrm{N}^*$. The arithmetic functional equation

$$t\varphi(n)+\varphi_2^2(n)=S\left(SL\left(n^{31}\right)\right). \tag{3.1}$$

has positive integer solutions, and the positive integer solutions are $(t,n)=(1,1),(2,40),(4,48),(6,27),(6,54),(10,24),(14,16),(15,12),(20,9),(20,18),(23,8),(31,2)$.

Proof: When $n=1$, $\varphi(1)=1$, $\varphi_2(1)=0$, $S(SL(1^{31})=S(1)=1$. Substituting into equation (3.1) yields $t=1$. Thus, the positive integer solution of equation (3.1) in this case is $(t,n)=(1,1)$. When $n=2$, $\varphi(2)=1$, $\varphi_2(2)=1$, $S(SL(2^{31})=S(2^{31})=32$. Substituting into equation (3.1) yields $t=31$. Thus, the positive integer solution of equation (3.1) in this case is $(t,n)=(31,2)$. When $n\geq 3$, let $n=p_1^{l_1}p_2^{l_2}\ldots p_s^{l_s}$ be the standard prime factorization of the positive integer n. By Lemma 2 and Lemma 3, we have:

$$SL(n^{31})=max\{p_1^{31l_1},p_2^{31l_2},\ldots,p_i^{31l_i}\}=p^{31l} \tag{3.2}$$

$$S\left(SL\left(n^{31}\right)\right)=S\left(p^{31l}\right)\leq 31lp. \tag{3.3}$$

Where p is a prime factor of n, and l is the exponent of p in the standard factorization of n. From equation (3.1) and Lemma 1, we obtain:

$$t\varphi(n)+\varphi_2^2(n)=t\varphi(n)+\frac{1}{4}\varphi^2(n)=S\left(SL\left(n^{31}\right)\right) \tag{3.4}$$

Then, by Lemma 4, we have:

$$t\varphi(n)+\frac{1}{4}\varphi^2(n)\geq t\sqrt{\frac{n}{2}}+\frac{1}{4}\left(\sqrt{\frac{n}{2}}\right)^2$$

$$=t\sqrt{\frac{n}{2}}+\frac{1}{8}n\geq t+\frac{1}{8}n. \tag{3.5}$$

Combining equations (3.3) and (3.5) yields:

$$31lp\geq t+\frac{1}{8}n.$$

Thus: $1\leq t\leq 31lp, 3\leq n\leq 248lp$. (3.6)

Let $n=p_1^{l_1}p_2^{l_2}\ldots p_s^{l_s}=p^l m$ with $l\geq\frac{3}{2}$, where $\gcd\left(p^l,m\right)=1$. Combining (3.3), (3.4) and Lemma 5, we obtain:

$$31lp\geq tp^{l-1}(p-1)\varphi(m)+\frac{1}{4}p^{2l-2}(p-1)^2\varphi^2(m)\geq\frac{1}{4}(p-1)^2p^{2l-2},$$

That is: $124l\geq(p-1)^2p^{2l-3}$. (3.7)

When $p=2$ we have: $124l\geq 2^{2l-3}$.

Solving this yields: $l\leq 6$.

if $l<\frac{3}{2}$, then $l\leq 6$ is also satisfied. We now discuss 1 by dividing it into six cases.

Case 1 If $l=1$, since p is a prime number, we have $p\geq 2$.

When $p=2$: From (3.6), we know $1\leq t\leq 62$ and $3\leq n\leq 496$. Thus, from (3.4), we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(2^{31}\right)=4\times 32=128$. Through calculation, we get the candidate pairs $(t,n)=(2,15),(2,16),(2,20),(2,24),(2,30),(7,5),(7,8),(7,10),(7,12)$. Substituting these pairs into equation (3.1) for verification shows that none of them are solutions to the equation.

When $p=3$: From (3.6), we know $1\leq t\leq 93$ and $3\leq n\leq 744$. Thus, from (3.4), we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(3^{31}\right)=4\times 66=264$. Through calculation, the equation has no solutions.

When $p=5$: From (3.6), we know $1\leq t\leq 155$ and $3\leq n\leq 1240$. Thus, from (3.4), we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(5^{31}\right)=4\times 125=500$. Through calculation, we get the candidate pairs $(t,n)=(10,11),(10,12),(62,3),(62,4),(62,6)$. Substituting these pairs into equation (3.1) for verification shows that none of them are solutions to the equation.

When $p=7$: From (3.6), we know $1\leq t\leq 217$ and $3\leq n\leq 1736$. Thus, from (3.4), we obtain: $4t\varphi(n)+$

$\varphi^2(n)=4S\left(7^{31}\right)=4\times196=784$.Through calculation, we get the candidate pairs $(t,n)=(48,5),(48,8),(48,10),(48,12)$. Substituting these pairs into equation (3.1) for verification shows that none of them are solutions to the equation.

When $p=11$:From (3.6), we know $1\le t\le341$ and $3\le n\le2728$. Thus, from (3.4), we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(11^{31}\right)=4\times319=1276$.Through calculation, we get the candidate pairs $(t,n)=(9,23),(9,46),(159,3),(159,4),(159,6)$. Substituting these pairs into equation (3.1) for verification shows that none of them are solutions to the equation.

When $p=13$:From (3.6), we know $1\le t\le403$ and $3\le n\le3224$. Thus, from (3.4), we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(13^{31}\right)=4\times377=1508$.Through calculation, we get the candidate pairs $(t,n)=(188,3),(188,4),(188,6)$. Substituting these pairs into equation (3.1) for verification shows that none of them are solutions to the equation.

When $p=17$:From (3.6), we know $1\le t\le527$ and $3\le n\le4216$. Thus, from (3.4), we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(17^{31}\right)=4\times510=2040$.Through calculation, the equation has no solutions.

When $p=19$, from (3.6) we know $1\le t\le589$ and $3\le n\le4721$. Thus, from (3.4) we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(19^{31}\right)=4\times570=2280$.After calculation, this equation has no solutions.

When $p=23$, from (6) we know $1\le t\le713$ and $3\le n\le5704$. Thus, from (3.4) we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(23^{31}\right)=4\times690=2760$.After calculation, this equation has no solutions.

When $p=29$, from (3.6) we know $1\le t\le899$ and $3\le n\le7192$. Thus, from (3.4) we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(29^{31}\right)=4\times870=3480$.After calculation, this equation has no solutions.

When $p=31$, from (3.6) we know $1\le t\le961$ and $3\le n\le7688$. Thus, from (3.4) we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(31^{31}\right)=4\times961=3844$.After calculation, we get $(t,n)=(480,3),(480,4),(480,6)$. Substituting these values of (t,n) into equation (3.1) for verification, we find that none of $(t,n)=(480,3),(480,4),(480,6)$ are solutions to equation (3.1).

When $p>31$, we have $1\le t\le31p$ and $3\le n\le248p$. By Lemma 3, weget: $4t\varphi(n)+\varphi^2(n)=4S\left(p^{31}\right)=124p$. Since $n=p^lm$ and $\gcd(p,m)=1$, it follows that $(p-1)\mid124$, hence $(p-1)\le124$ and $p\le125$. However, no prime p satisfying $<p\le125$ makes $(p-1)\mid124$ hold. Therefore, equation (3.1) has no positive integer solutions when $p>31$.

Case 2 If $l=2$, from equation (3.7) and the fact that p is a prime, we obtain $p\le5$.

When $p=2$, from (3.6) we know $1\le t\le124$ and $3\le n\le992$. Thus, from (3.4) we get: $4t\varphi(n)+\varphi^2(n)=4S\left(2^{31\times2}\right)=4\times64=256$ .After calculation, we obtain $(t,n)=(6,15),(6,16),(6,20),(6,24),(6,30),(15,5),(15,8),(15,10),(15,12)$. Substituting these values of (t,n) into equation (3.1) for verification, we find that $(t,n)=(15,12)$ is a solution to equation (3.1).

When $p=3$, from (3.6) we know $1\le t\le186$ and $3\le n\le1488$. Thus, from (3.4) we get: $4t\varphi(n)+\varphi^2(n)=4S\left(3^{31\times2}\right)=4\times129=516$ .After calculation, we obtain $(t,n)=(20,7),(20,9),(20,14),(20,18),(64,3),(64,4),(64,6)$. Substituting these values of (t,n) into equation (3.1) for verification, we find that $(t,n)=(20,9),(20,18)$ are solutions to equation (3.1).

When $p=5$, from (3.6) we know $1\le t\le310$ and $3\le n\le2480$. From (3.4) we get: $4t\varphi(n)+\varphi^2(n)=4S\left(5^{31\times2}\right)=4\times250=1000$.After calculation, this equation has no solutions.

Case 3 If $l=3$, from equation (3.7) and the fact that p is a prime, we obtain $p\le3$.

When $p=2$, from (3.6) we know $1\le t\le186$ and $3\le n\le1488$. Thus, from (3.4) we get: $4t\varphi(n)+\varphi^2(n)=4S\left(2^{31\times3}\right)=4\times96=384$ .After calculation, we obtain: $(t,n)=(2,17),(2,32),(2,40),(2,48),(2,60),(10,20),(5,13),(5,21),(5,26),(5,28),(5,36),(5,42),(10,15),(10,16),(10,24),(10,30),(23,5),(23,8),(23,10),(23,12)$.Substituting these values of (t,n) into equation (3.1) for verification, we find that $(t,n)=(2,40),(10,24),(23,8)$ are solutions to equation (3.1).

When $p=3$, from (3.6) we know $1\le t\le 279$ and $3\le n\le 2232$. Thus, from (3.4) we get: $4t\varphi(n)+\varphi^2(n)=4S\left(3^{31\times 3}\right)=4\times 189=756$ .After calculation, we obtain: $(t,n)=(6,19),(6,27),(6,38),(6,54),(30,7),(30,9),(30,14),(30,18),(94,3),(90,4),(94,6)$ .Substituting these values of (t,n) into equation (3.1) for verification, we find that $(t,n)=(4,48),(14,16)$ are solutions to equation (3.1).

Case 4 If $l=4$, it follows from Equation (3.7) and the fact that p is a prime number that $p\le 2$.

When $p=2$, Equation (3.6) gives $1\le t\le 248$ and $3\le n\le 1984$. Thus, from Equation (3.4), we obtain: $4t\varphi(n)+\varphi^2(n)=4S\left(2^{31\times 4}\right)=4\times 128=512$. After calculation, the pairs are: $(t,n)=(4,17),(4,32),(4,34),(4,40),(4,48),(4,60),(14,15),(14,16),(14,20),(14,24),(14,30),(31,5),(31,8),(31,10),(31,12)$. Substituting these (t,n) values into Equation (1) for verification, we find that $(t,n)=(4,48),(14,16)$ are solutions to the equation.

Case 5 If $l=5$, from equation (3.7) and the fact that p is a prime, we obtain $p\le 2$.

When $p=2$, from (3.6) we know $1\le t\le 310$ and $3\le n\le 2480$. Thus, from (3.4) we get: $4t\varphi(n)+\varphi^2(n)=4S\left(2^{31\times 5}\right)=4\times 160=640$ .After calculation, we obtain: $(t,n)=(3,25),(3,33),(3,44),(3,50),(3,66),(6,17),(6,32),(6,34),(6,40),(6,48),(6,60),(18,15),(18,16),(18,20),(18,24),(18,30),(39,5),(39,8),(39,10),(39,12)$. .Substituting these values of (t,n) into equation (3.1) for verification, we find that $(t,n)=(6,32)$ is a solution to equation (3.1).

Case 6 If $l=6$, from equation (3.7) and the fact that p is a prime, we obtain $p\le 2$.

When $p=2$, from (3.6) we know $1\le t\le 372$ and $3\le n\le 2976$. Thus, from (3.4) we get: $4t\varphi(n)+\varphi^2(n)=4S\left(2^{31\times 6}\right)=4\times 192=768$ .After calculation, we obtain $(t,n)=(192,5),(192,8),(192,10),(192,12)$. Substituting these values of (t,n) into equation (3.1) for verification, we find that none of $(t,n)=(192,5),(192,8),(192,10),(192,12)$ are solutions to equation (3.1).

In this paper, we utilize the definitions and properties of the *Smarandache* function $S(n)$, the *SmarandacheLCM* function $SL(n)$, the Euler totient function $\varphi(n)$, and the generalized Euler function $\varphi_2(n)$, along with elementary number theory methods, to study the solvability of the number-theoretic function equation $t\varphi(n)+\varphi_2^2(n)=S\left(SL\left(n^{31}\right)\right)$.In subsequent research, the method presented in this paper can be employed to investigate the positive integer solutions of similar types of number-theoretic function equations.

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