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Legendre Spectral Methods for Elliptic Eigenvalue Problems in Spherical Domains

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***Abstract*— *In this paper, we develop an efficient spectral method for second-order elliptic eigenvalue problems defined on spherical domains. By employing a spherical coordinate transformation, the original problem is reformulated as a sequence of mutually decoupled one-dimensional eigenvalue problems. For each resulting one-dimensional problem, we establish the corresponding weak formulation and discrete spectral scheme, and derive rigorous error estimates for the approximate eigenvalues. Finally, numerical experiments are provided to illustrate the efficiency of the proposed method and to confirm the theoretical convergence results.*

***Keywords*— *Second order elliptic eigenvalue problems, dimension reduction scheme, spectral method, error estimation, spherical domain.*

I. INTRODUCTION

Eigenvalue problems are widely used in scientific research and engineering, and their numerical solution methods have attracted much attention. Second-order elliptic eigenvalue problems are most commonly treated by one of three families of methods: the finite element method [1-3], the finite difference method [4, 5], and the spectral method [6-9]. The finite element method divides the computational domain into several subdomains and constructs piecewise interpolation polynomials on each element, converting the continuous equation into a discrete system [10]. The finite difference method approximates derivatives using difference quotients, transforming the continuous problem into a system of linear algebraic equations [11]. As a high-order numerical method with spectral accuracy, the spectral method exhibits exponential convergence when the solution is sufficiently smooth, with a convergence rate far exceeding the algebraic convergence of the finite element and finite difference methods. For eigenvalue problems on spherical domains, if the finite element method is directly programmed to solve them on the spherical domain, apart from the relatively complex meshing part, achieving high accuracy will consume a considerable amount of computing time and memory capacity. In [12], An et al. proposed a dimension-reduction-based hybrid spectral method for Maxwell transmission eigenvalue problems in spherical domains; theoretical analysis and numerical experiments demonstrate its high accuracy and efficiency.

Therefore, the purpose of this paper is to develop an efficient spectral method for second-order elliptic eigenvalue problems posed on spherical domains. By employing a spherical coordinate transformation, the original problem is reformulated as a sequence of mutually decoupled one-dimensional eigenvalue problems. For each resulting one-dimensional problem, we establish the corresponding weak formulation and discrete spectral scheme, and derive rigorous error estimates for the approximate eigenvalues. Finally, numerical experiments are provided to illustrate the efficiency of the proposed method and to confirm the theoretical convergence results.

The remainder of this paper is organized as follows. Section II introduces a reduced-dimensional formulation equivalent to the original problem. Section III derives the corresponding weak form and its discrete scheme. Section IV establishes error estimates for the approximate eigenvalues and eigenfunctions. Section V describes the algorithmic implementation of the discrete variational formulation. Section VI presents several numerical examples. Finally, Section VII provides concluding remarks.

II. DIMENSION REDUCTION SCHEME

As a model problem, we consider the following second-order elliptic eigenvalue problem:

$$-\Delta w(\mathbf{x}) = \lambda w(\mathbf{x}), \ \ \mathbf{x} \in \Omega, \qquad (2.1)$$

$$w(\mathbf{x}) = 0, \ \ \mathbf{x} \in \partial\Omega, \qquad (2.2)$$

where the domain $\Omega \subset \mathbb{R}^3$ is a ball, that is,

$$\Omega = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < R\},$$

with

$$|\mathbf{x}| = r = \sqrt{x^2 + y^2 + z^2}.$$

To facilitate the analysis, we introduce spherical coordinates (r, θ, ϕ) defined by

$$x = r\sin\theta\cos\phi, \ \ y = r\sin\theta\sin\phi, \ \ z = r\cos\theta,$$

and define the transformed unknown

$$\varphi(r, \theta, \phi) = w(x, y, z).$$

We define the differential operator in spherical coordinates by

$$\mathrm{L}\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\Delta_S\varphi, \qquad (2.3)$$

where Δ_S denotes the Laplace--Beltrami operator on the unit sphere S, given by

$$\Delta_S = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \qquad (2.4)$$

With this notation, the second-order eigenvalue problem (2.1)-(2.2) can be rewritten in spherical coordinates as

$$\mathrm{L}\varphi(R,\theta,\phi)=\lambda\varphi(R,\theta,\phi),(r,\theta,\phi)\in[0,R)\times[0,\pi]\times[0,2\pi),$$
(2.5)

$$\varphi(R,\theta,\phi)=0,(\theta,\phi)\in[0,\pi]\times[0,2\pi). \quad (2.6)$$

The spherical harmonics $\{Y_l^m\}$, normalized as in [13], are eigenfunctions of the Laplace--Beltrami operator Δ_S on the unit sphere S. They satisfy

$$\Delta_S Y_l^m \ =-l(l+1)Y_l^m, \qquad l\geq 0,|m|\leq l, \quad (2.7)$$

and form an orthonormal basis of $L^2(S)$:

$$\int_S Y_l^m Y_{l'}^{m'}dS \ =\delta_{ll'}\delta_{mm'}. \quad (2.8)$$

Accordingly, the functions φ and ψ admit the spherical harmonic expansions

$$\varphi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{|m|\leq l}\varphi_l^m(r)Y_l^m(\theta,\phi). \quad (2.9)$$

Substituting the spherical harmonic expansions (2.9) into the coupled system (2.5)- (2.6) and using the orthogonality of $\{Y_l^m\}$, the problem decouples into a sequence of one-dimensional eigenvalue problems for the radial functions $\varphi_l^m(r)$:

$$-\frac{1}{r^2}\left(\partial_r\left(r^2\partial_r\varphi_l^m(r)\right)-l(l+1)\varphi_l^m(r)\right)=\lambda_l\varphi_l^m(r), \qquad r\in[0,R), \quad (2.10)$$

$$\varphi_l^m(R)=0. \quad (2.11)$$

To map the radial problems onto a fixed reference interval, we introduce the linear transformation

$$r=\frac{t+1}{2}R, \qquad t\in[-1,1),$$

and define the rescaled unknown

$$\sigma_l^m(t)=\varphi_l^m\left(\tfrac{t+1}{2}R\right).$$

Under this transformation, the radial system (2.10)- (2.11) can be rewritten as

$$-\frac{1}{(t+1)^2}\partial_t\left((t+1)^2\partial_t\sigma_l^m(t)\right)+\frac{l(l+1)}{(t+1)^2}\sigma_l^m(t)=\lambda_l\frac{R^2}{4}\sigma_l^m(t),t\in[-1,1), \quad (2.12)$$

$$\sigma_l^m(1)=0. \quad (2.13)$$

III. WEAK FORM AND DISCRETE FORM

We introduce usual weighted Sobolev space as follow:

$$L_\omega^2(I):=\{u:\int_I \omega u^2dt<\infty\}$$

equipped with the corresponding inner product and norm

$$(u,v)_\omega=\int_I \omega uvdt,\|u\|_w=(\int_I \omega u^2dt)^{\frac{1}{2}},$$

where $\omega=(t+1)^2$ is a weight function defined on I=(-1,1). Next, we further define the non-uniformly weighted Sobolev space $H_{0,\omega,l}^1(I)$ as follow:

$$H_{0,\omega,l}^1(I):=\{u:\int_I \omega(u')^2+l(l+1)u^2dt<\infty,u(1)=0\}$$

equipped with the corresponding inner product and norm

$$(u,v)_{1,\omega,l}=\int_I \omega u'v'+l(l+1)uvdt,\|u\|_{1,\omega,l}=(u,u)_{1,\omega,l}^{\frac{1}{2}}.$$

Therefore, the weak form of (2.12)-(2.13) is: Find $(\lambda_l,\sigma_l^m)\in\mathbb{R}\times H_{0,\omega,l}^1(I)$, such that

$$\mathrm{A}_l(\sigma_l^m,h_l^m)=\frac{R^2}{4}\lambda_l\mathrm{B}_l(\sigma_l^m,h_l^m),\forall h_l^m\in H_{0,\omega,l}^1(I),$$
(3.1)

where

$$\mathrm{A}_l(\sigma_l^m,h_l^m)=\int_I \omega(\sigma_l^m)'(h_l^m)'+l(l+1)\sigma_l^m h_l^m dt$$

$$\mathrm{B}_l(\sigma_l^m,h_l^m)=\frac{R^2}{4}\int_I \omega\sigma_l^m h_l^m dt.$$

Define the approximation space $X_N=P_N\cap H_{0,\omega,l}^1$, where P_N denote a polynomial space of degree no more than N. Then the corresponding discrete form of (3.1) reads: Find $(\lambda_{lN},\sigma_{lN}^m)\in\mathbb{R}\times X_N$, such that

$$\mathrm{A}_l(\sigma_{lN}^m,h_{lN}^m)=\frac{R^2}{4}\lambda_{lN}\mathrm{B}_l(\sigma_{lN}^m,h_{lN}^m),\forall h_{lN}^m\in X_N, \quad (3.2)$$

IV. ERROR ESTIMATION

For brevity, we write $a\lesssim b$ to denote that $a\leq Cb$, where C is a positive constant independent of N.

Theorem 1. The bilinear form $\mathrm{A}_l(u,v)$ is continuous and coercive on

$$H_{0,\omega,l}^1(I)\times H_{0,\omega,l}^1(I),$$

that is,

$$|\mathrm{A}_l(u,v)|\lesssim\|u\|_{1,\omega,l}\|v\|_{1,\omega,l}, \qquad \mathrm{A}_l(u,u)\gtrsim\|u\|_{1,\omega,l}^2.$$

Proof. For any $u,v\in H_{0,\omega,l}^1(I)$, applying the Cauchy--Schwarz inequality yields

$$|\mathrm{A}_l(u,v)|=\left|\int_I\left(\omega u'v'+l(l+1)uv\right)dt\right|$$

$$\leq\left(\int_I\left(\omega(u')^2+l(l+1)u^2\right)dt\right)^{\frac{1}{2}}\left(\int_I\left(\omega(v')^2+l(l+1)v^2\right)dt\right)^{\frac{1}{2}}$$

$$=\|u\|_{1,\omega,l}\|v\|_{1,\omega,l}.$$

Hence, the bilinear form $\mathrm{A}_l(\cdot,\cdot)$ is continuous on

$$H_{0,\omega,l}^1(I)\times H_{0,\omega,l}^1(I)$$

Moreover,

$$\mathrm{A}_l(u,u)=\int_I\left(\omega(u')^2+l(l+1)u^2\right)dt=\|u\|_{1,\omega,l}^2,$$

which implies that $\mathrm{A}_l(\cdot,\cdot)$ is coercive.

This completes the proof.

Lemma 1. Let λ_l^k be the eigenvalues of problem (3.1), and let V_k be any k-dimensional subspace of $H^1_{0,\omega,l}(I)$. Assume that the eigenvalues are ordered as

$$\lambda_l^1 \le \lambda_l^2 \le \cdots \le \lambda_l^k \le \cdots.$$

Then the following minimax characterization holds:

$$\lambda_l^k = \min_{V_k \subset H^1_{0,\omega,l}(I)} \max_{\sigma \in V_k} \frac{\mathrm{A}_l(\sigma,\sigma)}{\mathrm{B}_l(\sigma,\sigma)}. \quad (4.1)$$

Proof. See Theorem 3.1 in [14].

Lemma 2. Let λ_l^i be the eigenvalues of weak form (3.1), arranged in ascending order, and define

$$E_{i,j} := \mathrm{span}\{\sigma_l^i, \ldots, \sigma_l^j\},$$

where σ_l^i denotes the eigenfunction associated with the eigenvalue λ_l^i. Then the following minimax characterizations hold:

$$\lambda_l^k = \max_{\sigma \in E_{m,k}} \frac{\mathrm{A}_l(\sigma,\sigma)}{\mathrm{B}_l(\sigma,\sigma)}, \qquad m \le k, \quad (4.2)$$

$$\lambda_l^k = \min_{\sigma \in E_{k,n}} \frac{\mathrm{A}_l(\sigma,\sigma)}{\mathrm{B}_l(\sigma,\sigma)}, \qquad k \le n. \quad (4.3)$$

Proof. See Lemma 3.2 in [14].

It is well known that the minimax principle remains valid for the discrete formulation (3.2) (see [14]).

Lemma 3 Let λ_{lN}^k denote the eigenvalues of (3.2), and let V_k be any k-dimensional subspace of X_N. Suppose that the eigenvalues are ordered as

$$\lambda_{lN}^1 \le \lambda_{lN}^2 \le \cdots \le \lambda_{lN}^k \le \cdots.$$

Then the following minimax characterization holds:

$$\lambda_{lN}^k = \min_{V_k \subset X_N} \max_{\sigma \in V_k} \frac{\mathrm{A}_l(\sigma,\sigma)}{\mathrm{B}_l(\sigma,\sigma)}. \quad (4.4)$$

Let $\Pi_N^{1,l} : H^1_{0,\omega,l}(I) \to X_N$ be the A_l-orthogonal projection defined by

$$\mathrm{A}_l\left(\sigma_l^m - \Pi_N^{1,l}\sigma_l^m, \phi\right) = 0, \qquad \forall \phi \in X_N.$$

Theorem 2. Let λ_{lN}^k be the k-th eigenvalue of the discrete formulation (3.2), which approximates the k-th eigenvalue λ_l^k of the weak form (3.1). Then the following estimate holds:

$$\lambda_l^k \le \lambda_{lN}^k \le \lambda_l^k \max_{\sigma \in E_{1,k}} \frac{\mathrm{B}_l(\sigma,\sigma)}{\mathrm{B}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma)}. \quad (4.5)$$

Proof. Since $X_N \subset H^1_{0,\omega,l}(I)$, it follows directly from (4.1) and (4.4) that

$$\lambda_l^k \le \lambda_{lN}^k.$$

Let $\Pi_N^{1,l} E_{1,k}$ denote the space spanned by

$$\Pi_N^{1,l}\sigma_l^1,\ \Pi_N^{1,l}\sigma_l^2,\ \ldots,\ \Pi_N^{1,l}\sigma_l^k.$$

Clearly, $\Pi_N^{1,l} E_{1,k}$ is a k-dimensional subspace of X_N. By the minimax principle, we obtain

$$\lambda_{lN}^k \le \max_{\sigma \in \Pi_N^{1,l} E_{1,k}} \frac{\mathrm{A}_l(\sigma,\sigma)}{\mathrm{B}_l(\sigma,\sigma)} = \max_{\sigma \in E_{1,k}} \frac{\mathrm{A}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma)}{\mathrm{B}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma)}.$$

Using the symmetry of $\mathrm{A}_l(\cdot,\cdot)$, we have

$$\mathrm{A}_l(\sigma,\sigma) = \mathrm{A}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma) + 2\mathrm{A}_l(\sigma - \Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma) + \mathrm{A}_l(\sigma - \Pi_N^{1,l}\sigma, \sigma - \Pi_N^{1,l}\sigma).$$

By the definition of the A_l-orthogonal projection $\Pi_N^{1,l}$,

$$\mathrm{A}_l(\sigma - \Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma) = 0,$$

and since $\mathrm{A}_l(\sigma - \Pi_N^{1,l}\sigma, \sigma - \Pi_N^{1,l}\sigma) \ge 0$, it follows that

$$\mathrm{A}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma) \le \mathrm{A}_l(\sigma,\sigma).$$

Therefore,

$$\lambda_{lN}^k \le \max_{\sigma \in E_{1,k}} \frac{\mathrm{A}_l(\sigma,\sigma)}{\mathrm{B}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma)} = \max_{\sigma \in E_{1,k}} \frac{\mathrm{A}_l(\sigma,\sigma)}{\mathrm{B}_l(\sigma,\sigma)} \frac{\mathrm{B}_l(\sigma,\sigma)}{\mathrm{B}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma)} \le \lambda_l^k \max_{\sigma \in E_{1,k}} \frac{\mathrm{B}_l(\sigma,\sigma)}{\mathrm{B}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma)}.$$

This completes the proof.

Next, we derive error estimates for the approximate eigenvalues. Introduce a class of non-uniformly weighted Sobolev spaces:

$$Y^s_{\omega^{-1,-1},l}(I) := \left\{u_l \in H^1_{*,\omega,l}(I) : \partial_t^j u_l \in L^2_{\omega^{-1+j,-1+j}}(I), 1 \le j \le s\right\},$$

equipped with the corresponding inner product and norm

$$(u_l, v_l)_{s,\omega^{-1,-1},l} = (u_l, v_l)_{1,\omega,l} + \sum_{j=1}^{s} \left(\partial_t^j u_l, \partial_t^j v_l\right)_{\omega^{-1+j,-1+j}},$$

$$\| u_l \|_{s,\omega^{-1,-1},l} = \sqrt{(u_l, u_l)_{s,\omega^{-1,-1},l}},$$

where the weight function is defined by

$$\omega^{\alpha,\beta}(t) = (1-t)^{\alpha}(1+t)^{\beta}.$$

From the Theorems 5-6 in [15], we have the following Lemma:

Lemma 4. There exists an operator $\pi_{*,N}^{1,l} : H^1_{*,\omega,l}(I) \to X_N$ such that for any $u_l \in Y^s_{\omega^{-1,-1},l}(I)$ with $s \ge 1$, there holds

$$\| \partial_t(\pi_{*,N}^{1,l} u_l - u_l) \| \hat{} \quad N^{1-s} \| \partial_t^s u_l \|_{\omega^{-1+s,-1+s}}.$$

Theorem 3. Let λ_{lN}^{k} be the k-th approximate eigenvalue of λ_l^k. Assume that $\{\sigma_l^i\}_{i=1}^k \subset Y^s_{\omega^{-1,-1},l}(I)$ with $s \geq k$. Then the following estimate holds:

$$|\lambda_{lN}^k - \lambda_l^k| \lesssim c(k) N^{2(1-s)} \max_{i=1,\cdots,k} \| \partial_t^s \sigma_l^i \|^2_{\omega^{s,s}},$$

Where $\mathrm{c(k)}$ is a constant independent of N.

Proof. For any $\sigma \in E_{1,k}$, it is clear that σ can be expressed as

$$\sigma = \sum_{i=1}^{k} \mu_i \sigma_l^i.$$

Therefore, we have

$$\frac{\mathrm{B}_l(\sigma,\sigma) - \mathrm{B}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma)}{\mathrm{B}_l(\sigma,\sigma)} \leq \frac{2\left|\mathrm{B}_l(\sigma, \sigma - \Pi_N^{1,l}\sigma)\right|}{\mathrm{B}_l(\sigma,\sigma)}$$

$$\leq \frac{2\sum_{i,j=1}^{k} |\mu_i| |\mu_j| \left|\mathrm{B}_l(\sigma_l^i - \Pi_N^{1,l}\sigma_l^i, \sigma_l^j)\right|}{\sum_{i=1}^{k} |\mu_i|^2}$$

$$\leq 2k \max_{i,j=1,\cdots,k} \left|\mathrm{B}_l(\sigma_l^i - \Pi_N^{1,l}\sigma_l^i, \sigma_l^j)\right|.$$

By the Cauchy-Schwarz inequality and the orthogonality property of the projection, we obtain

$$\left|\mathrm{B}_l(\sigma_l^i - \Pi_N^{1,l}\sigma_l^i, \sigma_l^j)\right| = \frac{1}{\lambda_l^j} \left|\lambda_l^j \mathrm{B}_l(\sigma_l^j, \sigma_l^i - \Pi_N^{1,l}\sigma_l^i)\right|$$

$$= \frac{1}{\lambda_l^j} \left|\mathrm{A}_l(\sigma_l^j, \sigma_l^i - \Pi_N^{1,l}\sigma_l^i)\right|$$

$$= \frac{1}{\lambda_l^j} \left|\mathrm{A}_l(\sigma_l^j - \Pi_N^{1,l}\sigma_l^j, \sigma_l^i - \Pi_N^{1,l}\sigma_l^i)\right|$$

$$\leq \frac{1}{\lambda_l^j} \Big(\mathrm{A}_l(\sigma_l^j - \Pi_N^{1,l}\sigma_l^j, \sigma_l^j - \Pi_N^{1,l}\sigma_l^j)\Big)^{1/2}$$

$$\times \Big(\mathrm{A}_l(\sigma_l^i - \Pi_N^{1,l}\sigma_l^i, \sigma_l^i - \Pi_N^{1,l}\sigma_l^i)\Big)^{1/2}.$$

Note that

$$\mathrm{A}_l(\sigma,\sigma) = \int_I \Big((t+1)^2(\sigma')^2 + l(l+1)\sigma^2 + \frac{R^2}{4}(t+1)^2\sigma^2\Big)dt$$

$$\lesssim \int_I \Big((t+1)^2(\sigma')^2 + \sigma^2\Big)dt \lesssim \int_I \Big((\sigma')^2 + \sigma^2\Big)dt = \|\sigma\|_1^2.$$

Consequently,

$$\left|\mathrm{B}_l(\sigma_l^i - \Pi_N^{1,l}\sigma_l^i, \sigma_l^j)\right| \lesssim \|\sigma_l^j - \pi_{*,N}^{1,l}\sigma_l^j\|_1 \|\sigma_l^i - \pi_{*,N}^{1,l}\sigma_l^i\|_1.$$

Finally, since

$$\frac{\mathrm{B}_l(\sigma,\sigma)}{\mathrm{B}_l(\Pi_N^{1,l}\sigma, \Pi_N^{1,l}\sigma)} \leq \frac{1}{1 - 2k \max_{i,j=1,\cdots,k} \left|\mathrm{B}_l(\sigma_l^i - \Pi_N^{1,l}\sigma_l^i, \sigma_l^j)\right|},$$

the desired estimate follows from Poincaré inequality and Lemma 4.

V. EFFICIENT IMPLEMENTATION OF THE ALGORITHM

In this section, we propose an efficient numerical algorithm to solve the discrete problem (3.2). To this end, we introduce the following basis functions:

$$\varphi_i(t) = L_i(t) - L_{i+2}(t), \quad i = 0, \ldots, N-2, \quad \varphi_{N-1}(t) = \frac{t-1}{2}, (5.1)$$

where $\mathrm{L}_k(\mathrm{t})$ denotes the Legendre polynomial of degree k. It is straightforward to verify that

$$X_N = \mathrm{span}\{\varphi_0(t), \ldots, \varphi_{N-1}(t)\}.$$

We define the matrix entries as

$$a_{ij} = \int_I \omega(t)\varphi_j'(t)\varphi_i'(t)dt, b_{ij} = \int_I \varphi_j(t)\varphi_i(t)dt, c_{ij} = \int_I \omega(t)\varphi_j(t)\varphi_i(t)dt.$$

The approximate solution σ_{lN}^m is sought in the form

$$\sigma_{lN}^m = \sum_{i=0}^{N-1} \sigma_i^l \varphi_i(t). \; (5.2)$$

Substituting (5.2) into (3.2), and choosing the test functions h_{lN}^m successively as all basis functions in X_N, we obtain the following generalized linear eigenvalue problem:

$$(\mathbf{A} + l(l+1)\mathbf{B})\mathbf{W}_1 = \frac{R^2}{4}\lambda_{lN}\mathbf{C}\mathbf{W}_1$$

where

$$\mathbf{A} = (a_{ij}), \mathbf{B} = (b_{ij}), \mathbf{C} = (c_{ij}), \mathbf{W}_1 = (\sigma_0^l, \ldots, \sigma_{N-1}^l)^T.$$

VI. NUMERICAL EXPERIMENTS

To demonstrate the convergence and efficiency of the proposed algorithm, we conducted a series of numerical experiments in MATLAB R2016b.

Example 1. We set $R=2$ and consider $\mathrm{l}=1,2,3,4$. The first four eigenvalues for various values of l and N are listed in Tables 1-4.

TABLE 1. First four eigenvalues for $l=1$ with varying N.

N	λ_{1N}^1	λ_{2N}^1	λ_{3N}^1	λ_{4N}^1
10	5.047682139106700	14.919879002903217	29.724995210584105	49.470272066822176
15	5.047682139106655	14.919878986027388	29.724967290910644	49.464452809348508
20	5.047682139106617	14.919878986027195	29.724967290906410	49.464452798343892
25	5.047682139106635	14.919878986027292	29.724967290906438	49.464452798344098
30	5.047682139106644	14.919878986027292	29.724967290906470	49.464452798344063

TABLE 2. First four eigenvalues for $l=2$ with varying N.

N	λ_{1N}^2	λ_{2N}^2	λ_{3N}^2	λ_{4N}^2
10	8.304365478569327	20.679808744037249	37.963771763712543	60.229735552133747
15	8.304365478567096	20.679807775373352	37.963718541340789	60.175726671892306

20	8.304365478567036	20.679807775373149	37.963718541016789	60.175726646353667
25	8.304365478567060	20.679807775373231	37.963718541016974	60.175726646353851
30	8.304365478567064	20.679807775373224	37.963718541017002	60.175726646353979

TABLE 3. First four eigenvalues for $l=3$ with varying N .

N	λ_{1N}^3	λ_{2N}^3	λ_{3N}^3	λ_{4N}^3
10	12.207798412598597	27.129094876147278	46.911451190933619	71.715808847443327
15	12.207798410904807	27.129089707539684	46.908959579540230	71.602242060449640
20	12.207798410904726	27.129089707538622	46.908959576737587	71.602239351336252
25	12.207798410904738	27.129089707538622	46.908959576737978	71.602239351335413
30	12.207798410904765	27.129089707538704	46.908959576737928	71.602239351335641

TABLE 4. First four eigenvalues for $l=4$ with varying N .

N	λ_{1N}^4	λ_{2N}^4	λ_{3N}^4	λ_{4N}^4
10	16.738578019825042	34.251223944506215	56.564406105239655	83.846846306284334
15	16.738577981276190	34.251212874313190	56.547878637387619	83.734023856199329
20	16.738577981276055	34.251212874278103	56.547878629381081	83.733992424170523
25	16.738577981276133	34.251212874278146	56.547878629381358	83.733992424167440
30	16.738577981276151	34.251212874278181	56.547878629381295	83.733992424167582

Tables 1-4 show that at least twelve significant digits of accuracy are achieved when $N \geq 20.$ To further illustrate the convergence behavior of the algorithm, the solution obtained with $\mathrm{N}=60$ is taken as a reference. Figures 1-2 plot the errors of the first four eigenvalues λ_{iN}^l(l=1,2,3,4; i=1,2,3,4) for various values of N. These results further demonstrate the effectiveness and spectral accuracy of the proposed algorithm.

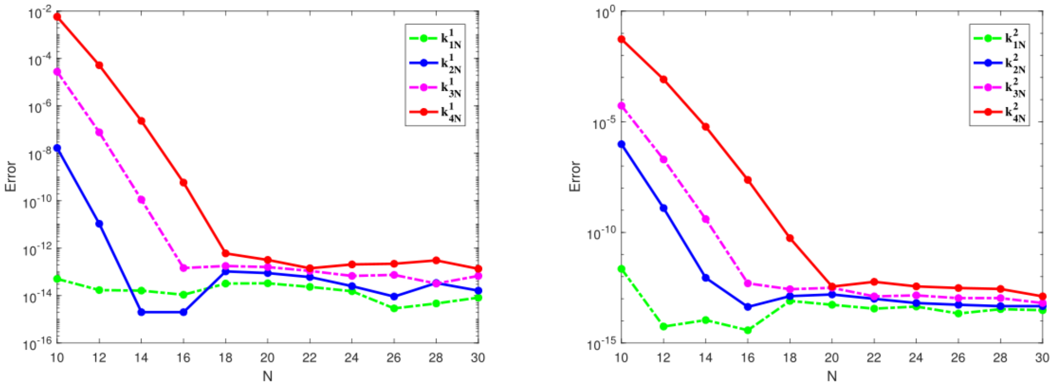

Fig. 1. Error between numerical and reference solutions for $l=1$ (left) and $l=2$ (right).

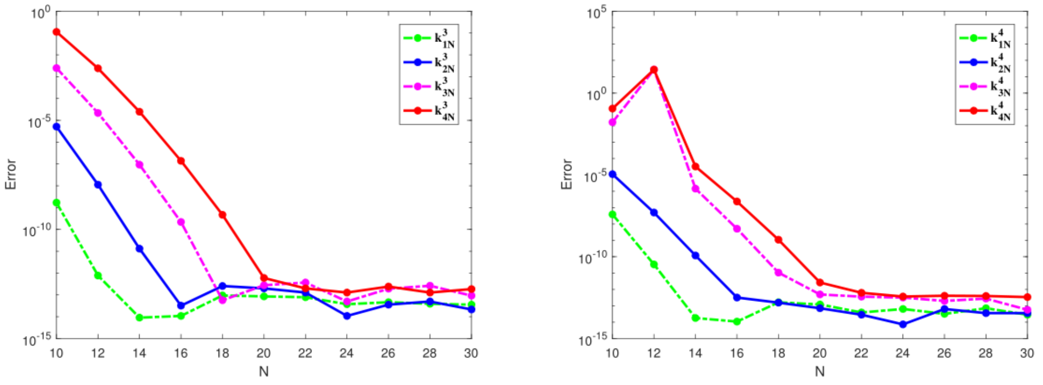

Fig. 2. Error between numerical and reference solutions for $l=3$ (left) and $l=4$ (right).

VII. CONCLUSIONS

In this paper, we present an efficient Legendre spectral method for solving second-order eigenvalue problems on spherical domains. By employing a spherical coordinate transformation, spherical harmonic expansion, and a variable separation technique, the original problem is reduced to a sequence of equivalent one-dimensional eigenvalue problems. These problems can be solved independently and in parallel, which significantly reduces computational cost and memory usage. Numerical experiments demonstrate the effectiveness and accuracy of the proposed method. Although this paper focuses on second-order eigenvalue problems on spherical domains, the approach can be readily extended to more complex fourth-order problems.

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