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The Solvability Problem of Number-Theoretic Functional Equations Related to Two Polygonal Numbers

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***Abstract*—*By using the definitions and properties of the Smarandache function* $S(n)$*, the generalized Euler function* $\varphi_e(n)$*, and the Smarandache LCM function* $SL(n)$*, together with elementary number theory methods, the solvability of the number-theoretic functional equation related to triangular and nonagonal numbers* $S\left(SL\left(n^{19}\right)\right)=k\varphi_2^2\left(\frac{n(n+1)}{2}\right)+t\varphi_2^2\left(\frac{n(7n-5)}{2}\right)$ *is studied. It is proved that all positive integer solutions of this number-theoretic functional equation are* $(k,t,n)=(4,2,2),(13,1,2),(10,2,3),(26,1,3)$.

Keywords—*Smarandache function* $S(n)$*; generalized Euler function* $\varphi_e(n)$*; Smarandache LCM function* $SL(n)$*; solvability.*

I. INTRODUCTION

The generalized Euler function $\varphi_e(n)$, the Smarandache LCM function $SL(n)$, and the Smarandache function $S(n)$ are all important functions in number theory. The function $\varphi(n)$ is defined as the number of integers in the sequence $1、2、3、\cdots n-1$ that are coprime to n. The function $\varphi_e(n)$ is defined as the number of integers in the sequence $1,2,\cdots\left[\frac{n}{e}\right]$ that are coprime to n. The Smarandache function is defined as $S(n)=\min\left\{m:m\in Z^+,n\mid m!\right\}$, and the Smarandache LCM function is defined as $SL(n)=\min\left\{m\in Z^+:n\mid[1,2,\cdots,m]\right\}$, where Z^+ is the set of positive integers. There have been many achievements in the study of the solvability of number-theoretic functional equations. For example, Reference [1] studied the non-trivial solution problem of the number-theoretic functional equation $\varphi(n)=S\left(n^k\right)$ and, using elementary number theory methods, provided the necessary and sufficient conditions for the equation to have positive integer solutions n with $n>1$, and derived the positive integer solutions of the equation. Reference [2] studied the solvability of $\varphi_2(N)=S\left(N^{16}\right)$.Based on the properties of the generalized Euler function $\varphi_2(N)$ and the Smarandache function $S(N)$, and using piecewise and elementary methods, it was proved that this number-theoretic equation has only 10 positive integer solutions: $N=847,972,1000,1092,1089,1372,1500,1694,2058,2178$. Reference [3] studied the solvability problem of the number-theoretic functional equation $S(SL(n^{11,12}))=\varphi_2(n)$ and, using elementary number theory methods, gave all positive integer solutions for two number-theoretic functional equations. Reference [4] studied the solvability of the number-theoretic functional equation $\varphi_2(n)=S(SL(n^k))$ where $k=15,17$. Using elementary number theory methods, all positive integer solutions for the two number-theoretic functional equations were given. Reference [5] used elementary number theory methods to study the solvability of the number-theoretic functional equation $(\varphi_2(n))^2=S(SL(n^{3,4}))$ and obtained the positive integer solutions for these two equations as $n=16$ and $n=24$, respectively. Reference [6] studied the number-theoretic functional equation $2\varphi(n)=\varphi_2(n)+S(n^{15})$ and, using elementary methods, derived all three positive integer solutions $n=867,1156,1734$. Reference [7] examined a number-theoretic functional equation related to hexagonal numbers $\gamma\varphi_2(2m^2-m)=S(SL(m^{10}))$ and obtained four sets of positive integer solutions. This paper uses elementary number theory methods to investigate a number-theoretic functional equation related to triangular and enneagonal numbers $S\left(SL\left(n^{19}\right)\right)=k\varphi_2^2\left(N(n,3)\right)+t\varphi_2^2\left(N(n,9)\right)$ and provides all positive integer solutions of this equation. Here, $k,t,n\in Z^+$, the expression for triangular numbers is $N(n,3)=\frac{n^2}{2}+\frac{n}{2}$ and the

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expression for enneagonal numbers is $N(n,9)=\frac{7}{2}n^2-\frac{5}{2}n$.

II. Main Lemma

Lemma 1 [1] When $n\ge 3$, there is $\varphi_2(n)=\frac{\varphi(n)}{2}$.

Lemma 2 [3] For any positive integer n, then $\varphi(n)\ge\sqrt{\frac{n}{2}}$.

Lemma 3 [2] For prime numbers q and positive integers k, there is $S(q^k)\le kq$; In particular, when $k<q$, there is $S(q^k)=kq$.

Lemma 4 [4] For arbitrary positive integers m and n, there is $\varphi(mn)=\frac{\gcd(m,n)\varphi(m)\varphi(n)}{\varphi(\gcd(m,n))}=\varphi(nm)$, especially when $\gcd(m,n)=1$, there is $\varphi(mn)=\varphi(m)\varphi(n)$.

Lemma 5 [7] sets the positive integer $n=q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_s^{\alpha_s}$, where $q_1<q_2<\cdots<q_s$ is a prime number, then

$$\varphi(n)=n\left[1-\frac{1}{q_1}\right]\left[1-\frac{1}{q_2}\right]\cdots\left[1-\frac{1}{q_s}\right],$$

$$S(n)=\max\left\{S(q_1^{\alpha_1}),S(q_2^{\alpha_2})\cdots,S(q_s^{\alpha_s})\right\},$$

$$SL(n)=\max\left\{q_1^{\alpha_1},q_2^{\alpha_2}\cdots,q_s^{\alpha_s}\right\}.$$

Lemma 6 [6] When $n\ge 2$, there is $\varphi(n)<n$; When $n\ge 2$, $\varphi(n)$ is even.

III. Theorem and its Proof

Theorem Suppose $k,t,n\in Z^+$, number theory function equation

$$S\left(SL\left(n^{19}\right)\right)=k\varphi_2^2\left(\frac{n(n+1)}{2}\right)+t\varphi_2^2\left(\frac{n(7n-5)}{2}\right), \quad (3.1)$$

There is a positive integer solution, and the positive integer solution is $(k,t,n)=(4,2,2),(13,1,2),(10,2,3),(26,1,3)$.

Proof: When $n=1$, there is $\varphi_2(1)=0, S\left(SL\left(1^{19}\right)\right)=S(1)=1$, substituting equation (3.1) shows that the equation has no solution in this case. When $n=2$, there are $\varphi_2(3)=1$, $\varphi_2(9)=3$, $S\left(SL\left(2^{19}\right)\right)=S\left(2^{19}\right)=22$, substituting equation (3.1) to get $(k,t,n)=(4,2,2),(13,1,2)$,then the positive solution of the equation. Let $n=p_1^{l_1}p_2^{l_2}\cdots p_s^{l_s}=p^l m\ge 3$, in this case $\gcd(p^l,m)=1$, known by lemmas 3 and lemmas 5: $SL\left(n^{19}\right)=\max\left\{p_1^{19l_1},p_2^{19l_2}\cdots,p_s^{19l_s}\right\}=p^{19l}$, (3.2)

$$S\left(SL\left(n^{19}\right)\right)=S\left(p^{19l}\right)\leqslant 19lp, \quad (3.3)$$

From lemmas 1, lemmas 2 ,lemmas 3 and (3.1) are:

$$S(SL(n^{19}))\ge\frac{n}{4}(k+t). \quad (3.4)$$

According to (3.3), (3.4) there are:

$$19lp\ge\frac{n}{4}(k+t),\text{ i.e. } 2\le k+t\le 76lp, \quad (3.5)$$

Therefore $1\le k\le 76lp, 1\le t\le 76lp, 3\le n\le 38lp$. (3.6)

When $l\ge 2$,form (3.1),(3.3) and lemma 1:

$$19lp\ge\frac{k}{4}\varphi^2\left(\frac{n(n+1)}{2}\right)+\frac{t}{4}\varphi^2\left(\frac{n(7n-5)}{2}\right). \quad (3.7)$$

(1)When n is an odd number, n and $\frac{n+1}{2}\in Z^+$, n and $\frac{7n-5}{2}\in Z^+$,so lemmas 4 and lemmas 6 are obtained:

$$\varphi\left(n\frac{n+1}{2}\right)=\frac{\gcd\left(n,\frac{n+1}{2}\right)\varphi(n)\varphi\left(\frac{n+1}{2}\right)}{\varphi\left(\gcd\left(n,\frac{n+1}{2}\right)\right)}\ge\varphi(n)\varphi\left(\frac{n+1}{2}\right),$$

$$\varphi\left(n\frac{7n-5}{2}\right)=\frac{\gcd\left(n,\frac{7n-5}{2}\right)\varphi(n)\varphi\left(\frac{7n-5}{2}\right)}{\varphi\left(\gcd\left(n,\frac{7n-5}{2}\right)\right)}\ge\varphi(n)\varphi\left(\frac{7n-5}{2}\right),$$

According to (3.7):

$$19lp\ge\frac{1}{2}p^{2l-2}(p-1)^2, \quad (3.8)$$

So $$38l\ge p^{2l-3}(p-1)^2\ge 2^{2l-3}, \quad (3.9)$$

Take the logarithm at both ends of equation (3.9) at the same time $\log 38+\log l\ge(2l-3)\log 2$, (3.10)

Solved by (3.10) $l\le 5$.

(2)When n is an even number, $\frac{n}{2}$ and $n+1\in Z^+$, $\frac{n}{2}$ and $7n-5\in Z^+$,so lemma 4 and lemma 6 are obtained:

$$\varphi\left(\frac{n}{2}(n+1)\right)=\frac{\gcd\left(\frac{n}{2},n+1\right)\varphi\left(\frac{n}{2}\right)\varphi(n+1)}{\varphi\left(\gcd\left(\frac{n}{2},n+1\right)\right)}\ge\varphi\left(\frac{n}{2}\right)\varphi(n+1),$$

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$$\varphi\left(\frac{n}{2}(7n-5)\right)=\frac{\gcd\left(\frac{n}{2},7n-5\right)\varphi\left(\frac{n}{2}\right)\varphi(7n-5)}{\varphi\left(\gcd\left(\frac{n}{2},7n-5\right)\right)}\geq\varphi\left(\frac{n}{2}\right)\varphi(7n-5)$$

According to (3.7)

$$19lp\geq\frac{1}{2}\varphi^2\left(\frac{p^l m}{2}\right), \quad (3.11)$$

When $p=2$,obtained by (3.11),

$$19lp\geq\frac{1}{2}\varphi^2\left(2^{l-1}m\right)=\frac{1}{2}2^{2l-4}\varphi^2(m),$$

Then $76l\geq 2^{2l-4}$, (3.12)

Take the logarithm at both ends of equation (3.12) at the same time $\log 76+\log l\geq(2l-4)\log 2$, (3.13)

Solved by (3.13) $l\leq 6$. When $p>2$,it is obtained by (3.11),

$$19lp\geq\frac{1}{2}p^{2l-2}(p-1)^2, \quad (3.14)$$

Then $38l\geq 2^{2l-3}$, (3.15)

Take the logarithm at both ends of equation (3.15)

$$\log 38+\log l\geq(2l-3)\log 2, \quad (3.16)$$

Solved by (3.16) $l\leq 5$ .To sum up, $l\leq 6$. When $l\leq 2$, $l\leq 6$ is still true, so there is always $l\leq 6$; The following is a discussion of the different values of p,l ,which are divided into 6 situations.

Case 1 When $l=1, p\geq 2$.

When $p=2$, at this time $1\leq k\leq 152, 1\leq t\leq 152, 3\leq n\leq 76$, according to (3.2), (3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(2^{19})=88$ can be obtained by calculation $(k,t,n)=(6,1,3)$. The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation. When $p=3$, at this time $1\leq k\leq 228,\ 1\leq t\leq 228, 3\leq n\leq 114$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(3^{19})=168$ can be obtained by calculation $(k,t,n)=(10,2,3)$, $(26,1,3)$. Substituting these two possible solutions into equation (3.1) for verification shows that $(10,2,3)$ and $(26,1,3)$ are solutions of the equation. In the same way, when $p=5, p=7, p=11, p=13, p=17$, $p=19$,all possible solutions are calculated to be substituted into the original equation (3.1) verification, and it can be seen that none of them are solutions to the equation. When $p>19$, then $1\leq k\leq 76lp, 1\leq t\leq 76lp, 3\leq n\leq 38lp$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(n^{19})=76p$. From $n=pm$,we analyze and discuss two cases:

(1) When n is an even number, according to lemma 4, we have $k\varphi^2(\frac{n}{2})\varphi^2(n+1)+t\frac{a^2\varphi^2\left(\frac{n}{2}\right)\varphi^2(7n-5)}{\varphi^2(a)}=76p$

where $a=\gcd\left(\frac{n}{2},7n-5\right)$, it follows that $a=1$ or 5.

①When $a=1$,

$$k(p-1)^2\varphi^2(\frac{m}{2})\varphi^2(n+1)+t(p-1)^2\varphi^2(\frac{m}{2})\varphi^2(7n-5)=76p$$

that is $(p-1)^2\mid 76$.

②When $a=5$,

$$k(p-1)^2\varphi^2(\frac{m}{2})\varphi^2(n+1)+t\frac{5^2(p-1)^2\varphi^2\left(\frac{m}{2}\right)\varphi^2(7n-5)}{4^2}=76p$$

that is,

$$16k(p-1)^2\varphi^2(\frac{m}{2})\varphi^2(n+1)+25t(p-1)^2\varphi^2(\frac{m}{2})\varphi^2(7n-5)=1216p$$

hence, $(p-1)^2\mid 1216$.

(2) When n is an odd number, according to lemma 4, we have $k\varphi^2(n)\varphi^2(\frac{n+1}{2})+t\frac{b^2\varphi^2(n)\varphi^2\left(\frac{7n-5}{2}\right)}{\varphi^2(b)}=76p$

where $b=\gcd\left(n,\frac{7n-5}{2}\right)$, it follows that $b=1$ or 5.

①When $b=1$,

$$k(p-1)^2\varphi^2(m)\varphi^2(\frac{n+1}{2})+t(p-1)^2\varphi^2(m)\varphi^2\left(\frac{7n-5}{2}\right)=76p$$

that is $(p-1)^2\mid 76$.

②When $b=5$,

$$k(p-1)^2\varphi^2(m)\varphi^2(\frac{n+1}{2})+t\frac{5^2(p-1)^2\varphi^2(m)\varphi^2\left(\frac{7n-5}{2}\right)}{4^2}=76p$$

that is,

$$16k(p-1)^2\varphi^2(m)\varphi^2(\frac{n+1}{2})+25t(p-1)^2\varphi^2(m)\varphi^2\left(\frac{7n-5}{2}\right)=1216p$$

hence, $(p-1)^2\mid 1216$. From ① and ② ,we obtain $(p-1)^2\mid 76$ or $(p-1)^2\mid 1216$,so equation (3.1) has no positive integer solutions.

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Case 2 If $l=2$, we can get $p \le 3$ from equation (3.9). When $p=2$, at this time $1 \le k \le 304, 1 \le t \le 304$, $3 \le n \le 152$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(2^{38})=160$ can be obtained by calculation $(k,t,n)=(8,2,3),(24,1,3)$. The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation. When $p=3$, at this time $1 \le k \le 456, 1 \le t \le 456, 3 \le n \le 228$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(3^{38})=324$ can be obtained by calculation $(k,t,n)=(1,5,3),(17,4,3),(33,3,3),(49,2,3),(65,1,3)$. The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation. In the same way, When $p=5$, at this time $1 \le k \le 760, 1 \le t \le 760$, $3 \le n \le 380$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(5^{38})=620$ can be obtained (k,t,n), and the nine sets of solutions $(11,9,3),(27,8,3),(43,7,3),(59,6,3),(75,5,3),(91,4,3),(107,3,3)$ $(123,2,3),(139,1,3)$ are substituted into the original equation (3.1) verification, which shows that none of them are solutions to the equation.

Case 3 If $l=3$, we can get $p \le 3$ from equation (3.9). When $p=2$, at this time $1 \le k \le 456, 1 \le t \le 456$, $3 \le n \le 228$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(2^{57})=248$ can be obtained by calculation $(k,t,n)=(14,3,3),(30,2,3),(46,1,3)$. The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation. When $p=3$, at this time $1 \le k \le 684, 1 \le t \le 684, 3 \le n \le 342$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(3^{57})=468$ can be obtained by calculation $(k,t,n)=(5,7,3),(5,7,3),(21,6,3),(37,5,3),(53,4,3),(69,3,3),(85,2,3),(101,1,3)$. The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation.

Case 4 If $l=4$, we can get $p \le 2$ from equation (3.9). When $p=2$, at this time $1 \le k \le 608, 1 \le t \le 608$, $3 \le n \le 304$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(2^{76})=320$ can be obtained by calculation $(k,t,n)=(16,4,3),(32,3,3),(48,2,3),(64,1,3)$. The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation.

Case 5 If $l=5$, we can get $p \le 2$ from equation (3.9). When $p=2$, at this time $1 \le k \le 760, 1 \le t \le 760$, $3 \le n \le 380$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(2^{95})=392$ can be obtained by calculation $(k,t,n)=(2,6,3),(18,5,3),(34,4,3),(50,3,3),(66,2,3),(82,1,3)$. The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation.

Case 6 If $l=6$, we can get $p \le 2$ from equation (3.9). When $p=2$, at this time $1 \le k \le 912, 1 \le t \le 912$, $3 \le n \le 456$, according to (3.2),(3.4) and lemma 3, $k\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(7n-5)}{2}\right)=4S(2^{114})=480$ can be obtained by calculation $(k,t,n)=(2,6,3),(18,5,3),(34,4,3),(50,3,3),(66,2,3),(82,1,3)$. The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation.

III CONCLUSION

Based on the properties and definitions of the generalized Euler function $\varphi_2(n)$, the Smarandache function $S(n)$, and the Smarandache LCM function $SL(n)$, this paper explores the fundamental situation of all positive integer solutions to the number-theoretic function equation $S\left(SL\left(n^{19}\right)\right)=k\varphi_2^2\left(\frac{n(n+1)}{2}\right)+t\varphi_2^2\left(\frac{n(7n-5)}{2}\right)$ where $k,t,n \in Z^+$. The methods employed in this research can provide a reference for subsequent studies on solving similar types of number-theoretic function equations.

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