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Exploration of the Solvability of Triangular and Pentagonal Number Theory Function Equations

Caiyu Xie[1,2], Caifa Xie[3,*], Yimei Yan[3]
[1]School of Mathematical Sciences, Guizhou Normal University, Guiyang, Guizhou 550025, China
[2]Guizhou Key Laboratory of NewGen Cyberspace Security, Guiyang 550001, China
[3]Zhenxiong County No. 2 Middle School, Zhenxiong , Yunnan 657200, China

Abstract**—***According to the characteristics of three types of number theory functions, namely Smarandache function $S(n)$, generalized Euler function $\varphi_2(n)$ and Smarandache LCM function $SL(n)$, the number theory function equations are studied by using the method of elementary number theory. In this paper, the solvability problem of the number theory function equation $k\varphi_2^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right)=2S\left(SL\left(n^{11}\right)\right)$ related to the number number of triangles and the number of pentagons is studied, where $k,t,n\in Z^+$ (Z^+ is a set of positive integers). It is concluded that there is a positive integer solution for the number theory function equation $k\varphi_2^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right)=2S\left(SL\left(n^{11}\right)\right)$, and all the positive integers of the equation are solved as $(k,t,n)=(k,2,1),(6,3,3),(12,1,2),(22,2,3),(38,1,3)$.*

Keywords**—***Three types of functions; Elementary methods; Heptagonal number; Regular integer solution.*

I. INTRODUCTION

The generalized Euler function $\varphi_e(n)$, the Smarandache LCM function $SL(n)$ and the Smarandache function $S(n)$ are all important functions in number theory, and it can be seen from the definition that $S(n)=\min\{m:m\in Z^+,n\mid m!\}$, $\varphi_e(n)$,$1,2,\cdots\left[\frac{n}{e}\right]$ is the number of numbers in the sequence n that are mutually prime with $SL(n)=\min\{m\in Z^+:n\mid[1,2,\cdots,m]\}$, where Z^+ is the set of positive integers. Many results have been made in the study of the solvability of number theory function equations. For example, Cai Junjun et al. in Ref. [1] constructed a general solution framework for the number theory function equations of a specific form such as $t\varphi(n)+\varphi_2(n)=S\left(SL\left(n^k\right)\right)$,and used elementary methods to classify and discuss the system, and finally comprehensively obtained all the solutions of the equation $t\varphi(n)+\varphi_2(n)=S\left(SL\left(n^{7,8}\right)\right)$.Ref. [2] Zhang Sibao studied the solvability of the number theory function equation $\varphi_2(n)=S\left(SL\left(n^k\right)\right)$, and obtained that the equation only has a positive integer solution when $k=15,17$. Ref. [3] Li Xinxin et al. used the method of elementary number theory to study the solvability of the number theory function equation $\left(\varphi_2(n)\right)^2=S\left(SL\left(n^{3,4}\right)\right)$, and obtained all the positive integer solutions of the equation. Ref. [4] Li Changji studied the positive integer solution of the number theory function equation $2\varphi(n)=\varphi_2(n)+S\left(n^{25}\right)$, and obtained all the positive integer solutions of the equation. Ref. [5] Zhang Sibao et al. discussed the solvability of equations such as $\gamma(\varphi_2(x))^k=S\left(SL\left(x^l\right)\right)$, and used the method of elementary number theory to give all positive integer solutions of two specific equations. Ref. [6] Zhang Sibao et al. studied the solvability of the number theory function equation $Y\varphi_2\left(2m^2-m\right)=S\left(SL\left(m^{10}\right)\right)$ and gave all the positive integer solutions by using the method of elementary number theory. Ref. [7] He Yanfeng et al. used the method of elementary number theory to study the situation that the number theory function equation $Z\left(n^2\right)=\varphi_e\left(SL\left(n^2\right)\right)$ has no positive integer solution at $e\in(3,4)$.

Based on the above research, this paper uses the elementary number theory method to study the solvability of the number theory function equation $k\varphi_2^2(N(n,3))+t\varphi^2(N(n,5))=2S\left(SL\left(n^{11}\right)\right)$ related to triangular numbers and heptagonal numbers, where triangular numbers can be

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expressed as $N(n,3)=\frac{n(n+1)}{2}$ ($n\in Z^{+}$) and pentagonal numbers can be expressed as $N(n,5)=\frac{3n^2-n}{2}$ ($n\in Z^{+}$), and all positive integer solutions of the equation are given.

II. Main Lemma

Lemma 1 [1] When $n\geq 3$, there is $\varphi_2(n)=\frac{\varphi(n)}{2}$.

Lemma 2 [3] For any positive integer n, then $\varphi(n)\geq\sqrt{\frac{n}{2}}$.

Lemma 3 [2] For prime numbers q and positive integers k, there is $S(q^k)\leq kq$; In particular, when $k<q$, there is $S(q^k)=kq$.

Lemma 4 [4] For arbitrary positive integers m and n, there is $\varphi(mn)=\frac{\gcd(m,n)\varphi(m)\varphi(n)}{\varphi(\gcd(m,n))}=\varphi(nm)$, especially when $\gcd(m,n)=1$, there is $\varphi(mn)=\varphi(m)\varphi(n)$.

Lemma 5 [7] sets the positive integer $n=q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_s^{\alpha_s}$, where $q_1<q_2<\cdots<q_s$ is a prime number, then

$$\varphi(n)=n\left[1-\frac{1}{q_1}\right]\left[1-\frac{1}{q_2}\right]\cdots\left[1-\frac{1}{q_s}\right],$$

$$S(n)=\max\left\{S(q_1^{\alpha_1}),S(q_2^{\alpha_2})\cdots,S(q_s^{\alpha_s})\right\},$$

$$SL(n)=\max\left\{q_1^{\alpha_1},q_2^{\alpha_2}\cdots,q_s^{\alpha_s}\right\}.$$

Lemma 6 [6] When $n\geq 2$, there is $\varphi(n)<n$; When $n\geq 2$, $\varphi(n)$ is even.

III. Theorem and its Proof

Theorem Suppose $k,t,n\in Z^{+}$, number theory function equation

$$k\varphi_2^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right)=2S\left(SL\left(n^{11}\right)\right), \quad (3.1)$$

There is a positive integer solution, and the positive integer solution is $(k,t,n)=(k,2,1),(6,3,3),(12,1,2),(22,2,3),(38,1,3)$.

Proof: When $n=1$, there is $\varphi_2(1)=0$, $\varphi(1)=1$, $S\left(SL\left(1^{11}\right)\right)=S(1)=1$, substituting equation (3.1) to get $(k,t,n)=(k,2,1)$, then the equation has a positive integer solution. When $n=2$, there are $\varphi_2(3)=1$, $\varphi(5)=4$, $S\left(SL\left(2^{11}\right)\right)=S\left(2^{11}\right)=14$, substituting equation (3.1) to get $(k,t,n)=(12,1,2)$, then the positive integer solution of the equation.

Let $n=q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_s^{\alpha_s}=q^{\alpha}m\geq 3$, in this case $\gcd(q^{\alpha},m)=1$, known by lemmas 3 and lemmas 5:

$$SL\left(n^{11}\right)=\max\left\{q_1^{11\alpha_1},q_2^{11\alpha_2}\cdots,q_s^{11\alpha_s}\right\}=q^{11\alpha}, \quad (3.2)$$

$$S\left(SL\left(n^{11}\right)\right)=S\left(q^{11\alpha}\right)\leq 11\alpha q, \quad (3.3)$$

From lemmas 1, lemmas 2 and (3.1) are:

$$\begin{aligned}2S\left(SL\left(n^{11}\right)\right)&=k\varphi_2^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right)\\&=\frac{k}{4}\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right)\\&\geq\frac{k}{4}\left(\sqrt{\frac{n(n+1)}{4}}\right)^2+t\left(\sqrt{\frac{n(3n-1)}{4}}\right)^2\\&=\frac{k}{4}\cdot\frac{n(n+1)}{4}+t\cdot\frac{n(3n-1)}{4}\\&\geq\frac{kn}{4}+tn=\frac{n}{4}(k+4t).\end{aligned} \quad (3.4)$$

According to (3.3), (3.4) there are:

$$22\alpha q\geq\frac{n}{4}(k+4t),\ \text{i.e.}\ 5\leq k+4t\leq 88\alpha q, \quad (3.5)$$

Therefore $1\leq k\leq 88\alpha q, 1\leq t\leq 22\alpha q, 3\leq n\leq 18\alpha q.$ (3.6)

When $\alpha\geq 2$, from (1), (3) and lemma 1:

$$\begin{aligned}22\alpha q&\geq k\varphi_2^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right)\\&=\frac{k}{4}\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right),\end{aligned} \quad (3.7)$$

(1) When n is an odd number, n and $\frac{n+1}{2}\in Z^{+}$, n and $\frac{3n-1}{2}\in Z^{+}$, so lemmas 4 and lemmas 6 are obtained:

$$\varphi\left(n\frac{n+1}{2}\right)=\frac{\gcd\left(n,\frac{n+1}{2}\right)\varphi(n)\varphi\left(\frac{n+1}{2}\right)}{\varphi\left[\gcd\left(n,\frac{n+1}{2}\right)\right]}\geq\varphi(n)\varphi\left(\frac{n+1}{2}\right),$$

$$\varphi\left(n\frac{3n-1}{2}\right)=\frac{\gcd\left(n,\frac{3n-1}{2}\right)\varphi(n)\varphi\left(\frac{3n-1}{2}\right)}{\varphi\left[\gcd\left(n,\frac{3n-1}{2}\right)\right]}\geq\varphi(n)\varphi\left(\frac{3n-1}{2}\right),$$

According to (3.7):

$$22\alpha q\geq\frac{k}{4}\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right)$$

$$\geq\frac{k}{4}\varphi^2(n)\varphi^2\left(\frac{n+1}{2}\right)+t\varphi^2(n)\varphi^2\left(\frac{3n-1}{2}\right)$$
$$\geq\frac{1}{4}\varphi^2(n)+\varphi^2(n)=\frac{5}{4}\varphi^2(n)$$
$$=\frac{5}{4}q^{2\alpha-2}(q-1)^2\varphi^2(m)\geq\frac{5}{4}q^{2\alpha-2}(q-1)^2, \quad (3.8)$$

So $$18\alpha\geq q^{2\alpha-3}(q-1)^2\geq 2^{2\alpha-3}, \quad (3.9)$$

Take the logarithm at both ends of equation (3.9)
$$\log 18+\log\alpha\geq(2\alpha-3)\log 2, \quad (3.10)$$
Solved by (3.10) $\alpha\leq 4$.

(2)When n is an even number, $\frac{n}{2}$ and $n+1\in Z^+$, $\frac{n}{2}$ and $3n-1\in Z^+$, so lemma 4 and lemma 6 are obtained:

$$\varphi\left(\frac{n}{2}(n+1)\right)=\frac{\gcd\left(\frac{n}{2},n+1\right)\varphi\left(\frac{n}{2}\right)\varphi(n+1)}{\varphi\left(\gcd\left(\frac{n}{2},n+1\right)\right)}\geq\varphi\left(\frac{n}{2}\right)\varphi(n+1),$$

$$\varphi\left(\frac{n}{2}(3n-1)\right)=\frac{\gcd\left(\frac{n}{2},3n-1\right)\varphi\left(\frac{n}{2}\right)\varphi(3n-1)}{\varphi\left(\gcd\left(\frac{n}{2},3n-1\right)\right)}\geq\varphi\left(\frac{n}{2}\right)\varphi(3n-1),$$

According to (3.7)
$$22\alpha q\geq\frac{k}{4}\varphi^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right)$$
$$\geq\frac{k}{4}\varphi^2\left(\frac{n}{2}\right)\varphi^2(n+1)+t\varphi^2\left(\frac{n}{2}\right)\varphi^2(3n-1), \quad (3.11)$$

When $q=2$, obtained by (3.11),
$$22\alpha q\geq\frac{5}{4}\varphi^2\left(\frac{q^\alpha m}{2}\right)=\frac{5}{4}\varphi^2\left(2^{\alpha-1}\right)\varphi^2(m)\geq\frac{5}{4}\left(2^{\alpha-1}\right)^2,$$

Then $$36\alpha\geq 2^{2\alpha-4}, \quad (3.12)$$

Take the logarithm at both ends of equation (3.12) at the same time $\log 36+\log\alpha\geq(2\alpha-4)\log 2$, (3.13)

Solved by (3.13) $\alpha\leq 5$ .When $q>2$, it is obtained by (3.11), and $22\alpha q\geq\frac{5}{4}\varphi^2(q^\alpha)\varphi^2\left(\frac{m}{2}\right)\geq\frac{5}{4}q^{2\alpha-2}(q-1)^2$ is

$$22\alpha\geq\frac{5}{4}q^{2\alpha-3}(q-1)^2\geq\frac{5}{4}2^{2\alpha-3}, \quad (3.14)$$
$$18\alpha\geq 2^{2\alpha-3}, \quad (3.15)$$

Take the logarithm at both ends of equation (3.15)
$$\log 18+\log\alpha\geq(2\alpha-3)\log 2, \quad (3.16)$$

Solved by (3.16) $\alpha\leq 4$ .To sum up, when $\alpha\leq 5$. is $\alpha<2$, $\alpha\leq 5$ is still true, so there is always $\alpha\leq 5$; The following is a discussion of the different values of q,α, which are divided into 5 situations.

Case 1 If $\alpha=1$, $q\geq 2$.

When $q=2$, at this time $1\leq k\leq 176, 1\leq t\leq 44$, $3\leq n\leq 36$, according to (3.1), (3.3) and lemma 1,
$$k\varphi^2\left(\frac{n(n+1)}{2}\right)+4t\varphi^2\left(\frac{n(3n-1)}{2}\right)=8S\left(2^{11}\right)=8\times 14=112,$$
can be obtained by calculation $(k,t,n)=(12,1,3)$. The solution of this group is verified by substituting the original equation (1), and it can be seen that it is not the solution of the equation.

When $q=3$, at this time $1\leq k\leq 264, 1\leq t\leq 66$, $3\leq n\leq 54$, according to (3.1), (3.3) and lemma 1,
$$k\varphi^2\left(\frac{n(n+1)}{2}\right)+4t\varphi^2\left(\frac{n(3n-1)}{2}\right)=8S\left(3^{11}\right)=8\times 27=216$$
can be obtained by calculation $(k,t,n)=(6,3,3),(22,2,3),(38,1,3)$.The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation. In the same way, when $q=5$、$q=7$、$q=11$, all possible solutions are calculated to be substituted into the original equation (3.1) verification, and it can be seen that none of them are solutions to the equation.

When $q>11$, then $1\leq k\leq 88q, 1\leq t\leq 22q$, $3\leq n\leq 18q$, according to (3.1), (3.3) and lemma 1,
$$k\varphi^2\left(\frac{n(n+1)}{2}\right)+4t\varphi^2\left(\frac{n(3n-1)}{2}\right)=8S\left(q^{11}\right)=88q,$$
because $n=qm$ and $\gcd(q,m)=1$, there is $(q-1)^2\mid 88$, so there is $(q-1)^2\leq 88$, and when $q>11$ the prime numbers do not satisfy $(q-1)^2\mid 88$, so there is no positive integer solution for equation (3.1).

Case 2 If $\alpha=2$, we can get $2\leq q\leq 4$ from equation (3.9).

When $q=2$, at this time $1\leq k\leq 352, 1\leq t\leq 88$, $3\leq n\leq 72$, according to (3.1), (3.3) and lemma 1,
$$k\varphi^2\left(\frac{n(n+1)}{2}\right)+4t\varphi^2\left(\frac{n(3n-1)}{2}\right)=8S\left(2^{11\times 2}\right)=8\times 24=192,$$
can be obtained by calculation $(k,t,n)=(16,2,3),(32,1,3)$. The solution of this group is verified by substituting the original equation (3.1), and it can be seen that it is not the solution of the equation. In the same way, when $q=3$, all possible solutions are substituted into the original equation (3.1) through calculation, and it can be seen that none of them are solutions to the equation.

Case 3 If $\alpha=3$, $q\leq 2$ can be obtained from Equation (3.9).

When $q=2$, at this time $1\leq k\leq 528, 1\leq t\leq 132$, $3\leq n\leq 108$, according to (3.1), (3.3) and lemma 1,

$$k\varphi^2\left(\frac{n(n+1)}{2}\right)+4t\varphi^2\left(\frac{n(3n-1)}{2}\right)=8S\left(2^{11\times 3}\right)=8\times 36=288,$$

the four sets of solutions are calculated by substituting $(k,t,n)=$ $(8,4,3),(24,3,3),(40,2,3),(56,1,3)$. into the original equation (3.1) verification, which shows that none of them are solutions to the equation.

Case 4 If $\alpha=4$, we can get $q\le 2$ from equation (3.9).

When $q=2$, at this time $1\le k\le 704, 1\le t\le 176$, $3\le n\le 144$, according to (3.1), (3.3) and lemma 1,

$$k\varphi^2\left(\frac{n(n+1)}{2}\right)+4t\varphi^2\left(\frac{n(3n-1)}{2}\right)=8S\left(2^{11\times 4}\right)=8\times 48=384$$

can be obtained (k,t,n), and the five sets of solutions $(16,5,3),(32,4,3),(48,3,3),(64,2,3),(80,1,3)$ are substituted into the original equation (3.1) verification, which shows that none of them are solutions to the equation.

Case 5 If $\alpha=5$, we can get $q\le 2$ from equation (3.9).

When $q=2$, at this time $1\le k\le 880, 1\le t\le 220$, $3\le n\le 180$, according to (3.1), (3.3) and lemma 1,

$$k\varphi^2\left(\frac{n(n+1)}{2}\right)+4t\varphi^2\left(\frac{n(3n-1)}{2}\right)=8S\left(2^{11\times 5}\right)=8\times 60=480$$

can be obtained (k,t,n), and the eight sets of solutions $(5,1,4),(8,7,3),(24,6,3),(40,5,3),(56,4,3),(72,3,3),(88,2,3),$ $(104,1,3)$ are calculated by substituting the original equation (3.1) to verify, and it can be seen that none of them are solutions to the equation.

In summary, all positive integer solutions of the number theory function equation $k\varphi_2^2\left(\frac{n(n+1)}{2}\right)+t\varphi^2\left(\frac{n(3n-1)}{2}\right)$ $=2S\left(SL\left(n^{11}\right)\right)$ are $(k,t,n)=(k,2,1),(6,3,3),(12,1,2),$ $(22,2,3),(38,1,3)$. This research method and results can further expand the exploration of the solution of the equation of the same type of number theory function.

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REFERENCES

[1] Cai Junjun, Huang Jing. Solvability study of number theory function equations $t\varphi(n)+\varphi_2(n)=S\left(SL\left(n^k\right)\right)$ [J].Journal of Hefei University,2025(05):14-19+28.

[2] Zhang Sibao. Solveability of number theory function equations $\varphi_2(n)=S\left(SL\left(n^k\right)\right)$ [J].Journal of Southwest University(Natural Science Edition),2020(04):65-69.

[3] Li Xinxin, Goryeo. Solveability of number theory function equations $\left(\varphi_2(n)\right)^2=S\left(SL\left(n^{3,4}\right)\right)$ [J].Journal of Hubei University(Natural Science Edition),2024(05):675-679.

[4] Li Changji. Positive integer solutions of the number-theoretic functional equation [Image][J]. Journal of Nanning Normal University (Natural Science Edition), 2019(4): 35-39.

[5] Zhang Sibao, Chen Yingying. Discussion on the solvability of equations of a class of number theory functions[J].Journal of Anhui University(Natural Science Edition),2025(04):1-7.

[6] Zhang Sibao, Jiang Lianxia, Chen Yingying. Positive integer solution of unilateral function equations related to hexagonal numbers[J].Journal of Guangxi University(Natural Science Edition),2025(04):917-925.

[7] HeYanfeng, Li Xie, Han Fan, Xue Yuanyuan. Research on the solvability of number theory function equations $Z\left(n^2\right)=\varphi_e\left(SL\left(n^2\right)\right)$ [J].Journal of Hubei University(Natural Science Edition),2024(05):667-674.