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# On Entire Solutions of Certain Nonlinear Delay-Differential Equation

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**Abstract**— *In this paper, we mainly study the existence and form of entire solutions of two classes delay-differential equation*

$$f^n(z)+\omega f^{n-1}(z)f'(z)+(\Delta_c f)^{(k)}=p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}$$

*and*

$$f^n(z)+\omega f^{n-1}(z)f^{(k)}(z)+q(z)e^{Q(z)}(\Delta_c f)^{(k)}=p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z},$$

*where* $n\ge 2$, $k\ge 0$ *are integers,* $\omega$, $p_1$, $p_2$, $\alpha_1$, $\alpha_2$ *are non-zero constants satisfying* $\alpha_1\ne\alpha_2$, *and* $q(z)\ne 0$ *is a polynomial,* $Q(z)$ *is a non-constant polynomial.*

**Keywords**— *Nevanlinna theory; Entire solution; Delay-differential equation; The exponent of convergence of zeros; Growth of order*.

## I. INTRODUCTION AND MAIN RESULTS

In modern science, the application of mathematical equations is one of the important directions, such as physics, chemistry, and quantum mechanics and mathematical models in economics basically rely on differential equations and difference equations, and with the development of the discipline, these equations involved in the discipline have become more and more complex, so many scholars have advanced themselves in many disciplines, such as physics began to further study the deeper equation problems, the more typical of which are differential-difference and delay-differential equations. With the establishment of the Nevanlinna theory, it has become a powerful tool for studying the properties of solution of equation. We assume that the reader is familiar with standard notation and basic results of Nevanlinna theory, see [1-3] for more details. Suppose $f$ is a meromorphic function in the complex plane $\mathbb{C}$, we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, outside of a possible exceptional set of finite logarithmic measure. If meromorphic function $a(z)$ satisfies $T(r,a(z))=S(r,f)$, then $a(z)$ is called small function of $f$. We define its shift by $f(z+c)$, $c\in\mathbb{C}\setminus\{0\}$.

**Definition 1.1.**[3] Let $f$ be a meromorphic function in $\mathbb{C}$, its difference operators are defined as

$$\Delta_c f=f(z+c)-f(z),\quad \Delta_c^l f(z)=\sum_{j=0}^{l}C_l^j(-1)^{l-j}f(z+jc)$$

where $l$ is a positive integer.

**Definition 1.2.**[1] Let $f$ be a meromorphic function in $\mathbb{C}$, the order of $f$, the hyper order of $f$ and the exponent of convergence of zeros of $f$, demoted as $\rho(f)$, $\rho_2(f)$, $\lambda(f)$, respectively, are defined as

$$\rho(f)=\limsup_{x\to\infty}\frac{\log^+T(r,f)}{\log r},$$

$$\rho_2(f)=\limsup_{x\to\infty}\frac{\log^+\log^+T(r,f)}{\log r},$$

$$\lambda(f)=\limsup_{x\to\infty}\frac{\log^+N(r,\frac{1}{f})}{\log r}.$$

In 2004, Yang and Li[4] obtained all the entire solutions of the nonlinear differential equation $f(z)^3+\frac{3}{4}f''=-\frac{1}{4}\sin 3z$. Because of $c\sin bz=\frac{c}{2i}e^{biz}-\frac{c}{2i}e^{-biz}$, in 2006, Li and Yang[5] further studied Tumura-Clunie differential equation, and obtained the following result.

**Theorem 1.3.**[5] Let $n\ge 4$ be an integer and $Q_d(z,f)$ be an algebraic differential polynomial in $f$ of degree $d\le n-3$. Let $p_1(z)$, $p_2(z)$ be two nonzero polynomials, $\alpha_1$ and $\alpha_2$ be two nonzero constants with $\frac{\alpha_1}{\alpha_2}$ not rational. Then the following differential equation

$$f^n(z)+Q_d(z,f)=p_1(z)e^{\alpha_1 z}+p_2(z)e^{\alpha_2 z}\qquad(1.1)$$

has no transcendental entire solution.

With the development of the difference analogues of Nevanlinna theory, the focus of research has gradually extended from Tumura-Clunie differential equation to Tumura-Clunie difference equation. In 2010, Yang and Laine[6] explored difference analogue of the above differential equation didn't admit entire solutions of finite order. In [7], Wen et al. classified the finite order entire solutions of the equation $f^n(z)+q(z)e^{Q(z)}f(z+c)=P(z)$, where $q(z)$, $P(z)$ and $Q(z)$ are polynomials, $n\ge 2$ is a integer.

By observing the above Tumura-Clunie differential equations, we find that the degree of polynomial $Q_d(z,f)$ in the equation is usually not higher than $n-1$. Naturally, scholars consider the existence of solution of Tumura-Clunie



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equation when $deg(Q_d(z,f)) = n$, while noting that $(f^n)' = nf^{n-1}f'$, and there are only two terms on the left side of the equation. Therefore, Chen et al.[8] studied the following delay-differential equation when the left side of the equation has 3 dominant terms, the following result is obtained.

**Theorem 1.4.[8]** If $f$ is a transcendental entire solution with finite order to equation

$$f^n(z)+wf^{n-1}(z)f'(z)+q(z)e^{Q(z)}f(z+c) = p_1e^{\lambda z}+p_2e^{-\lambda z}, \tag{1.2}$$

where $n$ is a positive integer, $c$, $\lambda$, $p_1$, $p_2$ are nonzero constants and $w$ is a constant, and $q(z)(\not\equiv 0)$, $Q(z)$ are polynomials such that $Q(z)$ is not a constant, then the following conclusions hold.

(i) If $n \geq 4$ for $w \neq 0$ and $n \geq 3$ for $w = 0$, then $f$ satisfies $\rho(f) = deg(Q) = 1$.

(ii) If $n \geq 1$ and $f$ belongs to $\Gamma_0$, then

$$f(z) = e^{\frac{\lambda z}{n}+B},\ Q(z) = -\frac{n+1}{n}\lambda z + b,$$

or

$$f(z) = e^{-\frac{\lambda z}{n}+B},\ Q(z) = \frac{n+1}{n}\lambda z + b,$$

where $b, B \in \mathbb{C}$ and $\Gamma_0 = \{e^{\alpha(z)} : \alpha(z) \text{ is a nonconstant polynomial}\}$.

In 2022, Hao and Zhang[9] replaced the dominant term $f^n(z)+wf^{n-1}(z)f'(z)$ in (1.2) with $bf^n(z)=af^{n-1}(z)f^{(k)}(z)$ and obtained the nonlinear differential-difference equation

$$bf^n(z)+af^{n-1}(z)f^{(k)}(z)+q(z)e^{Q(z)}f(z+c) = p_1e^{\lambda z}+p_2e^{-\lambda z} \tag{1.3}$$

admits a transcendental entire solution of finite order $f$ with $\delta_{1)}(0,f) = 1-\overline{\lim\limits_{r\to\infty}}\dfrac{N_{1)}\left(r,\frac{1}{f}\right)}{T(r,f)} > 0$, and $\rho(f) = deg(Q) = 1$.

Later on, Xiang et al.[10] considered solution of (1.3) when its right side is replaced by $p_1e^{\lambda_1 z}+p_2e^{\lambda_2 z}$, where $p_1$, $p_2$, $\lambda_1$, $\lambda_2$ are nonzero constants, they obtained following result.

**Theorem 1.5.[10]** Let $f$ is a transcendental entire solution with finite order to

$$f^n(z)+wf^{n-1}(z)f^{(k)}(z)+q(z)e^{Q(z)}f(z+c) = p_1e^{\lambda_1 z}+p_2e^{\lambda_2 z}, \tag{1.4}$$

then the following conclusions hold.

(1) If $n \geq 4$, then $\rho(f) = deg(Q) = 1$.

(2) If $n \geq 1$ and $\lambda(f) < \rho(f)$, then $q(z)$ degenerates into a constant, and

$$f(z) = \left(\frac{p_2n^k}{n^k+w\lambda_2^k}\right)^{\frac{1}{n}}e^{\frac{\lambda_2}{n}z},$$

$$Q(z) = \left(\lambda_1-\frac{\lambda_2}{n}\right)z+\log\frac{p_1}{q\left(\frac{p_2n^k}{n^k+w\lambda_2^k}\right)^{\frac{1}{n}}}-\frac{\lambda_2 c}{n},$$

or

$$f(z) = \left(\frac{p_1n^k}{n^k+w\lambda_1^k}\right)^{\frac{1}{n}}e^{\frac{\lambda_1}{n}z},$$

$$Q(z) = \left(\lambda_2-\frac{\lambda_1}{n}\right)z+\log\frac{p_2}{q\left(\frac{p_1n^k}{n^k+w\lambda_1^k}\right)^{\frac{1}{n}}}-\frac{\lambda_1 c}{n}.$$

In 2023, Zhu and Chen[11] extended the equation, in which $f^{(k)}(z+c)$ is replaced by $\Delta_c^l f$ and every exponential term on the right side of the equation became polynomial of degree $q$, and obtained the following result.

**Theorem 1.6. [11]** Let $l$, $n(\geq 2)$, $m$, $q$ be positive integers and $c$ be a nonzero complex number satisfying $\Delta_c^l f \not\equiv 0$. Suppose that $p_1,\ldots,p_m$ are polynomials in $z$ of degree $q$, whose leading coefficients $a_1,\ldots,a_m$ are distinct nonzero complex numbers. Let $P, H_0, H_1,\ldots,H_m$ be meromorphic functions of order less than $q$ such that $PH_1\ldots H_m \not\equiv 0$. If the equation

$$f^n(z)+p(z)\Delta_c^l f = H_0+\sum_{i=1}^{m}H_ie^{p_i(z)} \tag{1.5}$$

admits a meromorphic solution $f$ satisfying $\rho_2(f) < 1$ and $N(r,f) = S(r,f)$, then $\rho(f) = q$, and the following assertions hold.

(i) When $\lambda(f) < \rho(f)$, we have $H_0 \equiv 0$, $m = 2$,

$$f(z) = g_0(z)e^{\frac{p_1(z)}{n}},\ \frac{a_1}{a_2} = n,\ g_0^n(z) = H_1.$$

(ii) When $\lambda(f) = \rho(f)$, we have two possibilities:

(1) When $H_0 \not\equiv 0$, we have $n \leq m+2$.

(2) When $H_0 \equiv 0$, we have $n \leq m+1$.

Many researchers have been studied about the solvability and existence of solution of certain kind of non-linear delay-differential equations [6-11]. Motivated by Theorem 1.5 and Theorem 1.6, here we considered the properties of solutions of the following nonlinear delay-differential equation, in which a term $f(z+c)$ in the middle of (1.2) is replaced with delay-term $(\Delta_c f)^{(k)}$ , Our results are as follows.

**Theorem 1.7.** Suppose that $n \geq 4$, $w$, $p_1$, $p_2$, $a_1$, $a_2$ are nonzero constants, for the differential-difference equation

$$f^n(z)+wf^{n-1}(z)f'(z)+(\Delta_c f)^{(k)} = p_1e^{a_1 z}+p_2e^{a_2 z}, \tag{1.6}$$

then the following assertions hold:

(1) When $\frac{a_i}{a_j} \neq n$, $i \neq j$, we have the (1.6) has no finite order transcendental entire solutions;

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(2) when $a_i = na$, $a = a_j$, $i \neq j$, then (1.6) has finite order transcendental entire solution $f$, and it must be of $f(z) = Ce^{az}$ form, where $C$ are non-zero constant satisfying $C = \frac{1}{(1+aw)^{\frac{1}{n}}} p_i^{\frac{1}{n}}$ in for some $i = 1, 2$.

Observing (1.4) and (1.6), for a class of nonlinear delay-differential (1.6), we consider the first order differential term $wf^{n-1}(z)f'(z)$ of (1.6) is replaced by $wf^{n-1}(z)f^{(k)}(z)$ and $(\Delta_c f)^{(k)}$ is replaced by $(q(z)e^{Q(z)}(\Delta_c f)^{(k)}$, then we study nonlinear delay-differential equation

$$f^n(z) + wf^{n-1}(z)f^{(k)}(z) + q(z)e^{Q(z)}(\Delta_c f)^{(k)} = p_1 e^{a_1 z} + p_2 e^{a_2 z}, \tag{1.7}$$

where $n \geq 2$, $k \geq 0$ are integers, $w, p_1, p_2, a_1, a_2$ are non-zero constants satisfying $a_1 \neq a_2$, and $q(z) \neq 0$ is a polynomial, $Q(z)$ is a non-constant polynomial, and the following results are obtained.

**Theorem 1.8.** If $f$ is a finite order transcendental entire solution to (1.7), then the following conclusions hold:

(1) If $n \geq 4$, then $\rho(f) = \deg(Q) = 1$.

(2) If $n \geq 1$ and $\lambda(f) < \rho(f)$, then $q(z)$ degenerates into a constant, and

$$f(z) = \left(\frac{p_2 n^k}{n^k + w(a_2)^k}\right)^{\frac{1}{n}} e^{\frac{a_2}{n}z},$$

$$Q(z) = \log p_1 + \left(a_1 - \frac{a_2}{n}\right)z - \log q\left(\frac{p_2 n^k}{n^k + wa_2^k}\right)^{\frac{1}{n}} - \log\left(\frac{a_2}{n}\right)^k - \log\left(e^{\frac{a_2 c}{n}} - 1\right),$$

or

$$f(z) = \left(\frac{p_1 n^k}{n^k + w(a_1)^k}\right)^{\frac{1}{n}} e^{\frac{a_1}{n}z},$$

$$Q(z) = \log p_2 + \left(a_2 - \frac{a_1}{n}\right)z - \log q\left(\frac{p_1 n^k}{n^k + wa_1^k}\right)^{\frac{1}{n}} - \log\left(\frac{a_1}{n}\right)^k - \log\left(e^{\frac{a_1 c}{n}} - 1\right),$$

where $d_0$ is a constant.

## II. PRELIMINARY RESULTS

To prove our theorem, we need the following lemma gives proximity function, integral counting function and characteristic function of logarithmic derivative and shift of a meromorphic function $f$.

**Lemma 2.1.**[3] Let $f$ be a meromorphic function of $\rho_2(f) < 1$, $k$ be a positive integer and $c, c_1, c_2$ be nonzero complex numbers. Then

$$m\left(r, \frac{f^{(k)}(z+c)}{f(z)}\right) = S(r,f), \quad m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = S(r,f).$$

Next we give some properties of counting function and characteristic function about meromorphic function $f$.

**Lemma 2.2.**[3] Let $f$ be a meromorphic function of $\rho_2(f) < 1$, $k$ be a positive integer and $c$ be nonzero complex number. Then

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r,f),$$

$$N(r, f(z+c)) = N(r, f(z)) + S(r,f),$$

$$T(r, f(z+c)) = T(r, f(z)) + S(r,f).$$

Next lemma plays an important part in the study of uniqueness of meromorphic function.

**Lemma 2.3.**[13] suppose $f_1, f_2, \ldots, f_n (n \geq 2)$ be meromorphic functions and $h_1, h_2, \ldots, h_n$ be entire functions satisfying the following conditions:

(1) $\sum_{j=1}^{n} f_j(z)e^{h_j(z)} \equiv 0$;

(2) For $1 \leq j < k \leq n$, $h_j - h_k$ is not a constant;

(3) For $1 \leq j \leq n, 1 \leq t < k \leq n$, $T(r, f_j) = o\{T(r, e^{h_t - h_k})\}$, $r \to \infty$, $r \notin E$, where $E$ is the set of finite linear measure.

Then $f_j(z) \equiv 0, j = 1, \ldots, n$.

The following lemma plays a vital role in the study of complex dealy-differential equations and it can be seen in [6].

**Lemma 2.4.**[6] suppose that $f$ is a transcendental meromorphic solution of $\rho_2(f) < 1$ of the equation

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f)$, $Q(z,f)$ are dealy-differential polynomials, its derivatives and its shifts with small meromorphic coefficients. Such that the total degree of $Q(z,f)$ is less than or equal to $n$, then

$$m(r, P(z,f)) = S(r,f)$$

for all $r$ outside of a possible exceptional set of finite logarithmic measure.

**Lemma 2.5.**[13] Suppose that $f_j(z)$ $(j = 1, 2, 3, 4)$ are meromorphic function and $f_1(z), f_2(z)$ are not constants satisfying

$$\sum_{j=1}^{4} f_j(z) \equiv 1,$$

If

$$\sum_{j=1}^{4} N\left(r, \frac{1}{f_j}\right) + 3\sum_{j=1}^{4} \bar{N}(r, f_j) < (\lambda + o(1))T(r, f_k) (r \in I, k = 1, 2),$$

where $\lambda < 1$, then $f_3 \equiv 1$ or $f_4 \equiv 1$ or $f_3 + f_4 \equiv 1$.

**Lemma 2.6.**[11] Let $m$, $q$ be positive integers, $\omega_1, \omega_2 \cdots, \omega_m$ be distinct nonzero complex numbers, and $H_0, H_1, \cdots, H_m$ be meromorphic function of order less than $q$ such that $H_j \not\equiv 0$, $1 \le j \le m$, Set $\varphi(z) = H_0 + \sum_{j=1}^{n} H_j(z) e^{\omega_j z^q}$. Then the following results hold.

(1) There exist two positive numbers $d_1 < d_2$, such that for sufficiently large $r$, $d_1 r^q \le T(r, \varphi) \le d_2 r^q$.

(2) If $H_0 \equiv 0$, then $m(r, \frac{1}{\varphi}) = o(r^q)$.

## III. Proof of Theorem 1.7

**The proof of Theorem 1.7.** suppose $f$ be a finite order transcendental entire solution of (1.6).

Set $A = f^n(z) + \omega f^{n-1}(z) f'(z)$, $B = (\Delta_c f)^{(k)}$, so (1.6) can be written as

$$A + B = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{3.1}$$

Differentiating both sides of (3.1) and eliminate $e^{\alpha_1 z}$, we have

$$\alpha_1 A + \alpha_1 B - A' - B' = p_2(\alpha_1 - \alpha_2) e^{\alpha_2 z}, \tag{3.2}$$

again differentiating (3.2), then

$$\alpha_1 A' + \alpha_1 B' - A'' - B'' = p_2 \alpha_2 (\alpha_1 - \alpha_2) e^{\alpha_2 z} \tag{3.3}$$

we eliminate $e^{\alpha_2 z}$ from (3.2) and (3.3), then

$$\alpha_1 \alpha_2 A - (\alpha_1 + \alpha_2) A' + A'' = -\alpha_1 \alpha_2 B + (\alpha_1 + \alpha_2) B' - B'', \tag{3.4}$$

where

$$A = f^n + \omega f^{n-1} f'$$
$$A' = n f^{n-1} f' + \omega[(n-1) f^{n-2} (f')^2 + f^{n-1} f'']$$
$$A'' = n(n-1) f^{n-2} (f')^2 + n f^{n-1} f'' + (n-1)(n-2)\omega f^{n-3} (f')^3 + 3(n-1)\omega f^{n-2} f' f'' + \omega f^{n-1} f'''$$

then combining (3.4) and $A, A', A''$, we get

$$f^{n-3} F(z, f) = -\alpha_1 \alpha_2 B + (\alpha_1 + \alpha_2) B' - B'', \tag{3.5}$$

where

$$F(z, f) = \alpha_1 \alpha_2 f^3 + [\omega \alpha_1 \alpha_2 - n(\alpha_1 + \alpha_2)] f^2 f' + \omega f^2 f''' + [n - \omega(\alpha_1 + \alpha_2 - 1)] f^2 f'' + (n-1)[n - \omega(\alpha_1 + \alpha_2)] f (f')^2 + \omega(n-1)(n-2)(f')^3 + 3\omega(n-1) f f' f'', \tag{3.6}$$

Next give that $(n-3) \ge 1$, and $\alpha_1 \alpha_2 B - (\alpha_1 + \alpha_2) B' + B''$ is a delay-differential polynomial of $f$ and the total degree is at most 1. By Lemma 2.4, then

$$m(r, F) = S(r, f), \; m(r, fF) = S(r, f).$$

Next we considered $F(z, f) \not\equiv 0$ or $F(z, f) \equiv 0$ two cases.

**Case 1.** If $F(z, f) \not\equiv 0$, then

$$T(r, f) = m(r, f) = m(r, \frac{fF}{F}) \le m(r, fF) + m(r, F) \le S(r, f),$$

this is impossible.

**Case 2.** If $F(z, f) \equiv 0$, from (3.6), then

$$\alpha_1 \alpha_2 f^3 = -[\omega \alpha_1 \alpha_2 - n(\alpha_1 + \alpha_2)] f^2 f' - \omega f^2 f''' - [n - \omega(\alpha_1 + \alpha_2 - 1)] f^2 f'' - 3\omega(n-1) f f' f'' - \omega(n-1)(n-2)(f')^3 - (n-1)[n - \omega(\alpha_1 + \alpha_2)] f (f')^2 \tag{3.8}$$

From the above equation, suppose $f$ has infinite many zeros, then we can clearly get that the zeros of $f$ have multiplicity greater or equal to 2. suppose $z_0$ is a zero of $f$ with multiplicity $m \ge 2$, then right side of (3.8) has zero at $z_0$ with multiplicity at most $3m-3$, while left side of the same has zeros at $z_0$ with multiplicity $3m$, which is impossible. Hence $f$ has finitely many zeros, through applying Hadamard's factorization theorem, we get

$$f(z) = \mu(z) e^{h(z)} \tag{3.9}$$

Where $\mu(z)$ is the canonical product formed by zeros of $f$ such that $\rho(\mu) = \lambda(f) < \rho(f)$, and $h(z)$ is a polynomial with $\deg(h) = \rho(f) \ge 1$. Then (1.6) can be written as

$$\mu^n(z) + \omega \mu^{n-1}(z)(\mu'(z) + \mu(z) h'(z))] e^{nh(z)} + D(z) e^{h(z+c)} + E(z) e^{h(z)} = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} \tag{3.10}$$

where

$$D(z) = \mu^{(k)}(z+c) + \sum_{i=1}^{k} \mu^{(k-i)}(z+c)[\sum_{n=1}^{N_i} a_{ni} \prod_{j=1}^{i} (h^{(j)}(z+c))^{t_{nj}}],$$

$$E(z) = \mu^{(k)}(z) + \sum_{i=1}^{k} \mu^{(k-i)}(z)[\sum_{n=1}^{N_i} a_{ni} \prod_{j=1}^{i} (h^{(j)}(z))^{t_{nj}}], \; t_{nj} \in \mathbb{N},$$

$\sum_{j=1}^{i} j t_{nj} = i$, $a_{ni} \in \mathbb{C}$, $n = 1, 2, \cdots, N_i$, $j = 1, 2, \cdots, i$, $N_i$ is the number of groups satisfying $\sum_{j=1}^{i} j t_{nj} = i$ and the coefficient of the term $\mu(z)(h'(z))^k e^{h(z)}$ is $a_{nk}$, $n = 1, 2, \cdots, N_i$, and $n = k$.

If $\deg(h) = l \ge 2$, since $\rho(\omega \mu^{n-1}(z)(\mu'(z) + \mu(z) h'(z)) + \mu^n(z)) < \deg(h)$, then by definition of order of growth, it is clear that the order of growth of the left side of the above (3.10) be $l$ while right side of the same has 1 order of growth, this is impossible. Hence $\deg(h) = 1$, then assume $h(z) = az + b$, where $a \ne 0$ and $b$ are constant. After substitution the $h(z)$, (3.10) becomes

$$(\mu^n(z) + \omega \mu^{n-1}(z)(\mu'(z) + \mu(z) h'(z))) e^{n(az+b)} + D(z) e^{az+ac+b} + E(z) e^{az+b} - p_1 e^{\alpha_1 z} - p_2 e^{\alpha_2 z} = 0. \tag{3.11}$$

Next we study the following cases:

(1) If $na \ne \alpha_j$, $a \ne \alpha_i$, $i \ne j$, $i, j = 1, 2$; it follows from this and Lemma 2.3 that $p_1 = 0$ and $p_2 = 0$, which is a contradiction.

(2) If $na = \alpha_j$, $a \ne \alpha_i$, $i \ne j$, $i, j = 1, 2$; say $na = \alpha_1$ and $a \ne \alpha_2$, it again follows from this and Lemma 2.3 that

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$p_2=0$ and $p_1\neq 0$, which is a contradiction. By the same method, when $n\alpha=a_2$ and $\alpha\neq a_1$, we can deduce $p_1=0$ and $p_2\neq 0$, which is a contradiction.

(3) If $n\alpha=a_j$, $\alpha=a_i$, $i\neq j$, $i,j=1,2$; say $n\alpha=a_1$ and $\alpha=a_2$, we have

$$\begin{aligned}&[(m^n(z)+\omega m^{n-1}(z)(m'(z)+m(z)\eta'(z)))e^{n\beta}-p_1]e^{a_1z}\\&+(D(z)e^{\alpha c+\beta}+E(z)e^{\beta}-p_2)e^{a_2z}\equiv 0\end{aligned}\qquad(3.12)$$

again applying Lemma 2.3, we get

$$(m^n(z)+\omega m^{n-1}(z)(m'(z)+m(z)\eta'(z)))e^{n\beta}-p_1\equiv 0,\qquad(3.13)$$

then

$$[m^{n-1}(z)(m(z)+\omega(m'(z)+\alpha m(z))]e^{n\beta}\equiv p_1,$$

this give $m(z)$ must be a constant and $\alpha\neq-\frac{1}{\omega}$, let $m(z)=m_0$. Substituting this into (3.13), then $m_0=(1+\omega\alpha)^{-\frac{1}{n}}e^{-\beta}p_1^{\frac{1}{n}}$. Thus combining (3.9), we get $f(z)=(1+\omega\alpha)^{-\frac{1}{n}}p_1^{\frac{1}{n}}e^{\alpha z}$.

In a similar way, we get $f(z)=(1+\omega\alpha)^{-\frac{1}{n}}p_2^{\frac{1}{n}}e^{\alpha z}$, so the solution of (1.6) must be of $f(z)=Ce^{\alpha z}$ form, where $C$ are nonvanishing constant.

Thus Theorem 1.7 is proved.

## IV. Proof of Theorem 1.8

**The proof of Theorem 1.8.** suppose $f$ be a finite order transcendental entire solution of (1.7). First we prove Theorem 1.8 (1). According to proof of Theorem 1.7, to simplify (1.7), let $A=f^n(z)+\omega f^{n-1}(z)f^{(k)}(z)$, $B=q(z)(\Delta_c f)^{(k)}$, then

$$A+Be^{Q(z)}=p_1e^{a_1z}+p_2e^{a_2z}.\qquad(4.1)$$

Differentiating both sides of (4.1), then

$$A'+B_1e^{Q(z)}=p_1a_1e^{a_1z}+p_2a_2e^{a_2z},\qquad(4.2)$$

where

$B_1=B'+BQ'(z)=(q(z)(\Delta_c f)^{(k)})'+q(z)(\Delta_c f)^{(k)}Q'(z)$.

Combining (1.7), (4.2) and (4.1), we eliminate $e^{a_1z}$, $e^{a_2z}$, then proceeding to the similar as we have done in the proof of the theorem 1.7, we get

$$A''-(a_1+a_2)A'-a_1a_2A+(B_2'+B_2Q'-a_2B_2)e^{Q(z)}=0\qquad(4.3)$$

where $B_2=B_1-a_1B$.

Now we study the order of $f$ by the following two case.

**Case 1.** suppose $\rho(f)<1$, then by applying Lemma 2.3, we get

$$\begin{aligned}T(r,e^{Q(z)})&=T\Big(r,\frac{p_1e^{a_1z}+p_2e^{a_2z}-f^n-\omega f^{n-1}f^{(k)}}{q(\Delta_c f)^{(k)}}\Big)\\
&\le T(r,p_1e^{a_1z})+T(r,p_2e^{a_2z})+T(r,f^n)+T\Big(r,1+\omega\frac{f^{(k)}}{f}\Big)\\
&\quad+T(r,q(\Delta_c f)^{(k)})\\
&\le(a_1+a_2)\frac{r}{\pi}+nT(r,f)+(k+1)N(r,f)+N(r,f)\qquad(4.4)\\
&\quad+m\Big(r,\frac{(\Delta_c f)^{(k)}}{f}\Big)+m(r,f)+N(r,(\Delta_c f)^{(k)})+S(r,f)\\
&\le(a_1+a_2)\frac{r}{\pi}+(n+3k+5)T(r,f)+S(r,f)\\
&\le(a_1+a_2)\frac{r}{\pi}+o(r).\end{aligned}$$

It follows from this inequality that $\deg(Q)=1$. Let $Q(z)=\alpha z+\beta$, where $\alpha\neq 0$, $\beta$ are constant. Applying Lemma 2.3 to (4.3), then

$$B_2'+B_2Q'-a_2B_2=B_2'+(\alpha-a_2)B_2=0,\qquad(4.5)$$

where

$B_2=B_1-a_1B=(q(\Delta_c f)^{(k)})'+q(\Delta_c f)^{(k)}\alpha-a_1q(\Delta_c f)^{(k)}$.

For the above equation, we discuss $B_2\equiv 0$ or $B_2\not\equiv 0$.

**Subcase 1.1.** Suppose $B_2\equiv 0$, Since $q(\Delta_c f)^{(k)}\not\equiv 0$, by integration, we get

$$q(\Delta_c f)^{(k)}=c_1e^{(a_1-\alpha)z},$$

where $c_1$ is a nonvanishing constant. According to $f$ is a finite order transcendental entire function, then $a_1\neq\alpha$ and $\rho(\Delta_c f)=1$, Since $T(r,(\Delta_c f)^{(k)})\le(k+1)T(r,\Delta_c f)+S(r,f)\le(2k+2)T(r,f)+S(r,f)$, then $\rho(\Delta_c f)\le\rho(f)$, which contradicts $\rho(f)<1$.

**Subcase 1.2.** Suppose $B_2\not\equiv 0$, by integration (4.5), we get

$$\begin{aligned}B_2&=(q(\Delta_c f)^{(k)})'+(\alpha-a_1)q(\Delta_c f)^{(k)}\\&=c_2e^{(a_2-\alpha)z},c_2\in\mathbb{C}\setminus\{0\},\end{aligned}\qquad(4.6)$$

By routine computation of one order linear differential equation, we have

$$q(\Delta_c f)^{(k)}=c_3e^{(a_1-\alpha)z}+\frac{c_2}{a_2-a_1}e^{(a_2-\alpha)z},c_3\in\mathbb{C}\setminus\{0\},\qquad(4.7)$$

it is clear that $a_2\neq a_1$. From discuss of case 1, we get $\rho(\Delta_c f)<\rho(f)$, then if $\alpha=a_1$, we get

$$q(\Delta_c f)^{(k)}=\frac{c_2}{a_2-a_1}e^{(a_2-\alpha)z}+c_3,$$

by Lemma 2.6, then $\rho(\Delta_c f)=1$, which contradicts $\rho(f)<1$. If $\alpha=a_2$, we get $q(\Delta_c f)^{(k)}=c_3e^{(a_1-\alpha)z}$, from the same reason, then $\rho(\Delta_c f)=1$, which is a contradiction. If $\alpha\notin\{a_1,a_2\}$, as the same discuss above, we also get a contradiction. Then it is

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impossible that $\rho(f) < 1$.

**Case 2.** suppose $\rho(f) \geq 1$, we can rewrite (1.7), then

$$f^n(z) + wf^{n-1}(z)f^{(k)}(z) + Be^{Q(z)} = H(z), \quad (4.8)$$

where $B = q(z)(\Delta_c f)^{(k)}$, $H(z) = p_1e^{a_1z} + p_2e^{a_2z}$, and $\rho(H) = 1$ by applying Lemma 2.6.

So differentiating (1.7), we get

$$nf^{n-1}f' + w(n-1)f^{n-2}f'f^{(k)} + wf^{n-1}f^{(k+1)} + B_1e^Q = H' \quad (4.9)$$

Eliminating $e^Q$ from (4.8) and (4.9), then from the same method theorem 1.7, we have

$$f^{n-2}G(z,f) = HB_1 - H'B \quad (4.10)$$

where

$$G(z,f) = Bf^2 + wB_1ff^{(k)} - nBff' - Bwff^{(k+1)} - Bw(n-1)f'f^{(k)}.$$

Next we discuss $G(z,f) \equiv 0$ or $G(z,f) \not\equiv 0$, aim for contradiction.

**Subcase 2.1.** Suppose $G(z,f) \not\equiv 0$, give that $n-2 \geq 2$ and $HB_1 - H'B$ is a delay-differential of $f$ and the total degree is 2. Then applying Lemma 2.4 to (4.10), we get

$$m(r,G) = S(r,f),\ m(r,fG) = S(r,f).$$

Then

$$T(r,f) = m(r,f) = m(r,\frac{fG}{G}) \leq m(r,fG) + m(r,\frac{1}{G})$$
$$\leq T(r,G) + S(r,f) = S(r,f),$$

which is impossible.

**Subcase 2.2.** Suppose $G(z,f) \equiv 0$, from (4.10), we get $HB_1 - H'B \equiv 0$. This give

$$\frac{B'(z)}{B(z)} + Q'(z) - \frac{H'(z)}{H(z)} = 0,$$

Then

$$\frac{q'(z)}{q(z)} + \frac{(\Delta_c f)^{(k+1)}}{(\Delta_c f)^{(k)}} + Q'(z) - \frac{H'(z)}{H(z)} = 0.$$

On integrating above equation, we get

$$q(z)(\Delta_c f)^{(k)}e^{Q(z)} = \frac{1}{c_4}(p_1e^{a_1z} + p_2e^{a_2z}). \quad (4.11)$$

For the same reason and method of (3.8), it is clear that $f$ has finitely many zeros, hence applying Hadamard factorisation theorem, $f$ must be of the form

$$f(z) = g_0(z)e^{t(z)}, \quad (4.12)$$

where $t(z)$ is a polynomial such that $\rho(f) = \deg(t) \geq 1$ and $g(z)$ is the canonical product of zeros of $f(z)$ with $\lambda(f) = \rho(g) < \rho(f)$. Then

$$f^{(k)}(z) = g_k(z)e^{t(z)}, \quad (4.13)$$

where $g_k(z) = g'_{j-1} + t'g_{j-1},\ j = 1, 2, \ldots, k$.

Combing (4.11), (4.12) and (4.13), we get

$$[g_0^n + wg_0^{n-1}g_k]e^{nt} = (1 - \frac{1}{c_4})H. \quad (4.14)$$

By definition of the order of growth, we get that the order of growth of the left side is greater than 1, while the order of growth of the right side is exactly 1. This is a contradiction.

Thus based on Case 1 and Case 2, we get $\rho(f) = 1$. Applying Lemma 2.1 and Lemma 2.6 to (1.2), then

$$T(r,e^{Q(z)}) = T(r,\frac{p_1e^{a_1z} + p_2e^{a_2z} - f^n - wf^{n-1}f^{(k)}}{q(\Delta_c f)^{(k)}})$$
$$\leq T(r,p_1e^{a_1z} + p_2e^{a_2z}) + T(r,f^n(1 + w\frac{f^{(k)}}{f})) + T(r,q(\Delta_c f)^{(k)})$$
$$\leq (a_1 + a_2)\frac{r}{p} + (n + 3k + 5)T(r,f) + S(r,f)$$
$$\leq O(r) + S(r,f),$$

since $\deg(Q) \geq 1$, so we get $\deg(Q) = \rho(f) = 1$. Hence (i) of theorem 1.8 is proved.

Next we prove (ii) of theorem. Suppose $f$ is a finite transcendental entire solution of (1.7) with $\lambda(f) < \rho(f)$, then applying Hadamard factorisation theorem, from (4.12) and (4.13), we get

$$[g_0^n + wg_0^{n-1}g_k]e^{nt} + qg_k(z+c)e^{Q+t(z+c)} - qg_ke^{Q+t} = p_1e^{a_1z} + p_2e^{a_2z}. \quad (4.15)$$

By simplifying the above equation, then

$$f_1 + f_2 + f_3 = 1, \quad (4.16)$$

where

$$f_1 = -\frac{p_1}{p_2}e^{(a_1-a_2)z},$$

$$f_2 = \frac{g_0^n + wg_0^{n-1}g_k}{p_2}e^{nt-a_2z},$$

$$f_3 = \frac{q}{p_2}e^{Q+t-a_2z}(g_k(z+c)e^{t(z+c)-t(z)} - g_k),$$

Because $l_1 \neq l_2$, then $f_1$ is not a constant. Let $T(r) = \max\{T(r,f_1), T(r,f_2), T(r,f_3)\}$.

Now we study the order of $f$ by the following two case.

**Case 1.** suppose $\deg(t) = \rho(f) > 1$, it is clear that $g_k$ is a polynomial of $g_0$ and its derivatives with polynomial coefficients. Since $\rho(g_0) < \deg(t)$, by definition of the order of growth of $f$, we get $\rho((g_k(z+c)e^{t(z+c)-t(z)} - g_k)) \leq \deg(t)$, then $g_k(z+c)e^{t(z+c)-t(z)} - g_k$, $g_0^n + wg_0^{n-1}g_k$ and $e^{-a_2}$ are small functions of $e^t$. We have

$$N(r,\frac{1}{f_2}) = N(r,\frac{1}{g_0^n + wg_0^{n-1}g_k}) = S(r,e^t),$$

$$N(r,\frac{1}{f_3}) = N(r,\frac{1}{q(g_k(z+c)e^{t(z+c)-t(z)} - g_k)}) = S(r,e^t),$$

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Since $T(r) \geq T(r, f_2) = nT(r,e^t) + S(r,e^t)$, then applying Lemma 2.6, we get $f_2 \equiv 1$ or $f_3 \equiv 1$.

If $f_2 \equiv 1$, we have $(g_0^n + wg_0^{n-1}g_k)e^{nt-a_2z} = p_2$. By Lemma 2.3 and $\deg(h) > 1$, then $p_2 \equiv 0$, which is impossible.

If $f_3 \equiv 1$, then $f_1 + f_2 \equiv 0$, that is $(g_0^n + wg_0^{n-1}g_k)e^{nt} = p_2 e^{a_1 z}$, again applying Lemma 2.3, we get $p_2 \equiv 0$, which is a contradiction.

**Case 2.** suppose $\deg(t) = \rho(f) = 1$. then $\rho(g_k) < \deg(h) = \rho(e^{(a_1-a_2)}z)$ and $\rho(g_k(z+c)e^{t(z+c)-t(z)} - g_k) < \rho(e^{(a_1-a_2)}z)$, $T(r) \geq T(r, f_1) = T(r, e^{(a_1-a_2)z}) + S(r, e^{(a_1-a_2)z})$. Thus we obtain

$$N(r,\frac{1}{f_2}) = N(r,\frac{1}{g_0^n + wg_0^{n-1}g_k}) = S(r,e^{(a_1-a_2)z}),$$

$$N(r,\frac{1}{f_3}) = N(r,\frac{1}{q(g_k(z+c)e^{t(z+c)-t(z)} - g_k)}) = S(r,e^{(a_1-a_2)z}),$$

By Lemma 2.6, we get $f_2 \equiv 1$ or $f_3 \equiv 1$. Now we discuss these two situations.

**Subcase 2.1.** Suppose $f_2 \equiv 1$, we have $(g_0^n + wg_0^{n-1}g_k)e^{nh-a_2z} = p_2$. Otherwise by Lemma 2.6, we get $p_2 \equiv 0$. Thus

$$g_0^n + wg_0^{n-1}g_k = p_2, \tag{4.17}$$

it is clear that $g_0$ is a constant. Otherwise if $g_0$ is a nonconstant entire function, then it follows from (4.17) that 0 is a Picard exceptional value of $g_0$, applying Hadamard's factorization theorem, we know that $g_0$ can be expressed as the product of the canonical product formed by zeros of $g_0$ and a transcendental entire function, so we have $\rho(g_0) \geq 1$, which contradicts with $\rho(g_0) < 1$. Then $g_0$ is a nonzero constant. Thus $g_k = g_0((\frac{a_2}{n}))^k$ we can rewrite (4.17),

$$g_0^n\left(1 + (\frac{a_2}{n})^k\right) = p_2,$$

therefore $g_0 = \left(\frac{p_2 n^k}{n^k + wa_2^k}\right)^{\frac{1}{n}}$, then

$$f(z) = g_0 e^{t(z)} = \left(\frac{p_2 n^k}{n^k + wa_2^k}\right)^{\frac{1}{n}} e^{\frac{a_2 z}{n}}. \tag{4.18}$$

Since $f_2 \equiv 1$, we have $f_1 + f_3 \equiv 0$, that is

$$qg_0 e^{Q+\frac{a_2}{n}}(\frac{a_2 z}{n})^k (e^{\frac{a_2 c}{n}} - 1) = p_1 e^{a_1 z}.$$

From the same reason as $g_0$, $(q(z)$ degenerates into a constant.

By routine computation, we get

$$Q(z) = \log p_1 + (a_1 - \frac{a_2}{n})z - \log q\left(\frac{p_2 n^k}{n^k + wa_2^k}\right)^{\frac{1}{n}} - \log(\frac{a_2}{n})^k - \log(e^{\frac{a_2 c}{n}} - 1).$$

**Subcase 2.2.** Suppose $f_3 \equiv 1$, we have $f_1 + f_2 \equiv 0$, that is $(g_0^n + wg_0^{n-1}g_k)e^{nt-a_1z} = p_1$. From the same reason and method as in the Subcase 2.1. Then $g_0 = \left(\frac{p_1 n^k}{n^k + wa_1^k}\right)^{\frac{1}{n}}$, we get

$$f(z) = g_0 e^{t(z)} = \left(\frac{p_1 n^k}{n^k + wa_1^k}\right)^{\frac{1}{n}} e^{\frac{a_1 z}{n}},$$

Since $f_3 \equiv 1$, we have $qg_0 e^{Q+\frac{a_1 z}{n}}(\frac{a_1}{n})^k(e^{\frac{a_1 c}{n}} - 1) = p_2 e^{a_2 z}$. Then

$$Q(z) = \log p_2 + (a_2 - \frac{a_1}{n})z - \log q\left(\frac{p_1 n^k}{n^k + wa_1^k}\right)^{\frac{1}{n}} - \log(\frac{a_1}{n})^k - \log(e^{\frac{a_1 c}{n}} - 1).$$

Hence, this completes the proof of Theorem 1.8.

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