IJSES

International Journal of Scientific Engineering and Science
Volume 8, Issue 12, pp. 95-98, 2025. ISSN (Online): 2456-7361

Spectral Jacobi-Galerkin Method for Two-dimensional Fredholm Integral Equation

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Abstract— *This paper mainly focuses on the numerical solution of the two-dimensional second-kind Fredholm integral equation by the Jacobi-Legendre Spectral-Galerkin method. Based on the Gauss points related to the Jacobi weight function as the collocation points, and applying the Gauss orthogonal quadrature formula to deal with the integral term. When the kernel function and source function are smooth enough, according to the weighted discrete inner product form, the weak form of the numerical algorithm is further realized. Finally, numerical examples are given to prove the advantages of our algorithm, The results show that the effectiveness and spectral accuracy correctness of the proposed algorithm.*

Keywords— *Two-dimensional Fredholm integral equation, spectral Jacobi-Galerkin method, Gauss or-thogonal scheme, numerical experiment.*

I. Introduction

With the wide application of engineering disciplines, the importance of integral equations in modern society has become more and more prominent, and many models abstractly summarized in production and life practice are either integral equations or can be transformed into integral equations to solve. After years of development, the study of integral equations has entered a new height. The theory of integral equations was mainly established by Fredholm and Volterra at the end of the 19th century. Their work has deeply influenced the research of integral equations in the 20th century. Its research significance is not only reflected in mathematics but also has very high practical application value, such as in the fields of atmospheric physics, engineering mechanics, and image processing. Many mathematical model projects in science and engineering can also be described by integral equation models, such as population prediction models, biological population ecological models, nerve impulse propagation, medical scanning, abnormal diffusion problems, and heat conduction problems of memory materials. For references, see [1]-[6]. Therefore, many scholars pay attention to and study Fredholm integral equations. Han and Atkinson's work [7] introduce the relevant theories of integral equations and sort out the numerical algorithms of Fredholm integral equations, such as the collocation method, Galerkin method, iterative Galerkin method, iterative collocation method, Nystrom method, and product integration method.

Brambilla et al [8] described the implementation of one-dimensional integral equations in finite element models. Xiang Xinmin [9] gave the numerical analysis of spectral methods and introduced the stability and convergence theory of spectral methods and the latest progress in spectral method research in recent years. Shen and Tang et al. introduced in their works [10, 11] the theory of orthogonal polynomials closely combined with spectral methods, spectral collocation method and spectral Galerkin method, Fourier spectral method for solving periodic problems, and the numerical formats and proofs of convergence analysis of various equations. Maleknejad and Sohrabi used Legendre-like wavelets to give the numerical solution of the first kind of Fredholm integral equation [12]. Yin Yang and et al utilized the spectral collocation method to solve the second kind of Fredholm integral equations with weakly singular kernels[13]. Peng Guo proposed a high-precision approximation method based on the Gauss-Lobatto quadrature formula, analyzed the errors, and proved the effectiveness of the algorithm[14]. The research conducted by numerous authors sufficiently demonstrates that the accuracy of numerical algorithms for Fredholm integral equations is improving.

The rest of this article is arranged as follows. In Section 2, we demonstrate the implementation of the spectral Jacobi-Galerkin method for two-dimensional Fredholm integral equations. In Section 3, we will list the numerical results in the L norm and weighted l2-norm for two numerical examples. Finally, a summary and outlook are given for the full text.

II. The Spectral Galerkin Methods

In this paper, we consider defining the domain $\Omega=(-1,1)^2$, define element (x,y) in R^2, X_N is a space composed of Jacobi orthogonal polynomials defined on Ω, $\omega^{\alpha,\beta}(x,y)=(1-x^2)^\alpha(1-y^2)^\beta$ is the Jacobi weight function on Ω, where $-1<\alpha,\beta<1$. From this, let

$$X_N=\text{span}\left\{\phi_i(x)\phi_j(y),\quad i,j=0,1,\cdots,N\right\}.$$

We denote by $L^2_{\omega^{\alpha,\beta}}(\Omega)$ the space of the measurable functions $u:\Omega\to R$ such that

$$\int_\Omega |u(x,y)|^2\,\omega^{\alpha,\beta}(x,y)dxdy<\infty,$$

which is a Banach space endowed with the norm

$$\|u\|_{L^2_{\omega^{\alpha,\beta}}(\Omega)}=\left(\int_\Omega |u(x,y)|^2\,\omega^{\alpha,\beta}(x,y)dxdy\right)^{\frac{1}{2}},$$

and is a Hilbert space for the inner product

$$(u,v)_{\omega^{\alpha,\beta}}=\int_\Omega u(x,y)v(x,y)\omega^{\alpha,\beta}(x,y)dxdy.$$

$L^\infty(\Omega)$ is the Banach space of the measurable functions $u:\Omega\to R$ that are bounded outside a set of measure zero, equipped with the norm

International Journal of Scientific Engineering and Science
Volume 8, Issue 12, pp. 95-98, 2025. ISSN (Online): 2456-7361

$$\|u\|_{L^2_{\omega^{\alpha,\beta}}(\Omega)} = \operatorname{ess\,sup}_{(x,y)\in\Omega}|u(x,y)| .$$

Given a two-index $\alpha = (\alpha_1, \alpha_2)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \alpha_2$, and

$$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial_x^{\alpha_1}\partial_y^{\alpha_2}} ,$$

define

$$H^m_{\omega^{\alpha,\beta}}(\Omega) = \{u \in L^2_{\omega^{\alpha,\beta}}(\Omega): \text{ for each nonnegative two-index } \alpha \text{ with } |\alpha| \le m, \text{ the distributional derivative } D^\alpha u \text{ belongs to } L^2_{\omega^{\alpha,\beta}}(\Omega)\},$$

this is a Hilbert space for the inner product

$$(u,v)_{\omega^{\alpha,\beta},m} = \sum_{|\alpha|\le m}\int_\Omega D^\alpha u(x,y)D^\alpha v(x,y)\omega^{\alpha,\beta}(x,y)dxdy ,$$

it is convenient to introduce the seminorms

$$|u|_{H^{m,N}_{\omega^{\alpha,\beta}}(\Omega)} = \left(\sum_{|\alpha|\le m}\|D^\alpha u\|^2_{L^2_\omega(\Omega)}\right)^{\frac{1}{2}} .$$

The Two-dimensional Fredholm integral equation model is as follows

$$z(\bar{x},\bar{y}) + \int_0^a\int_0^b k(\bar{x},\bar{y},s,t)z(s,t)dsdt = f(\bar{x},\bar{y}), \quad (\bar{x},\bar{y}) \in [0,a]\times[0,b] . \quad (2.1)$$

To facilitate the use of Jacobi orthogonal polynomials, first perform variable substitutions on $\bar{x}$ and $\bar{y}$ in the Fredholm integral equation (2.1)

$$\bar{x} = \frac{a}{2}(1+x), \quad x = \frac{2\bar{x}}{a} - 1, \quad x \in [-1,1] ,$$

$$\bar{y} = \frac{b}{2}(1+y), \quad y = \frac{2\bar{y}}{a} - 1, \quad y \in [-1,1] ,$$

let

$$u(x,y) = z\left(\frac{a}{2}(1+x), \frac{b}{2}(1+y)\right), \quad g(x,y) = f\left(\frac{a}{2}(1+x), \frac{b}{2}(1+y)\right).$$

Similarly, in order to facilitate the use of Jacobi Gaussian quadrature rules, perform substitutions on the integration variables s and t

$$s = s(\theta) = \frac{a}{2}(1+\theta), \quad \theta \in [-1,1] ,$$

$$t = t(\varphi) = \frac{a}{2}(1+\varphi), \quad \varphi \in [-1,1] ,$$

let

$$\tilde{k}(x,y,s(\theta),t(\varphi)) = \frac{ab}{4}k(x,y,s,t) .$$

So Eq.(2.1) becomes

$$u(x,y) + \int_{-1}^1\int_{-1}^1 \tilde{k}(x,y,s(\theta),t(\varphi))u(s(\theta),t(\varphi))d\theta d\varphi = g(x,y), \quad (x,y)\in\Omega = (-1,1)^2 . \quad (2.2)$$

By introducing the integral operator T defined by

$$T \circ u(x,y) = \int_{-1}^1\int_{-1}^1 \tilde{k}(x,y,s(\theta),t(\varphi))u(s(\theta),t(\varphi))d\theta d\varphi ,$$

(2.2) can be reformulated as

$$u(x,y) + T\circ u(x,y) = g(x,y), \quad (x,y)\in\Omega . \quad (2.3)$$

Then, we can deduce a discrete scheme of (2.3) reads: Find a $\forall u_N \in X_N$, such that

$$(u_N,v_N)_\omega + (T\circ u_N,v_N)_\omega = (g,v_N)_\omega, \quad \forall v_N \in X_N . (2.4)$$

Select a set of orthogonal bases $\phi_{ij}(x,y) = \phi_i(x)\phi_j(y), \quad i,j = 0,1,\cdots,N$ in space X_N , approximate the Solution

$$u_N = \sum_{i,j=0}^N u_{ij}\phi_{ij}(x,y) = \sum_{i,j=0}^N u_{ij}\phi_i(x)\phi_j(y) , \quad (2.5)$$

Let

$$v_N = \phi_{pl}(x,y) = \phi_p(x)\phi_l(y) . \quad (2.6)$$

So on the region $[-1,1] \times [-1,1]$, we obtain the following linear equation

$$\sum_{i,j=0}^N\int_{-1}^1\int_{-1}^1 \phi_{ij}(x,y)\phi_{pl}(x,y)\omega^{\alpha,\beta}(x,y)dxdy\cdot u_{ij}$$
$$+\sum_{i,j=0}^N\int_{-1}^1\int_{-1}^1 \phi_{pl}(x,y)\omega^{\alpha,\beta}(x,y)\left(\int_{-1}^1\int_{-1}^1 \tilde{k}(x,y,s(\theta),t(\varphi))\phi_{ij}(s(\theta),t(\varphi))d\theta d\varphi\right)dxdy\cdot u_{ij}$$
$$= \int_{-1}^1\int_{-1}^1 g(x,y)\phi_{pl}(x,y)\omega^{\alpha,\beta}(x,y)dxdy.$$

(2.1)

Using the orthogonal rule of (N + 1)-Legendre Gauss points to calculate the integral of the above equation. Firstly, note that $\{\theta_m\}_{m=0}^N$, $\{\theta_n\}_{n=0}^N$ and $\{v_m^1\}_{m=0}^N$, $\{v_n^2\}_{n=0}^N$ are Legendre Gauss points and their corresponding Legendre weight functions, respectively, clearly obtained

$$T\circ u(x,y) \approx T_N \circ u(x,y) = \sum_{m=0}^N\sum_{n=0}^N \tilde{k}(x,y,s(\theta_m),t(\varphi_n))u_{ij}(s(\theta_m),t(\varphi_n))v_m^1 v_n^2 = \tilde{K}(x,y)$$

Next, we implement the discrete inner product

$$(u,v)_\omega \approx (u,v)_{\omega,N} = \sum_{d,h=0}^N u(x_d,y_h)v(x_d,y_h)\omega_d^1\omega_h^2 ,$$

where $\{x_d\}_{d=0}^N$, $\{y_h\}_{h=0}^N$ are $(N+1)$ -Jacobi Gauss points, and the corresponding weight functions are $\{\omega_d^1\}_{d=0}^N$, $\{\omega_h^2\}_{h=0}^N$. Thus the discrete form of equation (2.4) is as follows

$$(u_N,v_N)_{\omega,N} + (T_N\circ u_N,v_N)_{\omega,N} = (g,v_N)_{\omega,N}, \quad \forall v_N \in X_N . (2.8)$$

Substituting (2.5) and (2.6) into the above equation gives the discrete form of Eq.(2.7) as follows

$$\sum_{i,j=0}^N (\phi_{ij},\phi_{pl})_{\omega,N}u_{ij} + \sum_{i,j=0}^N (T_N\circ\phi_{ij},\phi_{pl})_{\omega,N}u_{ij} = (g,\phi_{pl})_{\omega,N} , \quad (2.9)$$

where

$$T_N\circ\phi_{ij} = \sum_{m=0}^N\sum_{n=0}^N \tilde{k}(x,y,s(\theta),t(\varphi))\phi_i(s(\theta_m))\phi_j(t(\varphi_n))v_m^1 v_n^2 = \tilde{K}_{ij}(x,y) ,$$

and

$$(T_N\circ\phi_{ij},\phi_{pl})_{\omega,N} = \sum_{d=0}^N\sum_{h=0}^N \tilde{K}_{ij}(x_d,y_h)\phi_p(x_d)\phi_l(y_h)\omega_d^1\omega_h^2 = \bar{K}_{ij}^{pl} ,$$

$$(\phi_{ij},\phi_{pl})_{\omega,N} = \sum_{d=0}^N\sum_{h=0}^N \phi_i(x_d)\phi_j(y_h)\phi_p(x_d)\phi_l(y_h)\omega_d^1\omega_h^2 = \bar{A}_{ij}^{pl} ,$$

$$(g,\phi_{pl})_{\omega,N} = \sum_{d=0}^N\sum_{h=0}^N g(x_d,y_h)\phi_p(x_d)\phi_l(y_h)\omega_d^1\omega_h^2 = \bar{F}_{pl} .$$

Hence, the discrete version of equation (2.9) can be represented in matrix form as shown below

$$(A+K)U = G ,$$

where

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$$A=\begin{bmatrix} \bar{A}_{00}^{00} & \bar{A}_{01}^{00} & \cdots & \bar{A}_{0N}^{00} & \bar{A}_{10}^{00} & \cdots & \bar{A}_{N0}^{00} & \cdots & \bar{A}_{NN}^{00} \\ \bar{A}_{00}^{01} & \bar{A}_{01}^{01} & \cdots & \bar{A}_{0N}^{01} & \bar{A}_{10}^{01} & \cdots & \bar{A}_{N0}^{01} & \cdots & \bar{A}_{NN}^{01} \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & \vdots \\ \bar{A}_{00}^{NN} & \bar{A}_{01}^{NN} & \cdots & \bar{A}_{0N}^{NN} & \bar{A}_{10}^{NN} & \cdots & \bar{A}_{N0}^{NN} & \cdots & \bar{A}_{NN}^{NN} \end{bmatrix},$$

$$K=\begin{bmatrix} \bar{K}_{00}^{00} & \bar{K}_{01}^{00} & \cdots & \bar{K}_{0N}^{00} & \bar{K}_{10}^{00} & \cdots & \bar{K}_{N0}^{00} & \cdots & \bar{K}_{NN}^{00} \\ \bar{K}_{00}^{01} & \bar{K}_{01}^{01} & \cdots & \bar{K}_{0N}^{01} & \bar{K}_{10}^{01} & \cdots & \bar{K}_{N0}^{01} & \cdots & \bar{K}_{NN}^{01} \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & \vdots \\ \bar{K}_{00}^{NN} & \bar{K}_{01}^{NN} & \cdots & \bar{K}_{0N}^{NN} & \bar{K}_{10}^{NN} & \cdots & \bar{K}_{N0}^{NN} & \cdots & \bar{K}_{NN}^{NN} \end{bmatrix},$$

$$U=\left[\bar{u}_{00} \quad \bar{u}_{01} \quad \cdots \quad \bar{u}_{0N} \quad \bar{u}_{10} \quad \cdots \quad \bar{u}_{N0} \quad \cdots \quad \bar{u}_{NN}\right],$$

$$F=\left[\bar{F}_{00} \quad \bar{F}_{01} \quad \cdots \quad \bar{F}_{0N} \quad \bar{F}_{10} \quad \cdots \quad \bar{F}_{N0} \quad \cdots \quad \bar{F}_{NN}\right].$$

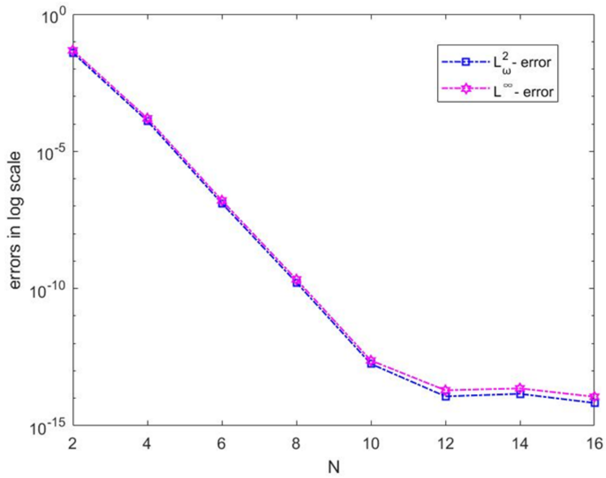
III. NUMERICAL EXPERIMENTS

In this section, illustrative examples are given to show the effectiveness of the method proposed in Section 2. We give numerical examples to confirm our analysis, $L^2_\omega(\Omega)$ and $L^\infty(\Omega)$ errors are employed to assess the efficiency of the method. All numerical calculations are implemented through MATLAB R2021a platform.

Example 1: Consider the following two-dimensional Fredholm integral equation

$$u(x,y)+\int_{-1}^{1}\int_{-1}^{1}(\text{tsin(x)+ys})u(s,t)\,dsdt = \text{xcos(y)}+\frac{4\sin(x)}{3}\text{-y}(1+\frac{4\sin(1)}{3}),$$

with exact solution is given by $u(x,y)=\text{x.cos(y)-y}$. By calculating the approximate solution, we obtain the $L^2_\omega(\Omega)$ and $L^\infty(\Omega)$ errors results of the spectral Jacobi-Legendre-Galerkin method: for different N, the errors between approximate solution and exact solutions are listed in Tables 1. To demonstrate the effectiveness of our algorithm intuitively, we present the images of both the exact and approximate solution in Figures 1. It is observed that the desired exponential rate of convergence is obtained.



Figures. 1. The error function under the L^2_ω and L^∞ norms

Observing the numerical examples, by calculating the approximate solution, we obtain the L^2_ω and the L^∞ error results of the spectral Jacobi-Galerkin method. For different N, the errors between the approximate solution and the exact solution are listed in Table 1. To intuitively demonstrate the effectiveness of our algorithm, we present the error images under L^2_ω and the L^∞ for $2 \le N \le 16$ in Figure 1. When $N \ge 12$, it can be intuitively seen that the accuracy of the approximate solution is approximately 10^{-14}.

Tables 1. The errors $\|u-\bar{u}_N\|_{L^2_\omega(\Omega)}$ and $\|u-\bar{u}_N\|_{L^\infty(\Omega)}$

N	2	4	6	8	10
L^2_ω-error	3.9100e-02	1.2779e-04	1.2710e-07	1.6591e-10	1.8020e-13
L^∞-error	4.7900e-02	1.5994e-04	1.6238e-07	2.1451e-10	2.3870e-13
N	**12**	**14**	**16**	**18**	**20**
L^2_ω-error	1.1759e-14	1.4569e-14	6.7628e-15	1.1601e-14	1.8941e-14
L^∞-error	1.9651e-14	2.2982e-14	1.1380e-14	3.3352e-14	3.6023e-14

IV. CONCLUDING REMARKS

In this paper, we propose a spectral Jacobi-Galerkin method based on the Jacobi-Legendre Gauss point to obtain the numerical solution of the two-dimensional Volterra integral equation, this method exhibits exponential decay in both the L^2_ω norm and the L^∞ norm, which is characteristic of the spectral method. In addition, it is expected that the method in this paper can be applied to integral equations that are nonlinear in two dimensions.

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International Journal of Scientific Engineering and Science
Volume 8, Issue 12, pp. 95-98, 2025. ISSN (Online): 2456-7361

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