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Efficient Spectral Method for Elliptic Equations Problems in Special Areas

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Abstract— *This paper proposes and studies a spectral method for solving elliptic problems in fan-shaped regions with Neumann boundary conditions. In theory, we transform the region to polar coordinates of* $[-1,1]\times[-1,1]$ *through transformation. We develop a unique and diverse style. and to construct a set of appropriate basis functions that are suitable for columnar conditions, form the problem into a matrix system, and verify its effectiveness, improving spectral accuracy through a large number of numerical examples.*

Keywords— *Elliptic problems, fan-shaped regions, spectral method, algorithm design, numerical experiment.*

I. INTRODUCTION

In many fields of science and engineering today, solving elliptic equations and their eigenvalues has always been a key research topic. Elliptical equations are widely used in multiple disciplines such as fluid mechanics, electromagnetics, and quantum mechanics, playing an irreplaceable role in accurately describing and understanding various physical phenomena and processes. This problem also has an important physical background and has a wide range of applications in quantum mechanics, fluid mechanics, modern science and technology, engineering, and other fields [1–5]. At present, there are many methods to solve eigenvalue problems [6–11], Such as general methods, finite element methods, and finite difference methods. Constructing a direct spectral method for solving complex geometric problems is a huge challenge, and there are basically two direct spectral methods: one is to embed complex geometric regions into a larger scale regular domain, called the virtual domain method; Another method is to use one of the mappings shown to map complex geometric regions into a regular domain. Gu and Shen [13] use the complete spectral method in the virtual domain to solve the elliptic differential equations in complex geometry. Levy [14] proposed an energetic method for solving partial differential equations in complex structural domains by introducing irregular regions into regular regions. For finite element methods, obtaining high-precision forced insights requires a large amount of storage capacity. The format construction of finite difference method is more flexible and the thinking method is relatively simple, but it becomes more difficult when the region is more complex. Therefore, using spectral methods to solve second-order elliptic equation problems on special regions is of great significance.

This article proposes an effective spectral method for solving second-order elliptic problems in a fanshaped region. The method converts Cartesian coordinates to polar coordinates of the fan-shaped region, and uses variable transformation to transform the function of the convex region in polar coordinates into a function of the $[-1,1]\times[-1,1]$ region. A set of effective basis functions is constructed, and an approximate solution is developed using this set of basis functions. The equation is discretized using the Lgendre spectral method. A discrete form equation is a linear characteristic system that can be effectively solved. This article provides a numerical example, and the numerical results show that the proposed method is effective and convergent.

The rest of this article is arranged as follows. In Section 2, We have defined a class of Sobolev spaces and established their variational and corresponding discrete formats. In Section 3, we focus on constructing the basis functions and implementing the algorithm. In Section 4, we present a numerical example to validate the theoretical findings and the effectiveness of the algorithm. Finally, we give in Section 5 a concluding remark.

II. WEAK FORM AND DISCRETE FORMAT

In this paper, we consider the second-order elliptic problem under Riemann boundary conditions as follows:

$$-\Delta u+\alpha u=f,\qquad in\ \Omega, \tag{2.1}$$

$$\frac{\partial u}{\partial n}=0,\qquad on\ \Omega, \tag{2.2}$$

here, Ω represents a sector-shaped region, while α represent positive constants, n is the outward normal
vector at boundary ∂Ω.

Introduce the usual Sobolev space:

$$L^2(\Omega):=\{u:\int_\Omega u^2dx<\infty\},\quad H^1(\Omega):=\{u:u\in L^2(\Omega)\},$$

endowed with the norms:

$$(u,v):=\int_{\Omega,\omega}uv\omega dx,\quad \|u\|_\omega=(u,v)^{\frac{1}{2}}.$$

Obviously, the weak form of (2.1)-(2.2) is: Find $u\in H^1(\Omega)$ such that

$$a(u,v)=F(v),\quad \forall v\in H^1(\Omega), \tag{2.3}$$

where

$$a(u,v)=\int_\Omega \nabla u\nabla v+\alpha uvdx,\quad F(v)=\int_\Omega fvdx. \tag{2.4}$$

Define transformation form as follows:

$$x=r\cos\theta,\quad y=r\sin\theta,\quad (r,\theta)\in[0,R]\times[\theta_1,\theta_2],$$

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$$r=\frac{(t+1)}{2}R,\quad \theta=\frac{(\theta_2-\theta_1)(s+1)}{2}+\theta_1,\quad (t,s)\in[-1,1]\times[-1,1]. \tag{2.5}$$

Let

$$\hat{u}(t,s)=u(r\cos\theta, r\sin\theta),$$
$$\hat{v}(t,s)=v(r\cos\theta, r\sin\theta),$$
$$\hat{f}(t,s)=f(r\cos\theta, r\sin\theta).$$

Then, we substitute (2.5) into (2.4) to obtain the following transformation form

$$\int_{\Omega}(t+1)\frac{\partial\hat{u}}{\partial t}\frac{\partial\hat{v}}{\partial t}+\frac{4}{(\theta_2-\theta_1)^2}\frac{1}{(t+1)}\frac{\partial\hat{u}}{\partial s}\frac{\partial\hat{v}}{\partial s}+\frac{\alpha R^2}{4}(t+1)\hat{u}\hat{v}dtds=\frac{R^2}{4}\int_{\Omega}(t+1)\hat{f}\hat{v}dtds.$$

where $\hat{\Omega}=[-1,1]\times[-1,1]$.

Define a class of weighted Sobolev space:

$$H_*^1(\hat{\Omega}):=\left\{u:\int_{\Omega}(t+1)\left|\frac{\partial\hat{u}}{\partial t}\right|^2+\frac{1}{(t+1)}\left|\frac{\partial\hat{u}}{\partial s}\right|^2+\frac{\alpha R^2}{4}(t+1)|\hat{u}|^2dtds<\infty,\frac{\partial\hat{u}}{\partial s}|t=-1=0\right\},$$

endowed with the norms:

$$(u,v)_* :=\int_{\Omega,\omega}\hat{u}\hat{v}\omega dx,\quad \|u\|_{1,*}=(\hat{u},\hat{v})_{1,*}^{\frac{1}{2}},$$

Then, obtain the corresponding weak form as:

$$a(\hat{u},\hat{v})=F(\hat{v}),\quad \forall\hat{v}\in H_*^1(\hat{\Omega}), \tag{2.6}$$

where

$$a(\hat{u},\hat{v})=\int_{\hat{\Omega}}(t+1)\frac{\partial\hat{u}}{\partial t}\frac{\partial\hat{v}}{\partial t}+\frac{4}{(\theta_2-\theta_1)^2}\frac{1}{(t+1)}\frac{\partial\hat{u}}{\partial s}\frac{\partial\hat{v}}{\partial s}+\frac{\alpha R^2}{4}(t+1)\hat{u}\hat{v}dtds,$$

$$F(\hat{v})=\frac{R^2}{4}\int_{\hat{\Omega}}(t+1)\hat{f}\hat{v}dtds.$$

Let P_N be a space of polynomials of degree N, and Define approximation space $X_N=H_*^1(\hat{\Omega})\cap(P_N\times P_N)$. Then, a spectral-Galerkin approximation of (2.6) reads: Find $u_N\in X_N$, such that

$$a(\hat{u}_N,\hat{v}_N)=F(\hat{v}_N),\quad \forall\hat{v}_N\in X_N, \tag{2.7}$$

where

$$a(\hat{u}_N,\hat{v}_N)=\int_{\hat{\Omega}}(t+1)\frac{\partial\hat{u}_N}{\partial t}\frac{\partial\hat{v}_N}{\partial t}+\frac{4}{(\theta_2-\theta_1)^2}\frac{1}{(t+1)}\frac{\partial\hat{u}_N}{\partial s}\frac{\partial\hat{v}_N}{\partial s}+\frac{\alpha R^2}{4}(t+1)\hat{u}_N\hat{v}_N dtds,$$

$$F(\hat{v}_N)=\frac{R^2}{4}\int_{\hat{\Omega}}(t+1)\hat{f}\,\hat{v}_N dtds.$$

III. THE ALGORITHMIC IMPLEMENTION

In this section, we will select appropriate basis functions to help us obtain effective solutions. Firstly,

we construct a set of appropriate basis functions. Let

$$\phi_k(x)=L_k(x)-L_{k+2}(x),\quad 0\le k\le N+2,$$
$$\phi_{N-1}(x)=\frac{x+1}{2},\ \phi_N(x)=1,$$

here, L_k is a k-th degree Legendre polynomial. It is obvious that

$$X_N=\{\phi_i(t)\phi_j(s),\quad i=0,1,...,N;\ J=0,1,...,N\}.$$

We have

$$u_N=\sum_{i=0}^{N}\sum_{j=0}^{N}u_{ij}\phi_i(t)\phi_j(s). \tag{3.1}$$

Specifically, u_{ij}represents the expansion coefficient, which is presented as follows

$$U=\begin{bmatrix}u_{00} & u_{01} & \cdots & u_{0N}\\ u_{10} & u_{11} & \cdots & u_{1N}\\ \vdots & \vdots & \ddots & \vdots\\ u_{N0} & u_{N1} & \cdots & u_{NN}\end{bmatrix}.$$

Taking $v_N=\phi_m(t)\phi_n(s)$. Then, the equivalent form of discrete format (2.4) is as follows

$$\sum_{i=0}^{N}\sum_{j=0}^{n}\Big(\int_{\hat{\Omega}}(t+1)\phi_i'(t)\phi_m'(t)\phi_j(s)\phi_n(s)\,dtds+\frac{4}{(\theta_2-\theta_1)^2}\int_{\hat{\Omega}}\frac{1}{(t+1)}\phi_i(t)\phi_m(t)\phi_j'(s)\phi_n'(s)\,dtds+\frac{\alpha R^2}{4}\int_{\hat{\Omega}}(t+1)\phi_i(t)\phi_m(t)\phi_j(t)\phi_n(s)\,dtds\Big)u_{ij}$$
$$=\frac{R^2}{4}\int_{\hat{\Omega}}(t+1)\hat{f}\phi_m(t)\phi_n(s)\,dtds.$$

Plugging the expressions of (3.1) into (2.4), we can obtain the following linear system:

$$AU=F. \tag{3.2}$$

where

$$A=A_1+A_2+A_3,\quad F=\frac{R^2}{4}\int_{\hat{\Omega}}(t+1)\hat{f}\ \phi_m(t)\phi_n(s)dtds,$$

$$U=[u_{00},\ u_{01},\ \cdots,\ u_{0N},\ u_{10},\ u_{11},\ \cdots,\ u_{1N},\ \cdots,\ u_{NN}],$$

$$A_1=\{a_{ij}\}_{i,j=0}^{N}=\sum_{i=0}^{N}\sum_{j=0}^{N}\int_{\hat{\Omega}}(1+t)\phi_i'(t)\phi_m'(s)\phi_j(t)\phi_n(s)dtds,$$

$$A_2=\{b_{ij}\}_{i,j=0}^{N}=\frac{4}{(\theta_2-\theta_1)^2}\int_{\hat{\Omega}}\frac{1}{(t+1)}\phi_i(t)\phi_m(s)\phi_j'(t)\phi_n'(s)dtds,$$

$$A_3=\{c_{ij}\}_{i,j=0}^{N}=\frac{\alpha R^2}{4}\int_{\hat{\Omega}}(t+1)\phi_i(t)\phi_m(t)\phi_j(s)\phi_n(s)dtds.$$

IV. NUMERICAL EXPERIMENT

To substantiate the theoretical analysis and the spectral accuracy of the algorithm, we will present numerous numerical examples in this section. We will conduct programmatic calculations on the MATLAB R2021a platform.

Example 1: We take $\Omega=[0,R]\times[\theta_1,\theta_2]$, $\alpha=1, R=1, \theta_1=0$ and $\theta_2=\frac{\pi}{2}$ as our example. We choose the exact solution $u(x,y)=x^2y^2(x^2+y^2-1)^2$. Obviously, the exact solution satisfies the boundary conditions (2.2). Then, we substitute the exact solution $u(x,y)$ into equation (2.1), and obtain $f(x,y)$. By calculating the approximate solution, we obtain: for different N, the errors between approximate solution and exact solutions are listed in Tables 1. To demonstrate the effectiveness of our algorithm intuitively, we present the images of both the exact and approximate solution in Figures 1. Additionally, we also plot the error plots between the exact and approximate solutions in Figures 2.

It can be observed from Table 1 that when $N>30$, the approximate solution $u(x,y)$ has an accuracy of about 10^{-15} in the L^2 norm and 10^{-13} in the H^1 norm.

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V. Concluding Remarks

This paper proposes and investigates an optimal Galerkin approximation based on the phase domain for elliptic problems under Neumann boundary conditions. Theoretically analyzed the variational and discrete forms of the problem. Finally, the effectiveness of the method was verified through numerical examples.

TABLE 1. Errors between $u(x,y)$ and $u_N(x,y)$ under different norms for various values of N.

N	10	20	30	40	50
$\|u(x,y) - u_N(x,y)\|$	6.5034e-09	1.6949e-14	6.4740e-15	3.9270e-15	2.0586e-15
$\|u(x,y) - u_N(x,y)\|_1$	9.4468e-08	3.4921e-13	1.9851e-13	1.5864e-13	1.0392e-13

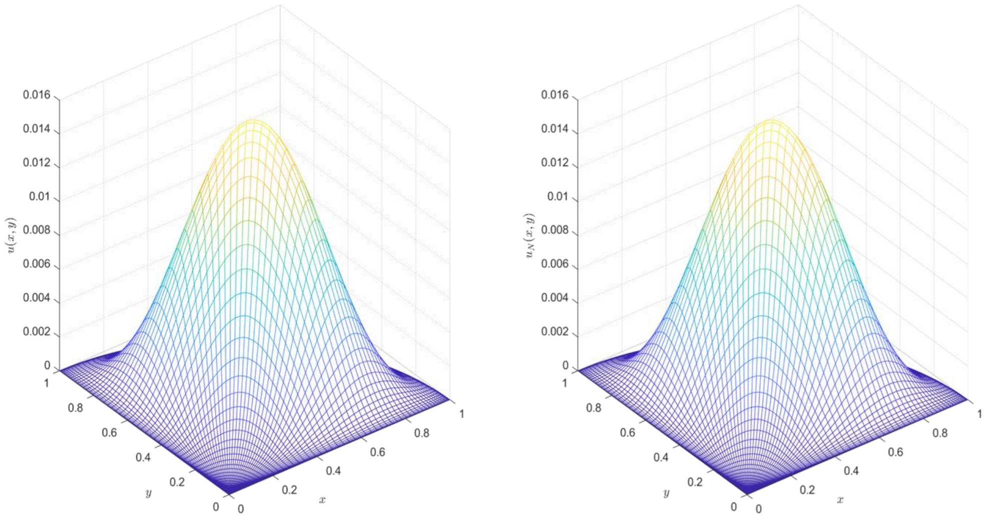

Fig. 1. Images of exact solution $u(x,y)$ (left) and approximate solution $u_N(x,y)$ (right) with N = 30.

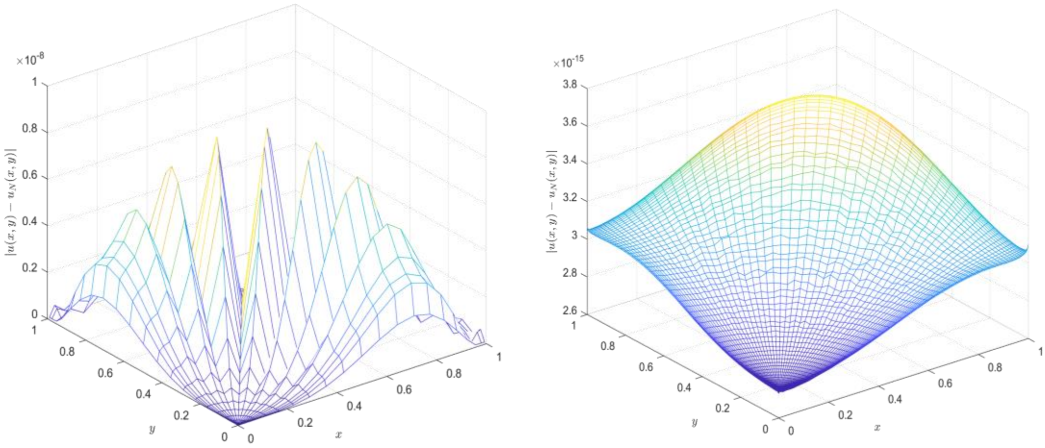

Fig. 2. Error images between $u(x,y)$ and $u_N(x,y)$ with N = 10 (life) and N = 30 (right).

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