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Random Attractors for Stochastic p-Laplacian Lattice System with Multi-delay Driven by Lévy Noise

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***Abstract*—*In this paper, we study the dynamical behavior of stochastic p-Laplacian lattice system with Lévy Noise and Multi-delay. We transform the lattice system with Multi-delay into a lattice system without delay. Then we prove the existence and uniqueness of solutions of system [1.1] by the uniform estimates of solutions. Finally, we choose a Hilbert space as a phase space to investigate the existence and uniqueness of weak pullback mean random attractors. The results of this article is new even the stochastic lattice system driven by single delay or Lévy Noise.*

***Keywords*—*Lévy noise, Multi-delay, p-Laplace , Attractors.*

I. INTRODUCTION

Nowadays, time delays often appears in various control systems, such as transmission phenomena, measurements, see [4,7]. Time-delay systems (TDSs) are the class of dynamic systems. In recent years, the analysis of TDSs have become a research hotspot. Many scholars study the delay lattice system, see [1].

Lévy noise as a non-Gaussian noise with heavy tails and jumps, which makes the application of TDISs noise more widespread [12, 6]. Therefore, it is necessary to study stochastic systems driven by Lévy noise.

Inspired by the above works, we study the dynamical behavior of multi-delay stochastic p-Laplacian porous medium lattice system defined on the integer set $\mathbb{Z}^k$ with Lévy noise:

$$\begin{cases} dv_i(t) + A_p(v(t))_i dt + av_i(t)dt \\ \quad = \epsilon_1 \sum_{k=1}^{\infty} (f^1_{k,i}(v_i(t-\rho_2)) + h^1_{k,i}(t)))dW_k(t) \\ \quad\quad + G_i\big(v_i(t-\rho_1)\big)dt + b_i(t)dt \\ + \epsilon_2 \sum_{k=1}^{\infty} \int_{|y_k|<1} \Big(f^2_{k,i}(v_i(t-\rho_3), y_k) + h^2_{k,i}(t)\Big) \tilde{L}_k(dt, dy_k), \\ \quad v_i(\tau) = v_{0,i}, v_i(s) = \phi_i(s-\tau), s \in (\tau-\rho, \tau), \end{cases} \tag{1.1}$$

where $k, i \in \mathbb{N}$; $t > \tau, \tau \in \mathbb{R}$; $a > 0$; $\epsilon_1, \epsilon_2 > 0$ are noise parameters; $\rho, \rho_1, \rho_2, \rho_3$ are delay parameters, $\rho = max\{\rho_1, \rho_2, \rho_3\}$; A_p is the discrete d-dimensional p-Laplace operator; $(G_i)_{i\in\mathbb{Z}^k}$, $(f^1_{k,i})_{k\in\mathbb{N},i\in\mathbb{Z}^k}$, $(f^2_{j,i})_{k\in\mathbb{N},i\in\mathbb{Z}^k}$ are three sequences of continuous functions with arbitrary order superlinear growth rate; $b = (b_i)_{i\in\mathbb{Z}^k}$, $h^1 = (h^1_{k,i}(t))_{k\in\mathbb{N},i\in\mathbb{Z}^k}$ and $h^2 = (h^2_{k,i}(t))_{k\in\mathbb{N},i\in\mathbb{Z}^k}$ are two ℓ^2 -valued stochastic processes($\| h^1 \|^2 = \sum_{j\in\mathbb{N}} \| h^1_j(t) \|^2 < \infty$ and $\| h^2 \|^2 = \sum_{j\in\mathbb{N}} \| h^2_j(t) \|^2 < \infty$). $(W_k)_{k\in\mathbb{N}}$ is a sequence of independent two-sided real-valued Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$; L_k is a Poisson random measure, ν_k is a Lévy measure with $\int_{\mathbb{R}-0} (|y|^2 \wedge 1)\nu_k(dy) < \infty$ and $\tilde{L}_k(dt, dy) = L_k(dt, dy) - \nu_k(dy)dt$. In this paper, we will prove the existence and uniqueness of solution of lattice system (1.1) with Multi-delay and Lévy noise, which is a meaningful and challenging work.

Moreover, the main purpose of this article is to study the existence and uniqueness of weak pullback mean random attractors of (1.1) . Due to the noise coefficient of the system is nonlinear and the Lévy noise is discontinuous, we currently do not have a method to convert the stochastic lattice system into a pathwise deterministic system.

And we cannot study the pathwise random attractors introduced by Crauel and Flandoli in [2]. Therefore, we study mean attractors proposed by Wang in [8]. We can apply the theory of weak mean random attractor in the reflexive Banach space. In order to solve this problem, we choose a product Hilbert space $L^2(\Omega, \mathcal{F}_\tau; \ell^2) \times L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2))$ as a phase space to investigate the existence and uniqueness of weak pullback mean random attractors.

Next, we present the main results of this article.

Theorem 1.1. Suppose (2.4), (2.5)-(2.6), (2.7)-(2.8) hold. Then for any$v_0 \in L^2(\Omega, \mathcal{F}_\tau; \ell^2)$and $\phi \in L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2))$, system (2.12) has a unique solution v in the meanning of Definition 3.1. For any $T > 0$, we get

$$\mathbb{E}[\| v \|^2_{C([\tau,\tau+T],\ell^2)}] \le K(\mathbb{E}[\| v_0 \|^2] + \int_{-\rho}^{0} \mathbb{E}[\| \phi(s) \|^2]ds + T + \int_{\tau}^{\tau+T} \mathbb{E}[\| I(s) \|^2]ds)e^{KT}, \tag{1.2}$$

where $K > 0$ is a constant independent of v_0, ϕ, ρ, t and T.

Theorem 1.2. Suppose (2.4), (2.5)-(2.6), (2.7)-(2.8), (4.3) hold. Then the mean random dynamical system Φ related to (2.12) has a unique weak $\mathcal{D}$-pullback mean random attractor $A = A(\tau): \tau \in \mathbb{R}\} \in \mathcal{D}$ in $L^2(\Omega, \mathcal{F}; \ell^2) \times L^2(\Omega, \mathcal{F}; L^2((-\rho, 0), \ell^2))$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$; this is,

(i) $A(\tau)$ is a weakly compact subset of $L^2(\Omega, \mathcal{F}_\tau; \ell^2) \times L^2\Big(\Omega, \mathcal{F}_\tau; L^2\big((-\rho, 0), \ell^2\big)\Big)$ $for\ every\ \tau \in \mathbb{R}$.

(ii) A is a weakly $\mathcal{D}$-pullback attracting set of Φ.

(iii) A is the minimal element of $\mathcal{D}$ with properties (i) and (ii).

II. THEORETICAL PREPARATION

The following some useful assumptions will be frequently applied throughout the entire paper.

In this article, we will use the following inequality many times,

$|x|x|^{r_1-2} - z|z|^{r_1-2} \le C_{r_1}(|x|^{r_1-2} + |z|^{r_1-2})|x - z|$,

$(\forall x, z \in \mathbb{R}, r_1 > 2)$. (2.1)

Suppose $\sigma \geq 0$, $r_2, r_3 \geq 0$ are constants, if $r_2 > r_3$, there exists a constant $\kappa > 0$ such that

$$\kappa - r_2 + r_3 e^{\kappa\sigma} < 0. \quad (2.2)$$

Let $\ell^r := \{u = (u_i)_{i\in\mathbb{Z}^k} : \sum_{i\in\mathbb{Z}^k} |u_i|^r < +\infty\}$ for $r \geq 1$, and the norm of ℓ^r is denoted by $\|\cdot\|_r$.

The norm and inner product of ℓ^2 are defined as $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

We define two operators B_d, $B_d^*: \ell^2 \to \ell^2$ by $(B_d v)_i = v_{i^{d+1}} - v_i$, $(B_d^* v)_i = v_{i^{d-1}} - v_i$, for $d \in [1,k] \cap \mathbb{N}$ and $v = (v_i)_{i\in\mathbb{Z}^k} \in \ell^2$.

Then the discrete d-dimensional p-Laplace operator $A_p v: \ell^2 \to \ell^2$ with $p \geq 2$ is defined by

$$(A_p v)_i = \sum_{d=1}^{k} (B_d^*(|B_d v|^{p-2} B_d v))_i = -\sum_{d=1}^{k} (|(B_d^* v)_i|^{p-2} \times (B_d^* v)_i + |(B_d v)_i|^{p-2} \times (B_d v)_i). \quad (2.3)$$

Next, we will make more assumptions: $b = (b_i)_{i\in\mathbb{Z}^k}$, $h_k^1 = (h_{k,i}^1)_{i\in\mathbb{Z}^k}$ and $h_k^2 = (h_{k,i}^2)_{i\in\mathbb{Z}^k}$ are ℓ^2-valued progressively measurable processes such that

$$\int_\tau^{\tau+T} \mathbb{E}[\| b(t) \|^2 + \sum_{k=1}^{\infty} \| h_k^1(t) \|^2 + \sum_{k=1}^{\infty} \| h_k^2(t) \|^2] dt = \int_\tau^{\tau+T} \mathbb{E}[\| I(t) \|^2] < \infty, \forall \tau \in \mathbb{R}, T > 0. \quad (2.4)$$

$G_i: \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous in $i \in \mathbb{Z}^k$; that is, for every bounded subset B of $\mathbb{R}$, there exist a constant $c_1 = c_1(B) > 0$ such that for any $z_1, z_2 \in \mathbb{R}$,

$$| G_i(z_1) - G_i(z_2) | \leq c_1 | z_1 - z_2 |. \quad (2.5)$$

For every $n \in \mathbb{N}$, there exist positive constants $\beta_i((\beta_i)_{i\in\mathbb{Z}^k} \in \ell^2)$ and θ_1 such that for any $z \in \mathbb{R}$,

$$| G_i(z) | \leq \beta_i + \theta_1 | z |. \quad (2.6)$$

For any $k \in \mathbb{N}$, $i \in \mathbb{Z}^k$, there exists $c_{k,i}(n) > 0$ such that $z_1, z_2 \in \mathbb{R}$ and $|z_1| \vee |z_2| \leq n$,

$$|f_{k,i}^1(z_1) - f_{k,i}^1(z_2)| \vee \int_{|y_k|<1} |f_{k,i}^2(z_1, y_k) - f_{k,i}^2(z_2, y_k)| \nu_k(dy_k) \leq c_{k,i}(n)|z_1 - z_2|^2, \quad (2.7)$$

where $\| c(n) \| = \sum_{k\in\mathbb{N}} \sum_{i\in\mathbb{Z}^k} |c_{k,i}(n)| < \infty$.

For any $k \in \mathbb{N}$ and $i \in \mathbb{Z}^k$, there exist $\delta_{k,i}, \theta_{k,i} > 0$ such that

$$|f_{k,i}^1(z)|^2 \vee \int_{|y_k|<1} |f_{k,i}^2(z, y_k)|^2 \nu_k(dy_k) < \delta_{k,i} + \theta_{k,i}|z|^2, \quad (2.8)$$

where $\| \delta \| = \sum_{k\in\mathbb{N}} \sum_{i\in\mathbb{Z}^k} |\delta_{k,i}| < \infty$, and $|\theta_{k,i}| = \sum_{k\in\mathbb{N}} \sum_{i\in\mathbb{Z}^k} |\theta_{k,i}| < \infty$.

We write $v = (v_i)_{i\in\mathbb{Z}^k}$, $G(v) = (G_i(v_i))_{i\in\mathbb{Z}^k}$, $f_k^1(v) = (f_{k,i}^1(v_i))_{i\in\mathbb{Z}^k}$, $f_k^2(v, y_k) = (f_{k,i}^2(v, y_k))_{i\in\mathbb{Z}^k}$, for $k \in \mathbb{N}$. By (2.4)-(2.5), we find that $G: \ell^2 \to \ell^2$ is locally Lipschitz continuous; that is, for every $n > 0$, there exists a constant $c_2 = c_2(n) > 0$ such that for all $u, v \in \ell^2$ with $\| u \| \leq n$ and $\| v \| \leq n$,

$$\| G(u) - G(v) \|^2 \leq c_2 \| u - v \|^2$$
$$\text{and } \| G(v) \|^2 \leq 2 \| \beta \|^2 + 2\theta_1^2 \| v \|^2. \quad (2.9)$$

By (2.7)-(2.8), we find that there exists $c_3 = c_3(n) > 0$ such that for all $u, v \in \ell^2$ with $\| u \| \leq n$ and $\| v \| \leq n$,

$$\sum_{k\in\mathbb{N}} \| f_k^1(u) - f_k^2(v) \|^2 \vee \sum_{k\in\mathbb{N}} \int_{|y_k|<1} |f_k^2(u, y_k) - f_k^2(v, y_k)| \vartheta_k(dy_k) \leq c_3 \| u - v \|^2,$$

$$\sum_{k\in\mathbb{N}} \| f_k^1(v) \|^2 \vee \sum_{k\in\mathbb{N}} \int_{|y_k|<1} \| f_k^2(v, y_k) \|^2 \nu_k(dy_k) < 2 \| \delta \|^2 + 2 \| \theta \|^2 \| v \|^2, \forall v \in \ell^2. \quad (2.10)$$

By [11], we know that $A_p: \ell^2 \to \ell^2$ is locally Lipschitz continuous; that is, for any $p \geq 2, \gamma > 1$, $u, v \in \ell^2$, $\| u \| \leq n$ and $\| v \| \leq n$, for every $n \in \mathbb{N}$, there exists $c_4 = c_4(n) > 0$ such that

$$\| A_p(u) - A_p(v) \|^2 \leq c_1(n) \| u - v \|^2$$
$$\text{and } \langle A_p(u) - A_p(v), u - v \rangle \geq 0. \quad (2.11)$$

Based on the above notation, the system (1.1) can be written the following system in ℓ^2 as

$$dv(t) + A_p(v(t))dt + av(t)dt = \epsilon_1 \sum_{k=1}^{\infty} \left(f_k^1(v(t - \rho_2)) + h_k^1(t)\right) dW_k(t) + G(v(t - \rho_1))dt + b(t)dt + \epsilon_2 \sum_{k=1}^{\infty} \int_{|y_k|<1} (f_k^2(v(t - \rho_3), y_k) + h_k^2(t))\tilde{L}_k(dt, dy_k),$$
$$v(\tau) = v_0, v(s) = \phi(s - \tau), s \in (\tau - \rho, \tau), \quad (2.12)$$

where $v_0 = (v_{0,i})_{i\in\mathbb{Z}^k}$ and $\phi = (\phi_i)_{i\in\mathbb{Z}^k}$.

III. Existence and Uniqueness of Solution: Theorem 1.1

Definition 3.1. Suppose that $v_0 \in L^2(\Omega, \mathcal{F}_\tau; \ell^2)$ and $\phi \in L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2))$. An ℓ^2-valued stochastic process $v(t), t > \tau - \rho$, is called a solution of system (2.12) if

(i) $v \in L^2(\Omega, \mathcal{F}_\tau; L^2((\tau - \rho, \tau), \ell^2))$ and $v_\tau = \phi$.

(ii) v is pathwise continuous on $[\tau, \infty)$, $\mathcal{F}_\tau$-adapted for all $t \geq \tau, v(\tau) = v_0$, and $v \in L^2(\Omega, C([\tau, \tau + T], \ell^2))$ *for all* $T > 0$.

(iii) For all $t \geq \tau$, the system (2.12) over (τ, t) holds, $\mathbb{P}$-a.s.

The proof of Theorem 1.1

Proof Now, we prove the existence of the solutions to (2.12) on $[\tau, \tau + \rho]$. By (2.9), for $\phi \in L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2))$, we get

$$\mathbb{E}[\int_\tau^{\tau+\rho} \| G(v(t - \rho_1)) \|^2 dt] \leq 2\rho \| \beta \|^2 + 2\theta_1^2 \mathbb{E}[\int_{-\rho_1}^{\rho-\rho_1} \| \phi(t) \|^2] dt < \infty. \quad (3.1)$$

By (2.10) and BDG's inequality, by [5, Lemma 8.22], for $t \in (\tau, \tau + \rho]$, we get

$$2\epsilon_1 \mathbb{E}[\sup_{\tau\leq r\leq t} |\sum_{k=1}^{\infty} \int_\tau^r \langle f_k^1(v(s - \rho_2)) + h_k^1(s), v(s) \rangle dW_k(s)|] + \mathbb{E}[\sup_{\tau\leq r\leq t} |\sum_{k=1}^{\infty} \int_\tau^r \int_{|y_k|<1} [\| v(s) + \epsilon_2(f_k^2(v(s - \rho_3), y_k) + h_k^2(s) \|^2 - \| v(s) \|^2] \tilde{L}_k(ds, dy_k)|] \leq \frac{1}{2}\mathbb{E}[\sup_{\tau\leq s\leq t} \| v(s) \|^2] + 16\epsilon_1^2 C_1^2 \| \theta \|^2 \mathbb{E}[\int_{-\rho_2}^{\rho-\rho_2} \| \phi(s) \|^2] ds + 4(4\epsilon_2^2 C_3^2 + \epsilon_2^2 C_2) \| \theta \|^2 \int_{-\rho_3}^{\rho-\rho_3} \mathbb{E}[\| \phi(s) \|^2] ds + C_4 \| \delta \|^2 \rho + C_5 \mathbb{E}[\int_\tau^t \sum_{k=1}^{\infty} (\| h_k^1(s) \|^2 + \|$$

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$h_k^2(s)\ \|^2)]ds,$ (3.2)

where C_1, C_2 and C_3 is a positive number.

$C\_4 = (16\epsilon_1^2 C_1^2 + 16\epsilon_2^2 C_3^2 + 4\epsilon_2^2 C_2, C_5 = (8\epsilon_1^2 C_1^2 + 8\epsilon_2^2 C_3^2 + 2\epsilon_2^2 C_2).$

By (3.1)-(3.2) system (2.12) on $[\tau, \tau+\rho]$ can be transformed into the following system without delay:

$$\begin{cases} dv(t) + A_p(v(t))dt + av(t)dt \\ = \epsilon_1 \sum_{k=1}^{\infty} (f_k^1(v(t-\rho_2-\tau)) + h_k^1(t)))dW_k(t) \\ +G(v(t-\rho_1-\tau))dt + b(t)dt \\ +\epsilon_2 \sum_{k=1}^{\infty} \int_{|y_k|<1} (f_k^2(v(t-\rho_3-\tau), y_k) \\ +h_k^2(t))\tilde{L}_k(dt, dy_k), \\ v(\tau) = v_0. \end{cases} \quad (3.3)$$

Then by [1, Theorem 3.1], by conditions (2.4), (2.5)-(2.6), (2.7)-(2.8), system (3.4) has a unique solution v defined on $[\tau, \tau+\rho]$ such that $\in L^2(\Omega, C([\tau, \tau+\rho], \ell^2))$.

Review this discussion, we can extend the solution v to the interval $[\tau, \infty)$ such that $v \in L^2(\Omega, C([\tau, \tau+T], \ell^2))$ for any $T > 0$ .Next, we start the uniform estimates of solutions. Applying Ito's formula to (2.12), we get

$$d\ \|\ v(t)\ \|^2 + 2\langle A_p(v(t)), v(t)\rangle dt + 2a\ \|\ v(t)\ \|^2\ dt$$
$$= 2\langle G(v(t-\rho_1)), v(t)\rangle dt + 2\langle b(t), v(t)\rangle dt$$
$$+\epsilon_2^2 \sum_{k=1}^{\infty} \int_{|y_k|<1} \|\ f_k^2(v(t-\rho_3), y_k)$$
$$+h_k^2(t)\ \|^2\ \nu_k(dy_k)dt + \epsilon_1^2 \sum_{k=1}^{\infty} \|\ f_k^1(v(t-\rho_2))$$
$$+h_k^1(t))\ \|^2\ dt + 2\epsilon_1 \sum_{k=1}^{\infty} \langle v(t), f_k^1(v(t-\rho_2)) + h_k^1(t)\rangle dW_k(t) + \sum_{k=1}^{\infty} \int_{|y_k|<1} [\|\ v(t) + \epsilon_2(f_k^2(v(t-\rho_3), y_k) + h_k^2(t)\ \|^2 - \|\ v(t)\ \|^2]\tilde{L}_k(dt, dy_k). \quad (3.4)$$

By (2.9), for all $t \in [\tau, \tau+T]$, we have

$$2\int_\tau^t \langle G(v(s-\rho_1)), v(s)\rangle ds \le (2\theta_1^2 + 1)\int_\tau^t \|\ v(s)\ \|^2\ ds + 2\ \|\ \beta\ \|^2\ (t-\tau) + 2\theta_1^2 \int_{-\rho}^0 \|\ \phi(s)\ \|^2\ ds. \quad (3.5)$$

By (3.4), (2.10)-(2.11), similar to (1.2)-(3.2) and (3.5), we get

$$\mathbb{E}[\sup_{\tau\le r\le t} \|\ v(r)\ \|^2] \le \mathbb{E}[\|\ v_0\ \|^2] + \frac{1}{2}\mathbb{E}[\sup_{\tau\le s\le t} \|\ v(s)\ \|^2] + C_6(t-\tau) + C_7\mathbb{E}[\int_{-\rho}^0 \|\ \phi(s)\ \|^2]ds + C_8\mathbb{E}[\int_\tau^t \|\ I(s)\ \|^2]ds + C_9\mathbb{E}[\int_\tau^t \|\ v(s)\ \|^2\ ds], \quad (3.6)$$

where $C_6 = (2\ \|\ \beta\ \|^2 + 4(4\epsilon_1^2 C_1^2 + 4\epsilon_2^2 C_3^2 + \epsilon_2^2 C_2 + \epsilon_1^2 + \epsilon_2^2)\ \|\ \delta\ \|^2), C_7 = (2\theta_1^2 + 4(4\epsilon_1^2 C_1^2 + 4\epsilon_2^2 C_3^2 + \epsilon_2^2 C_2 + \epsilon_1^2 + \epsilon_2^2)\ \|\ \theta\ \|^2), C_8 = (2(4\epsilon_1^2 C_1^2 + 4\epsilon_2^2 C_3^2 + \epsilon_2^2 C_2 + \epsilon_1^2 + \epsilon_2^2)\ \|\ \theta\ \|^2 + 1), C_9 = (2 + 2\theta_1^2 + 4(4\epsilon_1^2 C_1^2 + 4\epsilon_2^2 C_3^2 + \epsilon_2^2 C_2 + \epsilon_1^2 + \epsilon_2^2)\ \|\ \theta\ \|^2)$. By (3.6) and Gronwall's inequality, for all $t \in [\tau, \tau+T]$ *and* $T > 0$, we complete the proof. In order to prove Theorem (1.2), we first define a mean random dynamical system.

For all $\tau \in \mathbb{R}$ and $t \in \mathbb{R}^+$, let $\Phi(t,\tau)$ be a mapping from $L^2(\Omega, \mathcal{F}_\tau; \ell^2) \times L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2))$ to $L^2(\Omega, \mathcal{F}_{t+\tau}; \ell^2) \times (\Omega, \mathcal{F}_{t+\tau}; L^2((-\rho, 0), \ell^2))$ given by $\Phi(t,\tau)(v_0, \phi) = (v(t+\tau; \tau, v_0, \phi), v_{t+\tau}(\cdot; \tau, v_0, \phi))$,for any $(v_0, \phi) \in L^2(\Omega, \mathcal{F}_\tau; \ell^2) \times L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2))$, where $v(t; \tau, v_0, \phi)$ is the solution of (2.12), and $v_{t+\tau}(\eta; \tau, v_0, \phi) = v(t+\tau+\eta; \tau, v_0, \phi)$ for $\eta \in (-\rho, 0)$.

Then Φ is a mean random dynamical system on $L^2(\Omega, \mathcal{F}; \ell^2) \times L^2(\Omega, \mathcal{F}; L^2((-\rho, 0), \ell^2))$ over the

IV. .EXISTENCE OF WEAK PULLBACK MEAN RANDOM: THEOREM 1.1

Next, we present the existence and uniqueness of weak pullback mean random attractors of (2.12). For convenience, for every $\tau \in \mathbb{R}$, we set

$$H_\tau = L^2(\Omega, \mathcal{F}_\tau; \ell^2) \times L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2)).$$

And H_τ is a product Hilbert space with norm

$\|\ (v_0, \phi)\ \|_{H_\tau} = (\mathbb{E}[\|\ v_0\ \|_{\ell^2}] + \int_{-\rho}^0 \mathbb{E}[\|\ \phi(s)\ \|_{\ell^2}]ds)^{\frac{1}{2}}$, for all $(u_0, \vartheta) \in H_\tau$ hold.

Moreover, we assume ϵ_1, ϵ_2 in (1.1) are small enough satisfy $\|\ \theta\ \|^2\ (\epsilon_1^2 + \epsilon_2^2) < a - \sqrt{2}\theta_1$, then by (2.2) we find that there exist posive numbers ϖ, λ such that

$$\sqrt{2}\theta_1(1 + e^{\varpi\rho}) + 4\ \|\ \theta\ \|^2\ (\epsilon_1^2 + \epsilon_2^2)e^{\varpi\rho} + \lambda + \varpi - 2a < 0. \quad (4.1)$$

Let $U = \{U(\tau) \subseteq H_\tau : \tau \in \mathbb{R}\}$ be a family of nonempty bounded sets such that

$$\lim_{\tau\to-\infty} e^{\mu\tau}\ \|\ U(\tau)\ \|_{H_\tau}^2 = 0, \text{and}\ \|\ U(\tau)\ _{H_\tau} = \sup_{(v_0,\phi)\in H_\tau} \|\ (v_0, \phi)\ \|_{H_\tau}. \quad (4.2)$$

Denote by $\mathcal{D} = \{U = \{U(\tau) \subseteq H_\tau : \tau \in \mathbb{R}\} : U \text{ satisfies } (4.2)\}$.

We will prove that the system (2.12) has a unique weak $\mathcal{D}$-pullback mean random attractor. Therefore, for $\varpi > 0$, which is the constant as in (4.1). we further suppose that for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^\tau e^{\varpi s}\mathbb{E}[\|\ b(s)\ \|^2 + \sum_{k=1}^{\infty} (\|\ h_k^1(s)\ \|^2 + \|\ h_k^2(s)\ \|^2)]ds = \int_{-\infty}^\tau e^{\varpi s}\mathbb{E}[\|\ I(s)\ \|^2]ds < \infty. \quad (4.3)$$

Lemma 4.1 Suppose (2.4), (2.5)-(2.6), (2.7)-(2.8), (4.3) hold. Then for any $\tau \in \mathbb{R}$ and $U = \{U(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$,there exist $\tau \in \mathbb{R}, T = T(\tau, U) > \rho$ such that for all $t \ge T$,

$$\mathbb{E}[\|\ v(\tau; \tau-t, v_0, \phi)\ \|^2] + \int_{-\rho}^0 \mathbb{E}[\|\ v_\tau(s; \tau-t, v_0, \phi)\ \|^2]ds \le K_1(\int_{-\infty}^\tau e^{\varpi(s-\tau)}\mathbb{E}[\sum_{k=1}^{\infty} \|\ I(s)\ \|^2]ds, \quad (4.4)$$

where K_1 is a positive constant depending on $\theta_1, \varpi, \beta, \delta, \lambda, \epsilon_1, \epsilon_2$, but independent of τ, ρ *and* U.

Proof By (3.4), for any $t > 0$ and $r \in (\tau-t, \tau]$, we have

$$e^{\varpi r}\mathbb{E}[\|\ v(r)\ \|^2] = e^{\varpi(\tau-t)}\mathbb{E}[\|\ v_0\ \|^2] - 2\int_{\tau-t}^r e^{\varpi s}\mathbb{E}[\langle A_p(v(s)), v(s)\rangle]ds + (\varpi - 2a)\int_{\tau-t}^r e^{\varpi s}\mathbb{E}[\|\ v(s)\ \|^2]ds + 2\int_{\tau-t}^r e^{\varpi s}\mathbb{E}[\langle G(v(s-\rho_1)), v(s)\rangle]ds + 2\int_{\tau-t}^r e^{\varpi s}\mathbb{E}[\langle b(s), v(s)\rangle]ds + \epsilon_1^2 \sum_{k=1}^{\infty} \int_{\tau-t}^r e^{\varpi s}\mathbb{E}[\|\ f_k^1(v(s-\rho_2)) + h_k^1(s)\ \|^2]ds + \epsilon_2^2 \sum_{k=1}^{\infty} \int_{\tau-t}^r \int_{|y_k|<1} e^{\varpi s}\mathbb{E}[\|\ f_k^2(v(s-\rho_3), y_k) + h_k^2(s)\ \|^2]\nu_k(dy_k)ds = \sum_{i=1}^{7} I_i\ . \quad (4.5)$$

For the fourth term on the right-hand side of (4.5), by (2.9), we obtain

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$$I_4 \le \lambda \int_{\tau-t}^{r} e^{\varpi s}\mathbb{E}[\| v(s) \|^2]ds + \frac{1}{\lambda}\int_{\tau-t}^{r} e^{\varpi s} \mathbb{E}[\| b(s) \|^2]ds. \quad (4.6)$$

For the sixth and seventh terms on the right-hand side of (4.5), by (2.10), we have

$I_6 + I_7 \le 2(\epsilon_1^2 + \epsilon_2^2)\sum_{k=1}^{\infty} \int_{\tau-t}^{r} e^{\varpi s}\mathbb{E}[\| h_k^1(s) \|^2 + \| h_k^2(s) \|^2]ds + \frac{4(\epsilon_1^2+\epsilon_2^2)\|\delta\|^2}{\varpi}\left(e^{\varpi r} - e^{\varpi(\tau-r)}\right) + 4 \| \theta \|^2 (\epsilon_1^2 + \epsilon_2^2)e^{\varpi\rho} \int_{\tau-t}^{r} e^{\varpi s}\mathbb{E}[\| v(s) \|^2]ds + 4 \| \theta \|^2 (\epsilon_1^2 + \epsilon_2^2)e^{\varpi\rho}e^{\varpi(\tau-t)} \int_{-\rho}^{0} e^{\varpi s}\mathbb{E}[\| \phi(s) \|^2]ds.$(4.7)

By (4.5)-(4.7), (4.1), we get for all $r \in (\tau - t, \tau]$

$e^{\varpi r}\mathbb{E}[\| v(r) \|^2] \le e^{\varpi(\tau-t)}\mathbb{E}[\| v_0 \|^2] + C_8\left(e^{\varpi r} - e^{\varpi(\tau-r)}\right) + \left(2(\epsilon_1^2 + \epsilon_2^2) + \frac{1}{\lambda}\right)\sum_{k=1}^{\infty} \int_{\tau-t}^{r} e^{\varpi s}\mathbb{E}[\| I(s) \|^2]ds + \left(4 \| \theta \|^2 (\epsilon_1^2 + \epsilon_2^2) + \sqrt{2}\theta_1\right)e^{\varpi\rho}e^{\varpi(\tau-t)} \int_{-\rho}^{0} e^{\varpi s}\mathbb{E}[\| \phi(s) \|^2]ds,$ (4.8)

where $C_8 = \frac{\sqrt{2}\|\beta\|^2}{\varpi\theta_1} + \frac{4(\epsilon_1^2+\epsilon_2^2)\|\delta\|^2}{\varpi}$.

By (4.8), similar to [1,Lemma 3.1], for all $r \in (\tau - t, \tau], t \ge \rho$, we get

$\mathbb{E}[\| v(\tau;\tau - t, v_0, \phi) \|^2] + \int_{\tau-\rho}^{\tau} \mathbb{E}[\| v(s;\tau - t, v_0, \phi) \|^2]ds \le (1 + \rho e^{\mu\rho})(e^{-\varpi t}\mathbb{E}[\| v_0 \|^2] + (4 \| \theta \|^2 (\epsilon_1^2 + \epsilon_2^2) + \sqrt{2}\theta_1)e^{\varpi(\rho-t)} \int_{-\rho}^{0} \mathbb{E}[\| \phi(s) \|^2]ds + C_8 + (2(\epsilon_1^2 + \epsilon_2^2) + \frac{1}{\lambda})\sum_{k=1}^{\infty} \int_{\tau-t}^{r} e^{\varpi(s-r)}\mathbb{E}[I(s)]ds) = (1 + \rho e^{\mu\rho})\sum_{i=8}^{11} I_i.$ (4.9)

For the first and second terms on the right-hand side of (4.9), $(v_0, \phi) \in U(\tau - t)$, we find

$$I_8 + I_9 \le \left(e^{-\varpi\tau} + \left(4 \| \theta \|^2 (\epsilon_1^2 + \epsilon_2^2) + \sqrt{2}\theta_1\right)e^{\varpi(\rho-\tau)}\right)e^{\varpi(\tau-t)} \| U(\tau - t) \|^2 \to 0, as t \to \infty.$$

The proof of Theorem 1.2

Proof For every $\tau \in \mathbb{R}$, we define $Q(\tau) = \{(v, \phi) \in H_\tau: \| (v, \phi) \|_{H_\tau}^2 \le Q(\tau)\}$, and $Q(\tau) = K_1(\int_{-\infty}^{\tau} e^{\varpi(s-\tau)} \mathbb{E}[\sum_{k=1}^{\infty} \| I(s) \|^2 ds)$, where $K_1 > 0$ is the same number as in (4.4).

Because $Q(\tau)$ is a bounded closed convex subset of H_τ, it is weakly compact in H_τ. Then by (4.3), Lemma 4.1 and [9, Theorem 2.13], similar to the poof of [1 Theorem 3.1], we can complete the proof.

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