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hp Local Discontinuous Galerkin Finite Element Method Based on Steklov Eigenvalue Problem

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***Abstract*—*The flexibility and applicability of the finite element method can be used to solve the problem of the solution of elements with different shapes and properties. When there are very complex factors, such as uneven material properties, arbitrary boundary conditions, complex geometric shapes, etc., the finite element method can be flexibly processed and solved. The local discontinuous Galerkin method is used to solve the Steklov eigenvalue problem, in which the discontinuous Galerkin method can effectively solve the eigenvalue error problem. We propose a complete error estimation, and the hp prior error estimation can analyze the two-dimensional unstructured mesh with suspended nodes. The resulting mesh size h is optimal, and the degree p of the polynomial is suboptimal.*

***Keywords*—** *Steklov eigenvalues, local discontinuous Galerkin method, hp prior error analysis.*

I. INTRODUCTION

The Steklov eigenvalue problem has a wide range of applications in physics and engineering. The dynamics of isotropic elastic media is combined with some general conclusions to solve the free motion of particles or the constrained motion of particles [1]; the problem of determining the lower bound of the lowest frequency of the vibration of a rigid metal pendulum composed of a pendulum suspended on a steel wire is studied by means of integral equation method and composition method [2]; the lateral motion of an elastic string with mass at one end and the model of the transmission line tilting towards the circuit are investigated [3]; an approximate finite element analysis of structural vibration modes of coplanar incompressible fluids and a numerical solution of spectral problems in fluid-solid interaction are analyzed [4]; the eigenvalue problem of tangent plane on collinear fault system is considered under the sliding friction law [5]. The approximation solution of the boundary element of the Steklov eigenvalue problem of Laplace operator is obtained by reducing the eigenvalue problem [6]; the incongruous finite element approximation of Steklov eigenvalue problem over two dimensional concave Angle domain is investigated [7]. Use extrapolation or split extrapolation to improve the accuracy of the approximate solution of Steklov eigenvalue problem [8]. The discrete eigenvalues and eigenvectors are quickly calculated by constructing a set of appropriate basis functions of Legendre polynomials [9]. Fast Fourier-Galerkin method is used to solve Steklov eigenvalue problem [10]. A finite element method for effective 4-order Steklov eigenvalue problems over spherical regions is obtained by dimensionality reduction [11]. By using the local discontinuous Galerkin method, hp analysis of the convection diffusion equation is carried out to obtain the conclusion that the diameter of the partition element is optimal and the polynomial degree is suboptimal [12]. The best convergence order is obtained by using the discontinuous Galerkin method for a prior and a posterior estimation of Steklov eigenvalue problems [13].

II. THEORETICAL PREPARATION

$\Omega \subset R^2$ be a bounded domain with Lipshitz boundary $\partial\Omega$. and let **n** be the outward normal to $\partial\Omega$, consider the Steklov eigenvalue problem: Find $\lambda \in R$ and a nontrivial function $u \in H^1(\Omega)$, such that

$$\begin{cases} -\Delta u + u = 0, & in\ \Omega \\ \frac{\partial u}{\partial n} = \lambda u, & on\ \partial\Omega, \end{cases} \tag{2.1}$$

If $\mathbf{q} = \nabla u$, take the Green integral transformation of (2.1) to obtain the corresponding weak form, and define a continuous bilinear form $a(u, v)$, such that.

$$a(u,v) = (\nabla u, \nabla v) + (u,v), \forall u, v \in H^1(\Omega)\ ,$$

where $(u,v) = \int_\Omega uvdx$, under these assumptions, there exist two positive constants unrelated to u , v two positive constants independent of B and C , such that the bilinear form $a(\cdot,\cdot)$ satisfies

$$\begin{aligned} |a(u,v)| \leqslant B\|u\|_{1,\Omega}\|v\|_{1,\Omega}, \qquad \forall u, v \in H^1(\Omega) \\ |a(v,v)| \geqslant C\|v\|_{1,\Omega}^2, \qquad \forall v \in H^1(\Omega) \end{aligned} \tag{2.2}$$

The weak form of (2.1) is: Find $(\lambda, u) \in R \times H^1(\Omega), u \neq 0$, such that the following equation is true

$$a(u,v) = \lambda < u, v >, \forall v \in H^1(\Omega). \tag{2.3}$$

where $< u, v >= \int_{\partial\Omega} uvds$.

Let $\mathcal{T}_h = \{\kappa\}$ be a shape-regular mesh of Ω. The diameter of a face e (an edge when d=2) is denoted by h_e, the diameter of a cell $\kappa \in \mathcal{T}_h$ is denoted by h_κ, The set of faces of cells $\Gamma_h = \Gamma_h^i \cup \Gamma_h^b$ where Γ_h^i denotes the interior faces set and Γ_h^b denotes the set of faces lying on the boundary $\partial\Omega$.

$p_\kappa \geqslant 1$ indicates the highest degree of polynomial in unit $\kappa \in \mathcal{T}_h$, where $\underline{p} = \{p_\kappa\}_{\kappa\in\mathcal{T}_h}$, the hp finite element space is defined as

$$S^{\underline{p}}(\mathcal{T}_h) = \{u \in L^2(\Omega): u|_\kappa \in S^{p_\kappa}(\kappa), \forall\kappa \in \mathcal{T}_h\}$$

when the unit κ is a triangle, $S^{p_\kappa}(\kappa)$ is the polynomial space $p^{p_K}(\kappa)$ over κ. Introduce the space of piecewise H^s functions space of degree s:

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega): v|_\kappa \in H^s(\kappa), \forall\kappa \in \mathcal{T}_h\}.$$

The auxiliary variable q = ∇u is introduced, then (2.1) can be rewritten as follows.

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$$\begin{cases}-\nabla\cdot\mathbf{q}+u=0, & in\ \Omega\\ \frac{\partial u}{\partial n}=\lambda u, & on\ \partial\Omega,\end{cases} \quad (2.4)$$

$V_h=S^{\underline{p}}(\mathcal{T}_h)$ and $Q_h=S^{\underline{p}}(\mathcal{T}_h)^2$ represent the hp local discontinuous finite element space, then the hp-ldg format of the approximation problem of (2.5), find $(\lambda_h,u_h)\in C\times S^{\underline{p}}(\mathcal{T}_h)$, $u_h\neq 0$, for all $\kappa\in\mathcal{T}_h$, $\forall v\in V_h$, $\mathbf{t}\in\mathbf{Q}_h$,such that

$$\int_\kappa \mathbf{q}_h\cdot\nabla v dx-\int_{\partial\kappa}\hat{\mathbf{q}}_h\cdot\mathbf{n}_\kappa v ds+\int_\kappa u_h v dx=0, \quad (2.5)$$

$$\int_\kappa \mathbf{q}_h\cdot\mathbf{t}dx-\int_{\partial\kappa}\hat{\mathrm{u}}_h\cdot\mathbf{n}_\kappa v ds+\int_\kappa u_h\nabla\cdot\mathbf{t}dx=0, \quad (2.6)$$

where $v\in V_h$, the n_κ is the unit outward normal vector of $\partial\kappa$, $\hat{\mathbf{u}}$ and $\hat{\mathbf{q}}$ are the numerical fluxes of $\mathbf{u}$ and $\mathbf{q}$, $\mathbf{u}$ and $\mathbf{q}$ are approximations of traces on $\partial\Omega$, defines the mean and jump of v over e :

$$\{\{v\}\}=\frac{1}{2}(v^++v^-), \qquad \{\{\mathbf{r}\}\}=\frac{1}{2}(\mathbf{r}^++\mathbf{r}^-).$$

$$[\![v]\!]=\frac{1}{2}(v^+\mathbf{n}_\kappa^++v^-\mathbf{n}_\kappa^-), \qquad [\![\mathbf{r}]\!]=\mathbf{r}^+\mathbf{n}_\kappa^++\mathbf{r}^-\mathbf{n}_\kappa^-.$$

Where e is a surface consisting of two neighboring faces of κ^+ and κ^- common interior faces.v and $\mathbf{r}$ are smooth functions on $\kappa^\pm$ and $v^\pm$ and $\mathbf{r}^\pm$ are traces on the boundaries of $\partial\kappa^\pm$, defines the mean and jump of v and $\mathbf{r}$ on e, $v_+=v|_{\kappa+}$, $v^-=v|_{\kappa^-}$, where $e=\partial\kappa^+\cap\partial\kappa^-$, the n_κ is the outward normal vector from κ^+ to κ^-, then we have

$$\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}v\mathbf{q}\cdot\mathbf{n}ds=\int_{\mathcal{E}}\{\{\mathbf{q}\}\}[\![v]\!]\,ds+\int_{\mathcal{E}_I}[\![\mathbf{q}]\!]\{\{v\}\}\,ds.$$

the definition of numerical flux of $\hat{\mathbf{u}}$ and $\hat{\mathbf{q}}$ are as follows:

$$\hat{\mathbf{u}}|_e=\begin{cases}\{\{u\}\}+\eta[\![u]\!] & e\subset\Gamma_h^i,\\ u & e\subset\Gamma_h^b,\end{cases}$$

$$\hat{\mathbf{q}}|_e=\begin{cases}\{\{\mathbf{q}\}\}-\alpha[\![u]\!]-\eta[\![\mathbf{q}]\!] & e\subset\Gamma_h^i,\\ \mathbf{q} & e\subset\Gamma_h^b,\end{cases}$$

where the parameters α and η are chosen appropriately, and to define the parameters, a function of h and p are introduced into the relative local unit size and approximation degree in $L^\infty(\mathcal{E})$, where $\eta=h^{-1}p^2$, such that $\|\eta\|_{\infty,\mathcal{E}_J}\leqslant\beta$, where $\alpha>0$ and $\beta>0$ are constants independent of the mesh size, the

$$h=h(x)=\begin{cases}\min\{h_{\kappa^+},h_{\kappa^-}\}, & \kappa\in e_{\kappa^+\cap\kappa^-},\\ h_\kappa & \kappa\in e_{\kappa\cap\Omega},\end{cases}$$

$$p=p(x)=\begin{cases}max\{p_{\kappa^+},p_{\kappa^-}\}, & \kappa\in e_{\kappa^+\cap\kappa^-},\\ p_\kappa, & \kappa\in e_{\kappa\cap\Omega},\end{cases}$$

where $e_{\kappa^+\cap\kappa^-}=\mathrm{int}(\partial\kappa^+\cap\partial\kappa^-)$, $e_{\kappa\cap\Omega}=\mathrm{int}\,(\partial\kappa\cap\partial\Omega)$.

Define the lifting operator $\Phi(u)\in\mathbf{Q}_h$, $\mathbf{t}\in\mathbf{Q}_h$, $u\in V(h)+H^1(\Omega)$, such that

$$\int_\Omega\Phi(u)\mathbf{t}dx=\sum_{\kappa\in\mathcal{T}_h}\int_{\Gamma_h^i}[\![u]\!]\{\{\mathbf{t}\}\}-\eta[\![u]\!][\![\mathbf{t}]\!]ds.$$

Since $\mathbf{q}=\nabla u$, then

$$\int_\Omega\mathbf{q}\cdot\mathbf{t}dx=\int_\Omega\nabla_h u\mathbf{t}dx-\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}(u-\hat{u})\mathbf{t}\cdot n_\kappa ds$$

$$=\int_\Omega\nabla_h u\mathbf{t}dx-\sum_{\kappa\in\mathcal{T}_h}\int_{\Gamma_h^i}[\![u]\!]\{\{\mathbf{t}\}\}-\eta[\![u]\!][\![\mathbf{t}]\!]ds. \quad (2.8)$$

For the source problem, using Green's formula, (2.5) and the definition of numerical flux, $\mathbf{q}\in\mathbf{Q}_h$, which yields

$$\int_\Omega\mathbf{q}\cdot\nabla v dx+\int_\Omega uv dx-\int_{\Gamma_h^i}(\{\{\mathbf{q}\}\}-\alpha[\![u]\!]-\eta[\![\mathbf{q}]\!]\cdot[\![v]\!])ds=\int_{\Gamma_h^b}\lambda uv ds, \quad (2.9)$$

By (2.7), (2.8), (2.9) and the definition of numerical flux, $u\in \mathrm{V}_h$, there are

$$\int_\Omega(\nabla_h u-\Phi(u))\nabla_h v dx-\int_{\Gamma_h^i}(\{\{\mathbf{q}\}\}-\alpha[\![u]\!]-\eta[\![\mathbf{q}]\!])[\![v]\!]ds+\int_\Omega uv dx$$

$$=\int_\Omega(\nabla_h u-\Phi(u))\nabla_h v dx-\int_{\Gamma_h^i}(\{\{\mathbf{q}\}\}-\eta[\![\mathbf{q}]\!])[\![v]\!]ds+\int_{\Gamma_h^i}\alpha[\![u]\!][\![v]\!]ds+\int_\Omega uv dx$$

$$=\int_\Omega(\nabla_h u-\Phi(u))(\nabla_h v-\Phi(v))dx+\int_{\Gamma_h^i}a[\![u]\!][\![v]\!]ds+\int_\Omega uv dx$$

$$=\int_{\Gamma_h^b}\lambda uv dx, \quad (2.10)$$

therefore

$$a_h(\mathrm{u},v_h):=\int_\Omega(\nabla_h u-\Phi(u))(\nabla_h v-\Phi(v))dx+\int_{\Gamma_h^i}a[\![u]\!][\![v]\!]ds+\int_\Omega uv dx=\int_{\Gamma_h^b}\lambda uv dx. \quad (2.11)$$

The finite element approximation of (2.3) is given by: Find$(\lambda_h,u_h)\in C\times S^{\underline{p}}(\mathcal{T}_h)$ and $u_h\neq 0$, such that

$$a_h(u_h,v_h)=\lambda_h<u_h,v_h>,\forall v_h\in S^h. \quad (2.12)$$

The source problem of (2.3) is given by: Find $w\in H^1(\Omega)$, such that

$$a(w,v)=<f,v>,\quad\forall v\in H^1(\Omega). \quad (2.13)$$

The local discontinuous finite element approximation of (2.12) is given by: Find $w_h\in V_h$, such that

$$a_h(w_h,v_h)=<f,v_h>,\quad\forall v_h\in V_h. \quad (2.14)$$

Define the linear bounded operator $T:L^2(\Omega)\to H^1(\partial\Omega)$ satisfying

$$a(Tf,v)=<f,v>,\ \forall f\in L^2(\partial\Omega),\ v\in H^1(\Omega). \quad (2.15)$$

Then the equivalent operator of (2.4) is the form

$$Tu=\frac{1}{\lambda}u. \quad (2.16)$$

From (2.13), the corresponding discrete solution operator $T_h:L^2(\partial\Omega)\to V_h$ satisfies

$$a_h(T_hf,v)=<f,v>,\quad\forall f\in L^2(\partial\Omega),\ \forall v\in V_h. \quad (2.17)$$

The equivalent operator form of (2.12) as follow:

$$T_hu_h=\frac{1}{\lambda_h}u_h. \quad (2.18)$$

The dual problem of (2.4) is given by: Find$(\lambda^*,u^*)\in C\times H^1(\Omega)$ and $u^*\neq 0$, such that

$$a(v,u^*)=\lambda^*<v,u>,\quad\forall v\in H^1(\Omega). \quad (2.19)$$

The source problem of (2.18) is given by: Find $w^*\in H^1(\Omega)$, such that

$$a(v,w^*)=<v,g>,\quad\forall v\in H^1(\Omega). \quad (2.20)$$

Define the linear bounded operator $T^*:L^2(\Omega)\to H^1(\partial\Omega)$ such that

$$a(v,T^*g)=<v,g>,\quad\forall g\in L^2(\partial\Omega),\ v\in H^1(\Omega). \quad (2.21)$$

The finite element approximation of (2.18) is given by:

Find $(\lambda_h^*, u_h^*) \in C \times V_h$ and $u_h^* \neq 0$, such that

$$(v_h, w_h^*) =< v_h, g >, \quad \forall v_h \in V_h.$$

Then the equivalent operator of (2.18) is

$$T^* u^* = \frac{1}{\lambda^*} u^*. \qquad (2.22)$$

The finite element approximation of (2.18) is given by:
Find $(\lambda_h^*, u_h^*) \cdot C \times V_h$ and $u_h^* \neq 0$, such that

$$a_h(v_h, u_h^*) = \lambda_h^* < v_h, u_h^* >, \quad \forall v_h \in V^h. \qquad (2.23)$$

The local discontinuous finite element approximation of (2.19) is given by: Find $w_h^* \in V_h$, such that

$$a_h(v_h, w_h^*) =< v_h, g >, \quad \forall v_h \in V_h. \qquad (2.24)$$

The sum space $V(h) = V_h + H^1(\Omega)$ is introduced which assigns a local discontinuous finite element norm, where the energy norm is:

$$\|v\|_h^2 = \sum_{\kappa \in \mathcal{T}_h} \left(\|\nabla_h v\|_{0,\kappa}^2 + \|v\|_{0,\kappa}^2 \right) + \sum_{e \in \Gamma_h^i} \left\| h^{-\frac{1}{2}} p [\![v]\!] \right\|_{0,\mathcal{E}_i}^2. \qquad (2.25)$$

Galerkin orthogonality is:

$$a_h(w - w_h, v_h) = 0, \forall v_h \in V_h, \qquad (2.26)$$

$$a_h(v_h, w^* - w_h^*) = 0, \forall v_h \in V_h. \qquad (2.27)$$

Proof. Substituting the numerical flux into (2.5), combining with (2.8)

$$\begin{aligned} a_h(w_h, v_h) = & \sum_{\kappa \in \mathcal{T}_h} \int_\kappa (\nabla w_h \nabla v_h + w_h v_h) dx \\ & + \sum_{e \in \Gamma_h^i} (\int_e - ([\![w_h]\!] \{\{\nabla v_h\}\} \\ & + \{\{\nabla w_h\}\} [\![v_h]\!]) ds + \eta \int_e [\![w_h]\!] [\![\nabla v]\!] \\ & + [\![\nabla w_h]\!] [\![v_h]\!] ds + \int_e \alpha [\![w_h]\!] [\![v_h]\!]) ds. \end{aligned} \qquad (2.28)$$

Let w be a solution of the source problem, and $w \in H^1$, we can get $[\![w]\!] \big|_{\Gamma_h^i} = 0$, $[\![\nabla w]\!] \big|_{\Gamma_h^i} = 0$, we set

$$a_h(\mathrm{w}, v_h) = P_1 - P_2, \qquad (2.29)$$

where

$$P_1 = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa (\nabla \mathrm{w} \nabla v_h + \mathrm{w} v_h) dx,$$

$$P_2 = \sum_{e \in \Gamma_h^i} \{\{\nabla \mathrm{w}\}\} [\![v_h]\!] ds,$$

for the equation P_1, using the Green's formula and (2.1), we obtain

$$\begin{aligned} P_1 = & \sum_{\kappa \in \mathcal{T}_h} \left(\int_\kappa (-\Delta \mathrm{w} v_h + \mathrm{w} v_h) dx + \int_{\partial\kappa} \nabla \mathrm{w} \cdot \mathbf{n}_\kappa v_h ds \right) \\ = & \sum_{\kappa \in \mathcal{T}_h} \int_\kappa (-\Delta \mathrm{w} v_h + \mathrm{w} v_h) dx + \sum_{e \in \Gamma_h^i} \int_\mathrm{e} \nabla \mathrm{w} \cdot \mathbf{n}_\kappa v_h \\ & + \sum_{e \in \Gamma_h^D} \int_\mathrm{e} \nabla \mathrm{w} \cdot \mathbf{n}_\kappa v_h ds \\ = & \sum_{e \in \Gamma_h^i} \int_\mathrm{e} \nabla \mathrm{w} \cdot \mathbf{n}_\kappa v_h + \int_{\partial\Omega} \mathrm{f}\, v_h\, ds \\ = & \sum_{e \in \Gamma_h^i} \int_\mathrm{e} \{\{\nabla \mathrm{w}\}\} [\![v_h]\!] ds + \sum_{e \in \Gamma_h^i} [\![\nabla \mathrm{w}]\!] \{\{v_h\}\} ds \\ & + \int_{\partial\Omega} \mathrm{f}\, v_h\, ds \\ = & \sum_{e \in \Gamma_h^i} \int_\mathrm{e} \{\{\nabla \mathrm{w}\}\} [\![v_h]\!] ds + \int_{\partial\Omega} \mathrm{f}\, v_h ds \qquad ds, \end{aligned}$$

combine (2.29) to get

$$a_h(\mathrm{w}, v_h) = \int_{\partial\Omega} f v_h dx, \forall v_h \in \mathrm{V(h)}, \qquad (2.30)$$

similarly

$$a_h(v_h, w^*) = \int_{\partial\Omega} g v_h dx, \forall v_h \in \mathrm{V(h)}. \qquad (2.31)$$

Then (2.26) directly from (2.30) and (2.14), (2.27) directly from (2.31) and (2.14).

Continuity and coercivity of $a_h(u, v)$ as follow:

$$|a_h(u_h, v_h)| \lesssim \|u_h\|_h \|v_h\|_h, \forall u_h, v_h \in V(h), \qquad (2.32)$$

$$\|u_h\|_h^2 \lesssim |a_h(u_h, u_h)|, \forall u_h \in V_h. \qquad (2.33)$$

Proof. using (2.7), the trace inequality as well as the Cauchy-Schwarz inequality, which yields

$$\begin{aligned} \|\phi(v)\|_{0,L^2(\Omega)} = & \sup_{r \in L^2(\Omega)} \frac{(\phi(v), r)}{\|r\|_{L^2(\Omega)}} = \sup_{r \in L^2(\Omega)} \frac{\int \phi(v) r dx}{\|r\|_{L^2(\Omega)}} \\ \lesssim & \sup_{r \in L^2(\Omega)} \frac{\int_{\Gamma_h^i} (\{\{r\}\} - \eta[[r]])[[v]] ds}{\|r\|_{L^2(\Omega)}} \\ \lesssim & \sup_{r \in L^2(\Omega)} \frac{\|\{\{r\}\}\|_{L^2(\Gamma_h^i)} \|[[v]]\|_{L^2(\Gamma_h^i)}}{\|r\|_{L^2(\Omega)}} \\ \lesssim & \sup_{r \in L^2(\Omega)} \frac{\|\{\{r\}\}\|_{L^2(\Gamma_h^i)} \|[[v]]\|_{L^2(\Gamma_h^i)}}{\|r\|_{L^2(\Omega)}} \\ \lesssim & \sup_{r \in L^2(\Omega)} h^{-\frac{1}{2}} p \|[v]]\|_{L^2(\Gamma_h^i)}, \end{aligned} \qquad (2.34)$$

therefore,

$$\| \phi(v \|_{L^2(\Omega)} \lesssim \| h^{-\frac{1}{2}} p [[v]] \|_{0,\Gamma_h^i}. \qquad (2.35)$$

Using (2.11), we let

$$A_h(u_h, v_h) = \int_\Omega (\nabla_h u_h - \Phi(u_h))(\nabla_h v_h - \Phi(v_h)) dx,$$

$$I_h(u_h, v_h) = \int_{\Gamma_h^i} a [\![u_h]\!] [\![v_h]\!] ds + \int_\Omega u_h v_h dx,$$

thus

$$\begin{aligned} |A_h(u_h, v_h)| \lesssim & \| \nabla_h u_h \|_{0,\Omega} \| \nabla_h v_h \|_{0,\Omega} \\ & + \| \nabla_h u_h \|_{0,\Omega} \| \Phi(v_h) \|_{0,\Omega} \\ & + \| \Phi(u_h) \|_{0,\Omega} \| \nabla_h v_h \|_{0,\Omega} \\ & + \| \Phi(u_h) \|_{0,\Omega} \| \Phi(v_h) \|_{0,\Omega}, \end{aligned}$$

$$|I_h(u_h, v_h)| \lesssim a \| h^{-\frac{1}{2}} p [\![u_h]\!] \|_{0,\Gamma_h^i} \| h^{-\frac{1}{2}} p [\![v_h]\!] \|_{0,\Gamma_h^i} + \| u_h \|_{0,\Omega} \| v_h \|_{0,\Omega},$$

using Cauchy Schwarz inequality and (2.35) yields

$$\begin{aligned} |A_h(u_h, v_h) + I_h(u_h, v_h)| \lesssim & |A_h(u_h, v_h)| + |I_h(u_h, v_h)| \\ \lesssim & (\|\nabla_h u_h\|_{0,\Omega}^2 + \|u_h\|_{0,\Omega}^2 + a \left\| h^{-\frac{1}{2}} p [\![u_h]\!] \right\|_{0,\mathcal{E}_i}^2)^{\frac{1}{2}} \\ & (\|\nabla_h v_h\|_{0,\Omega}^2 + \|v_h\|_{0,\Omega}^2 + a \left\| h^{-\frac{1}{2}} p [\![v_h]\!] \right\|_{0,\mathcal{E}_i}^2)^{\frac{1}{2}} \\ = & \|u_h\|_h \|v_h\|_h. \end{aligned} \qquad (2.36)$$

Therefore, the proof of (2.32) is complete. Next, we proof (2.33), when γ satisfies $\frac{1}{1+\eta} < \gamma < 1$,

$$|A_h(v_h, v_h) + I_h(v_h, v_h)| \gtrsim$$

$$\left|\int_\Omega (1-\gamma)(\nabla_h v_h)^2 dx + (1-\frac{1}{\gamma})(\Phi(v_h))^2 dx\right| +$$
$$+\left|\int_{\Gamma_h^i} a[\![u_h]\!][\![v_h]\!] ds + \int_\Omega u_h v_h dx\right|$$
$$\gtrsim (\|\nabla_h v\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 + \alpha \left\|h^{-\frac{1}{2}} p[[v]]\right\|^2)$$
$$= \| v_h \|_h^2. \quad (2.37)$$

the proof of (2.33) is complete.

Lemma 2.1. Let w be a solution of equation (2.13), $w \in H^{1+r}(\Omega)(r < \frac{1}{2})$, $f \in L^2(\partial\Omega)$, the regularity estimate is as follows

$$\| w \|_{1+r} \leq c_\Omega \| f \|_{0,\partial\Omega}. \quad (2.38)$$

Using lemma 2.1[13], $e \in \partial\kappa$, for any $\phi \in H^{1+\xi}(\kappa)(1 < \xi < \frac{1}{2})$, $\Delta\phi \in L^2(\kappa)$, there are

$$\| \nabla\phi \cdot \mathbf{n} \|_{\xi-\frac{1}{2},e} \lesssim \left(\| \nabla\phi \|_{\xi,\kappa} + h_\kappa^{1-\xi} \| \triangle \phi \|_{0,\kappa}\right), \quad (2.39)$$

where ψ is the solution of $a_h(v,\psi) = (v,g)$, $\forall v \in H^1(\Omega)$, $g \in L^2(\Omega)$, exists $w \in H^{1+\beta}(\Omega)$ $(\beta > \frac{1}{2})$, we have

$$\| w \|_{1+\beta} \lesssim \| g \|_{0,\Omega}, \quad (2.40)$$

where $\psi^I \in H^{1+\beta}(\Omega)$ is the interpolating function of ψ on $\mathcal{T}_h$.

Lemma 2.2. Refer to Proposition 4.9[14], where $v \in H^{s_\kappa}(\kappa)$ $(s_\kappa \geqslant 1)$, then there exists $\Pi_{p_\kappa}^{h_\kappa} v \in S^{p_\kappa}, p_\kappa = 1,2,\cdots$, $(0 \leqslant m \leqslant s_\kappa)$ satisfying

$$\| v - \Pi_{p_\kappa}^{h_\kappa} v \|_{m,\kappa} \lesssim h_\kappa^{\min(p_\kappa+1,s_\kappa)-m} p_\kappa^{m-s_\kappa} \|v\|_{s_\kappa,\kappa}, \quad (2.41)$$

$$\| v - \Pi_{p_\kappa}^{h_\kappa} v \|_{0,\partial\kappa} \lesssim h_\kappa^{\min(p_\kappa+1,s_\kappa)-\frac{1}{2}} p_\kappa^{\frac{1}{2}-s_\kappa} \|v\|_{s_\kappa,\kappa}. \quad (2.42)$$

The global discontinuous interpolation operator is: Π_p^h: $H_0^1(\Omega) \to V_h$, such that $\Pi_p^h(u)|_\kappa = \Pi_{p_\kappa}^{h_\kappa}(u|_\kappa)$ for a vector-valued function $\mathrm{r} = (\mathrm{r}_1, \mathrm{r}_2, \cdots, \mathrm{r}_d)$, define $\Pi_p^h(\mathrm{r})|_\kappa = (\Pi_p^h \mathrm{r}_1, \Pi_p^h \mathrm{r}_2, \cdots, \Pi_p^h \mathrm{r}_d)$.

Lemma 2.3. Let w and w_h be the solutions of (2.13) and (2.14) respectively, $w|_\kappa \in H^{1+s}(\kappa)$, then there holds

$$\|w - w_h\|_h \lesssim \inf_{v_h \in V_h} \|w - v_h\|_h, \quad (2.43)$$

$$\|w - w_h\|_h \lesssim \sum_{\kappa\in\mathcal{T}_h} (h^{\min(p_\kappa,s_\kappa)} p_\kappa^{\frac{1}{2}-s_\kappa} \| w \|_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}}. \quad (2.44)$$

Proof. We first prove (2.43), using (2.33), $v \in S^h$, which yields

$$\| w_h - v \|_h^2 \leqslant a_h(w_h - v, w_h - v)$$
$$= a_h(w - v, w_h - v) + a_h(w_h, w_h - v) - a_h(w, w_h - v)$$
$$= a_h(w - v, w_h - v) + b_h(f, w_h - v) - a_h(w, w_h - v),$$

when $\| w_h - v \|_h \neq 0$, from (2.45) and lemma 3.2[12], we can obtain

$$\| w_h - v \|_h \leqslant \| w - v \|_h + \frac{a_h(w, w_h - v) - b_h(f, w_h - v)}{\| w_h - v \|_h}$$
$$\leqslant \| w - v \|_h + \sum_{\kappa\in\mathcal{T}_h} (h^{\min(p_\kappa,s_\kappa)+1} p_\kappa^{-s_\kappa} \| w \|_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}}, \quad (2.45)$$

using the triangle inequality, we get

$$\|w - w_h\|_h \leqslant \|w - v + v - w_h\|_h$$
$$\leqslant \|w - v\|_h + \|v - w_h\|_h, \quad (2.46)$$

the proof of (2.43) can be obtained by combining (2.45) and (2.46) when h is small enough.

Next, we proof (2.44), from (2.25), let $E_h(w) = w - \Pi_p^h w$, we have

$$\| E_h(w) \|_h^2 \lesssim \left(\sum_{\kappa\in\mathcal{T}_h} \left(\|\nabla_h E_h(w)\|_{0,\kappa}^2 + \|E_h(w)\|_{0,\kappa}^2\right)\right)$$
$$+ \sum_{e\in\Gamma_h^i} \left\|h^{-\frac{1}{2}} p[\![E_h(w)]\!]\right\|_{0,\mathcal{E}_i}^2$$
$$\lesssim \sum_{\kappa\in\mathcal{T}_h} \|\nabla_h E_h(w)\|_{0,\kappa}^2 + \|E_h(w)\|_{0,\kappa}^2$$
$$+ \sum_{\kappa\in\mathcal{T}_h} (\sum_{e\in\Gamma_h^i} \left\|h^{-\frac{1}{2}} p[\![\ E_h(w)]\!]\right\|_{0,e}^2)$$
$$\lesssim \sum_{\kappa\in\mathcal{T}_h} (h_\kappa^{\min(p_\kappa,s_\kappa)} p_\kappa^{1-s_\kappa} \| w \|_{1+s_\kappa,\kappa})^2 +$$
$$\sum_{\kappa\in\mathcal{T}_h} (\ h^{-\frac{1}{2}+\min(p_\kappa+1,1+s_\kappa)-\frac{1}{2}} p_\kappa^{1+\frac{1}{2}-1-s_\kappa} \| w \|_{1+s_\kappa,\kappa})^2$$
$$\lesssim \sum_{\kappa\in\mathcal{T}_h} (\ h^{\min(p_\kappa,s_\kappa)} p_\kappa^{\frac{1}{2}-s_\kappa} \| w \|_{1+s_\kappa,\kappa})^2. \quad (2.47)$$

from (2.47), we have

$$\| w - \Pi_p^h w \|_h \lesssim (\sum_{\kappa\in\mathcal{T}_h} (h^{\min(p_\kappa,s_\kappa)} p_\kappa^{\frac{1}{2}-s_\kappa} \| w \|_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}}. \quad (2.48)$$

Using error estimation and interpolation error estimation

$$\inf_{v_h\in V_h} \|w - v_h\| \lesssim \| w - \Pi_p^h w \| \quad (2.49)$$

(2.44) can be proved by (2.43), (2.48) and (2.49).

Theorem 2.1. If w and w_h are the solutions of (2.13) and (2.14) respectively and $w\ |_\kappa \in H^{1+s_\kappa}(\kappa)(s_\kappa > \frac{1}{2})$, then there holds

$$\| w - w_h \|_{0,\Omega} \lesssim\ h^{\min(p_\kappa,\beta)} p_\kappa^{\frac{1}{2}-\beta} \| w - w_h \|_h, \quad (2.50)$$

$$\| w - w_h \|_{0,\Omega} \lesssim (\sum_{\kappa\in\mathcal{T}_h} (h^{\min(p_\kappa,s_\kappa)+r} p_\kappa^{1-r-s_\kappa} \| w \|_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}} \quad (2.51)$$

Proof. We first prove (2.50), consider the dual problem of the source problem of (2.1) $a(v,w^*) = \langle v,g \rangle$, for $g \in L^2(\Omega)$, using the consistency, (2.27)and (2.32), we obtain

$$\langle w - w_h, g \rangle = a_h(w - w_h, w^*) = a_h(w - w_h, w^* - w_h^*)$$
$$\lesssim \| w - w_h \|_h \| w^* - w_h^* \|_h. \quad (2.52)$$

Using (2.44) and regularity, let $g = w - w_h$, we have

$$\|w^* - w_h^*\|_h \lesssim h^r p^{\frac{1}{2}-r} \|w^*\|_{1+r,\Omega} \lesssim h^r p^{\frac{1}{2}-r} \|g\|_{0,\Omega}. \quad (2.53)$$

From (2.52) and (2.53)

$$\|w - w_h\|_{0,\Omega} \lesssim \sup_{g\in L^2(\Omega)} \frac{|\ w - w_h, g\ |}{\| g \|_{0,\Omega}}$$
$$\lesssim \frac{\| w - w_h \|_h, \| w^* - w_h^* \|_h}{\| g \|_{0,\Omega}}$$
$$\lesssim h^r p^{\frac{1}{2}-r} \| w - w_h \|_h. \quad (2.54)$$

i.e., (2.50) is valid.

Next prove (2.51), from (2.44) and (2.50)

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$$\| w-w_h \|_{0,\Omega} \lesssim h^r p^{\frac{1}{2}-r} \| w-w_h \|_h$$
$$\lesssim (\sum_{\kappa\in\mathcal{T}_h} (h^{\min(p_\kappa,s_\kappa)+r} p_\kappa^{1-r-s_\kappa} \| w \|_{1+s_\kappa,\kappa})^2)^{\frac{1}{2}}.$$

the proof is completed.

Theorem 2.2. Let w and w_h are the solutions of (2.13) and (2.14) respectively, $w|_\kappa \in H^{1+s_\kappa}(\kappa)(0<s_\kappa<\frac{1}{2})$, then there holds

$$\|w-w_h\|_{0,\partial\Omega} \lesssim h^r p^{\frac{1}{2}-r} \|w-w_h\|_h \quad s>\frac{1}{2}, \qquad (2.55)$$
$$\| w-w_h \|_{0,\partial\Omega} \lesssim h^{r+s} p_\kappa^{1-r-s} \| f \|_{0,\partial\Omega} \; r\le s<\frac{1}{2}. \qquad (2.56)$$

Proof. We first prove (2.55), considering the dual equation of (2.20), for any fixed $f,g\in L^2(\partial\Omega)$, using regularity and (2.28), we obtain

$$\langle g, w-w_h\rangle = a_h(w-w_h,w^*) = a_h(w-w_h, w^*-w^{*I})$$
$$= \sum_{\kappa\in\mathcal{T}_h} \int_\kappa (\nabla(w-w_h)\nabla(w^*-w^{*I}) + (w-w_h)(w^*-w^{*I})\,dx) + \sum_{e\in\Gamma_h^i} (\int_e -([\![w-w_h]\!]\{\{\nabla(w^*-w^{*I})\}\} + \{\{\nabla(w-w_h)\}\}[\![w^*-w^{*I}]\!])ds + \eta\int_e [\![w-w_h]\!][\![\nabla(w^*-w^{*I})]\!] + [\![\nabla(w-w_h)]\!][\![w^*-w^{*I}]\!]ds + \int_e \alpha[\![w-w_h]\!][\![w^*-w^{*I}]\!])ds +$$
$$= I_1+I_2+I_3. \qquad (2.57)$$

where

$$I_1 = \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \nabla(w-w_h)\nabla(w^*-w^{*I}) + (w-w_h)(w^*-w^{*I})dx + \sum_{e\in\Gamma_h^i}\int_e \alpha[\![w-w_h]\!][\![w^*-w^{*I}]\!])ds,$$
$$I_2 = \sum_{e\in\Gamma_h^i}\int_e ([\![w-w_h]\!]\{\{\nabla(w^*-w^{*I})\}\} + \{\{\nabla(w-w_h)\}\}[\![w^*-w^{*I}]\!])ds,$$
$$I_3 = \sum_{e\in\Gamma_h^i}\eta\int_e [\![w-w_h]\!][\![\nabla(w^*-w^{*I})]\!] + [\![\nabla(w-w_h)]\!][\![w^*-w^{*I}]\!]ds.$$

When $s>\frac{1}{2}$, for I_1, using (2.44), let $s_\kappa = r$, we get

$$I_1 \lesssim \| w-w_h \|_h \| w^*-w^{*I} \|_h$$
$$\lesssim h^r p^{\frac{1}{2}-r} \| w^* \|_{1+\mathrm{r}} \|w-w_h\|_h \| w^* \|_{1+\mathrm{r}}. \qquad (2.58)$$

Using inverse estimation, (2.39) and (2.41), there are

$$I_2 \lesssim \sum_{\kappa\in\mathcal{T}_h} (\| [\![w-w_h]\!] \|_{\frac{1}{2}-r,e} \| \{\{\nabla(w^*-w^{*I})\}\} \|_{r-\frac{1}{2},e} + \eta \| [\![w-w_h]\!] \|_{\frac{1}{2}-r,e} \| [\![\nabla(w^*-w^{*I})]\!] \|_{r-\frac{1}{2},e}$$
$$\lesssim \sum_{\kappa\in\mathcal{T}_h} h^r p_\kappa^{1-3r} \| h^{-\frac{1}{2}} p [\![w-w_h]\!] \|_{0,e} \| w^* \|_{1+r}. \qquad (2.59)$$

Similarly

$$I_3 \lesssim \sum_{\kappa\in\mathcal{T}_h} \eta \| [\![w-w_h]\!] \|_{\frac{1}{2}-r,e} \| [\![\nabla(w^*-w^{*I})]\!] \|_{r-\frac{1}{2},e}$$
$$\lesssim \sum_{e\in\Gamma_h^i} \left(\eta h^{-\frac{1}{2}} p \| [\![(w-w_h)]\!] \|_{0,e}\right) h^r p_\kappa^{1-3r} \| w^* \|_{1+r}. \qquad (2.60)$$

Substituting (2.58), (2.59) and (2.60) into (2.57), the proof of (2.55) is completed.

When $0<s<\frac{1}{2}$, let $s_\kappa = r$, using (2.44) and the regularity

$$I_1 \lesssim \| w-w_h \|_h \| w^*-w^{*I} \|_h$$
$$\lesssim h^r p^{\frac{1}{2}-r} \| w^* \|_{1+\mathrm{r}} \|w-w_h\|_h. \qquad (2.61)$$

Using inverse estimates, (2.39), (2.41) and (2.44)

$$I_2 = \sum_{e\in\Gamma_h^i} \| [\![w-w_h]\!] \|_{\frac{1}{2}-r,e} \| \{\{\nabla(w^*-w^{*I})\}\} \|_{r-\frac{1}{2},e}$$
$$\lesssim h^{r+s} p^{\frac{1}{2}-s+1-3r} \| f \|_{0,\partial\Omega} \| g \|_{0,\partial\Omega}. \qquad (2.62)$$

Similarly

$$I_3 = \sum_{e\in\Gamma_h^i} \eta \| [\![w-w_h]\!] \|_{\frac{1}{2}-r,e} \| [\![\nabla(w^*-w^{*I})]\!] \|_{r-\frac{1}{2},e}$$
$$\lesssim \eta h^{r+s} p^{\frac{1}{2}-s+1-3r} \| f \|_{0,\partial\Omega} \| g \|_{0,\partial\Omega}. \qquad (2.63)$$

Combining the above three formula, we get

$$\| w-w_h \|_{0,\partial\Omega} \lesssim h^{r+s} p_\kappa^{1-r-s} \| f \|_{0,\partial\Omega}. \qquad (2.64)$$

From (2.44) and the regularity, we have

$$\|A_h f\|_h \lesssim \|A_h f - Af\|_h + \|Af\|_h$$
$$\lesssim \|A_h f - Af\|_h + \|Af\|_1$$
$$\lesssim h^r p^{\frac{1}{2}-r} \|Af\|_{1+r} + \|Af\|_1$$
$$\lesssim \|f\|_{0,\partial\Omega}. \qquad (2.65)$$

The proof is completed.

III. A Priori Error Estimation

A. A priori error analysis of the Steklov eigenvalue problem

Assume λ is the jth eigenvalue of (2.3) with the algebraic multiplicity q and the ascent α, $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+q-1}$. When $\|T_h - T\|_{0,\partial\Omega} \to 0$, q eigenvalues $\lambda_{j,h}, \cdots \lambda_{j+q-1,h}$ of (2.6) will converge to λ . Let $M(\lambda)$ be the space of generalized eigenvectors of (2.3) associated with λ, and $M_h(\lambda)$ be the direct sum of the generalized eigenspace of (2.12) associated with λ_h that converge to λ.

Given two closed subspaces V and U, The gap between these two subspaces is denoted as

$$\delta(U,V) = \sup_{u\in V,\|u\|_{0,\Omega}=1} \inf_{v\in U} \| u-v \|_{0,\Omega}, \hat{\delta}(U,V) = \max\{\delta(U,V),\delta(V,U)\}.$$

And denote the arithmetic mean $\hat{\lambda}_h = \frac{1}{q}\sum_{i=j}^{j+q-1} \lambda_{i,h}$.

Theorem 3.1. Assume that $M(\lambda)\subset H^{1+r}(\Omega)\left(s>\frac{1}{2}\right)$, $\tau = min\{p_{\kappa,s}\}$ then there holds

$$\hat{\delta}(M(\lambda), M_h(\lambda)) \lesssim h^{\tau+r} p^{1-\tau} \qquad (3.1)$$
$$\left|\hat{\lambda}_h - \lambda\right| \lesssim h^{2\tau} p^{2\tau} \qquad (3.2)$$
$$\left|\lambda_h - \lambda \lesssim h^{2\tau/\alpha} p^{2\tau/\alpha}\right| \qquad (3.3)$$

Let $u_h \in M_h(\lambda)$ be the direct sum of the generalized eigenvector spaces of (2.12), then there exists an eigenvalue function u of (2.3) such that

$$\|u-u_h\|_{0,\partial\Omega} \lesssim h^{(\tau+r)/\alpha}p^{(1-\tau)/\alpha}, \quad (3.4)$$

$$\|u-u_h\|_h \lesssim h^{\tau}p_\kappa^{-\tau} + h^{(\tau+r)/\alpha}p^{(1-\tau)/\alpha}, \quad (3.5)$$

$$\|u-u_h\|_0 \lesssim h^r p_\kappa^{-r} \parallel u-u_h \parallel_h + \|\lambda u-\lambda_h u_h\|_{0,\partial\Omega}. \quad (3.6)$$

If we set $\alpha=1$, then

$$\|u-u_h\|_{0,\partial\Omega} \lesssim hp^{-1}\|u-u_h\|_h. \quad (3.7)$$

Proof. Note that $Tf=w$ and $T_h f = w_h$, Combining the regularity with (2.56), we obtain

$$\begin{aligned}\parallel T-T_h \parallel_{0,\partial\Omega} &= \sup_{0\neq f\in L^2(\partial\Omega)} \frac{\parallel Tf-T_h f \parallel_{0,\partial\Omega}}{\parallel f \parallel_{0,\partial\Omega}} \\ &= \sup_{0\neq f\in L^2(\partial\Omega)} \frac{\parallel w-w_h \parallel_{0,\partial\Omega}}{\parallel f \parallel_{0,\partial\Omega}} \\ &\lesssim \sup_{0\neq f\in L^2(\partial\Omega)} \frac{h^{r+s}p^{1-r-s}\parallel f \parallel_{0,\partial\Omega}}{\parallel f \parallel_{0,\partial\Omega}} \\ &\lesssim h^{r+s}p^{1-r-s} \to 0, (h\to 0, p\to\infty).\end{aligned}$$

From Theorem 7.1, Theorem 7.2, Theorem 7.3 and Theorem 7.4 [15], there are

$$\hat{\delta}\big(M(\lambda), M_h(\lambda)\big) \lesssim \parallel (T-T_h)|_{M(\lambda)} \parallel_{0,\partial\Omega}, \quad (3.8)$$

$$\left|\lambda-\hat{\lambda}_h\right| \lesssim \sum_{i,l=j}^{j+q-1} |\langle (T-T_h)\varphi_i, \varphi_l^*\rangle| + \parallel (T-T_h)|_{M(\lambda)} \parallel_{0,\partial\Omega} \parallel (T^*-T_h^*)|_{M(\lambda^*)} \parallel_{0,\partial\Omega}, \quad (3.9)$$

$$|\lambda-\lambda_h| \lesssim \left\{ \sum_{n,i=j}^{j+q-1} |\langle (T-T_h)\varphi_i, \varphi_l^*\rangle| + \parallel (T-T_h)|_{M(\lambda)} \parallel_{0,\partial\Omega} \parallel (T^*-T_h^*)|_{M(\lambda)} \parallel_{0,\partial\Omega} \right\}^{\frac{1}{\alpha}} \quad (3.10)$$

$$|u-u_h|_{0,\partial\Omega} \lesssim \parallel (T-T_h)|_{M(\lambda)} \parallel_{0,\partial\Omega}^{\frac{1}{\alpha}}, \quad (3.11)$$

where $\{\varphi_i\}_{i=j}^{j+q-1}$ is the basis of $M(\lambda)$, $\{\varphi_l^*\}_{l=j}^{j+q-1}$ is a dual basis.

From theorem 2.1 and the theorem 2.2, we derive

$$\begin{aligned}\parallel (T-T_h)|_{M(\lambda)} \parallel_{0,\partial\Omega} &= \sup_{f\in M(\lambda),\|f\|_{0,\partial\Omega}=1} \parallel Tf-T_h f \parallel_{0,\partial\Omega} \\ &\lesssim \sup_{f\in M(\lambda),\|f\|_{0,\partial\Omega}=1} h^{\tau+r}p^{1-r-\tau} \parallel Tf \parallel_{\tau+1,\Omega}. \quad (3.12)\end{aligned}$$

Similarly, we have

$$\begin{aligned}\parallel (T^*-T_h^*)|_{M(\lambda)} \parallel_{0,\partial\Omega} &\lesssim \sup_{f\in M(\lambda),\|f\|_{0,\partial\Omega}=1} \parallel T^*f-T^*{}_h f \parallel_{0,\partial\Omega} \\ &\lesssim \sup_{f\in M(\lambda),\|f\|_{0,\partial\Omega}=1} h^{\tau+r}p^{1-r-\tau} \parallel T^*f \parallel_{\tau+1,\Omega}.\end{aligned}$$

Substituting (3.12) into (3.8) and (3.11), we can obtain (3.1) and (3.4) respectively. Using the operator properties, regularity, (2.20) , (2.27) and (2.32), we have

$$\begin{aligned}\langle (T-T_h)\varphi_i, \varphi_l^*\rangle &= a_h(A\varphi_i - A_h\varphi_i, A^*\varphi_l^*) \\ &= a_h(A\varphi_i - A_h\varphi_i, A^*\varphi_l^* - A_h^*\varphi_l^*) \\ &\lesssim \parallel A\varphi_i - A_h\varphi_i \parallel_h \parallel A^*\varphi_l^* - A_h^*\varphi_l^* \parallel_h \\ &\lesssim h^{\tau}p^{\frac{1}{2}-\tau} \parallel T\varphi_i \parallel_{\tau+1} h^{\tau}p^{\frac{1}{2}-\tau} \parallel T^*\varphi_l^* \parallel_{\tau} \\ &\lesssim h^{2\tau}p^{1-2\tau}. \quad (3.14)\end{aligned}$$

Substituting (3.14), (3.12) and (3.3) into (3.9) and (3.10). we get (3.2) and (3.3) respectively.

Since $u=\lambda Tu$, $u_h = \lambda_h T_h u_h$, using the triangle inequality, (2.65), (3.3) and (3.4), we deduce

$$\begin{aligned}\|u-u_h\|_h - \|u-\lambda T_h u\|_h &\lesssim \|u_h - \lambda T_h u\|_h \\ &= \|T_h(\lambda_h u_h - \lambda u)\|_h \\ &\lesssim \|\lambda_h u_h - \lambda u\|_{0,\partial\Omega} \\ &\lesssim h^{(\tau+r)/\alpha}p^{(1-2\tau)/\alpha}. \quad (3.15)\end{aligned}$$

together with (2.43), which yields (3.5).

$$\begin{aligned}\|u-u_h\|_0 &= \|\lambda Au - \lambda A_h u\|_0 + \|\lambda A_h u - \lambda_h A_h u_h\|_0 \\ &\lesssim h^{\min(p,\beta)}p^{\frac{1}{2}-\beta}\|\lambda Au - \lambda A_h u\|_h + \|\lambda A_h u - \lambda_h A_h u_h\|_h \\ &\lesssim h^{\min(p,\beta)}p^{\frac{1}{2}-\beta} \inf_{v_h\in V_h} \|\lambda Au - v_h\|_h + \|\lambda u - \lambda_h A_h u_h\|_{0,\partial\Omega}. \quad (3.16)\end{aligned}$$

Since $\inf_{v_h\in V_h} \|\lambda Au - v_h\|_h \lesssim \parallel u-u_h \parallel_h$, (3.6) is valid.

When $\alpha=1$, from spectral approximation principle, (2.55), and (2.43), we have

$$\begin{aligned}\|u-u_h\|_{0,\partial\Omega} &\lesssim \|Tu - T_h u\|_{0,\partial\Omega} \\ &\lesssim h^{\min(p,\tau)}p^{\frac{1}{2}-\tau}\|Tu-T_h u\|_h \\ &\lesssim h^{\min(p,\tau)}p^{\frac{1}{2}-\tau} \inf_{v_h\in V_h} \|Tu - v_h\|_h \\ &\lesssim h^{\min(p,\tau)}p^{\frac{1}{2}-\tau}\|u-u_h\|_h,\end{aligned}$$

the proof is completed.

B. Numerical experiments

In this section, a series of numerical experiments will be conducted to verify the effectiveness of the hp local discontinuous finite element method of Steklov eigenvalue problem by compiling the code under the IFEM package, and the computed results will be sorted in descending order to obtain the data.

In this experiment, the test domain are set to be the L-shape domain $\Omega_L = [0,1]\times[0,\frac{1}{2}] \cup [0,\frac{1}{2}]\times[\frac{1}{2},1]$ and square $\Omega_S = [0,1]^2$ respectively.

TABLE I. About the region Ω_S numerical solution results for the first fourth eigenvalues

h	P	dof	λ_1	λ_2	λ_3	λ_4
1/4	1	24	0.241985120396731	1.557082494331800	1.567981965130290	2.766574268815750
1/4	2	48	0.240079724549933	1.496947672301450	1.499428975046800	2.082787917163070
1/4	3	80	0.240079112305256	1.492322541686640	1.492334691413000	2.082653128244820
1/8	1	96	0.240603412309877	1.515427100614430	1.519205201504950	2.252320633910260
1/8	2	192	0.240079142099497	1.492630666254280	1.492794826780140	2.082659643988760
1/8	3	320	0.240079085865350	1.492303508371800	1.492303745768920	2.082647158468340
1/16	1	384	0.240215724296756	1.498834475702410	1.500067866013550	2.125958682038990
1/16	2	768	0.240079089485034	1.492324336253270	1.492334931269400	2.082647939864130
1/16	3	1280	0.240079085433790	1.492303140728370	1.492303144658960	2.082647055702750
1/32	1	1536	0.240113813327580	1.494002217106130	1.494356189460460	2.093634847726540
1/32	2	3072	0.240079085696021	1.492304473625110	1.492305148150550	2.082647112024300
1/32	3	5120	0.240079085425170	1.492303134470372	1.492303134692798	2.082647053903782

			+ 0.00000000000000i	+ 0.00000000000000i	+ 0.000000000000000i	+ 0.000000000000000i
1/64	1	6144	0.240087824313781	1.492733483605300	1.492827836095860	2.085414539962150
1/64	2	12288	0.240079085442566	1.492303218530770	1.492303261109540	2.082647057751940
1/64	3	20480	0.240079085419030 + 0.000000000000000i	0.768274571229386 - 0.842808219666222i	0.768274571229386 + 0.842808219666222i	-1.157380518255964 +0.000000000000000i

TABLE II. About the region Ω_L numerical solution results for the first fourth eigenvalues

h	P	dof	λ_1	λ_2	λ_3	λ_4
1/2	1	18	0.184370548231648	0.980232299404417	1.82906530795754	4.13808322383428
1/2	2	36	0.182980279501687	0.918525278982131	1.699156800429130	3.275523798631990
1/2	3	60	0.182966075088768	0.900855678534774	1.689327741901200	3.220087676768690
1/4	1	72	0.183358727346188	0.931174919280224	1.73919672226841	3.56578249426117
1/4	1	144	0.182966480133728	0.900805354854160	1.689905958347010	3.223623216518960
1/4	2	240	0.182964513803436	0.896396671029967	1.688714315870450	3.217910283979370
1/8	1	288	0.183069281417831	0.908401968715363	1.70330292648774	3.32131530982049
1/8	1	576	0.182964567302139	0.896233781046824	1.688763930472870	3.218267571713410
1/8	2	960	0.182964279921273	0.894722861609305	1.688617360337970	3.217860903736770
1/16	1	1152	0.182991309607356	0.899267392795164	1.692530493140910	3.245737724595320
1/16	2	2304	0.182964287235214	0.894643891263804	1.688622521043840	3.217886398047420
1/16	3	3840	0.182964243597903	0.894070589896526	1.688603059166600	3.217859838784520
1/32	1	4608	0.182971105542950	0.895785659995388	1.689615048411320	3.225034155288780
1/32	2	9216	0.182964244673090	0.894037759700364	1.688603653469660	3.217861497450140
1/32	3	15360	0.182964237949173 +0.00000000000000i	0.893813548474596 +0.000000000000000i	1.688600895976938 +0.000000000000000i	-2.349206210258648 -1.759265727335553i

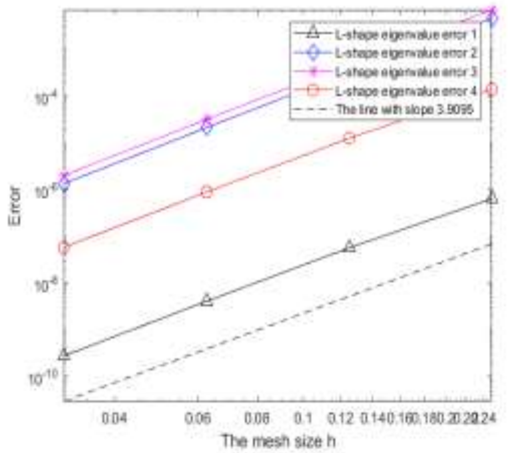

Fig. 1. The error curves of the approximation for the first fourth eigenvalues on Ω_S with P = 2.

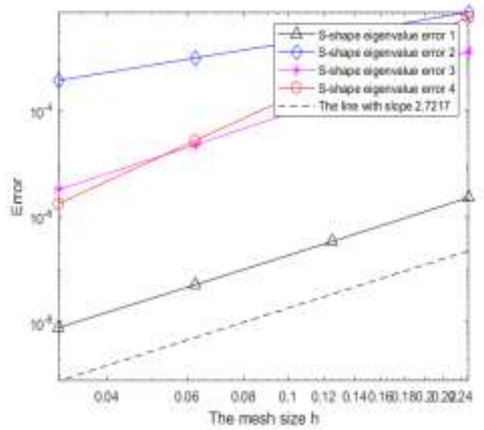

Fig. 2. The error curves of the approximation for the first fourth eigenvalues on Ω_L with P = 2.

IV. CONCLUSION

Steklov eigenvalue problem is widely used in physics. In this paper, a hp locally discontinuous Galerkin method is used to obtain the optimal order of convergence, where h is optimal and p is suboptimal, and the optimal order error is estimated in the L-shape domain and square domain respectively under the iFEM package. The convergence of the Dirichlet operator is superior on the continuous Ω region, which shows that the numerical experiment is effective and feasible, so it has good application value.

REFERENCES

[1] Bergman S, Schiffer M, “Kernel functions and elliptic differential equations in mathematical physics,” *Courier Corporation*, 2005.

[2] Ahn H J, Hyun, “Vibrations of a pendulum consisting of a bob suspended from a wire: the method of integral equations,” *Quarterly of Applied Mathematics,* vol. 39, no.1, pp.109-117, 1981.

[3] Hinton, D B , J. K. Shaw, “Differential Operators with Spectral Parameter Incompletely in the Boundary Conditions (eng) ,” *Funkcialaj Ekvacioj,* vol. 33, no. 3, pp. 363-385, 1990.

[4] Bermudez A, Rodriguez R, Santamarina D, “A finite element solution of an added mass formulation for coupled fluid-solid vibrations,” *Numerische Mathematik,* vol. 87, pp. 201–227, 2000.

[5] Bucur DIonescu, I. R, “Asymptotic analysis and scaling of friction parameters,” *ZAMP: Zeitschrift fur Angewandte Mathematik und Physik: = Journal of Applied Mathematics and Physics: = Journal de Mathematiques et de Physique Appliquees,* vol. 57, no. 6, 2006

[6] Han Houde, Kwan Chi, “Boundary element approximation for Steklov eigenvalue problems (in Chinese),” *Applied Mathematics A Journal of Chinese Universities, vol. 2, pp. 128-135,* 1994

[7] Li Qin, Yang Yidu, “A note on incongruous element approximation for Steklov eigenvalue problems (in Chinese),” *Journal of Guizhou Normal University(Natural Sciences),* vol. 27, no. 3, pp. 4, 2009.

[8] Cheng Pan, “High precision algorithm of boundary integral equations for Steklov eigenvalue problems (in Chinese),” *University of Electronic Science and technology*.2011.

[9] An Jing, “An efficient Legendre-Galerkin spectral approximation for Steklov eigenvalue problems (in Chinese),” *Science China Mathematics,* vol. 45, no. 1, pp. 83-92, 2015.

[10] Jie Rong, Chan Kin-jun, Zhu Xiaoling, “Fast Fourier-Galerkin method for Steklov eigenvalue problems (in Chinese),” *Journal of Nanning Normal University(Natural Science Edition),* vol. 37, no. 1, pp. 8-17, 2020.

Volume 8, Issue 3, pp. 93-100, 2024.
ISSN (Online): 2456-7361

[11] An Jing, Jiang Jiantao, "An efficient finite element method for 4-order Steklov eigenvalue problem based on dimensionality reduction scheme (in Chinese)," *Journal of Guizhou Normal University(Natural Sciences)*, vol. 39, no. 4, pp. 10, 2021.
[12] Perugia I, Schötzau D, "An hp-analysis of the local discontinuous Galerkin method for diffusion problems," *Journal of Scientific Computing*, vol. 17, pp. 561-571, 2002.
[13] Cai Z, Ye X, Zhang S, "Discontinuous Galerkin finite element methods for interface problems: a priori and a posteriori error estimations," *SIAM journal on numerical analysis*, vol. 49, no. 5, pp. 1761-1787, 2011.
[14] Perugia I, Sch ¨otzau D, "The hp-local discontinuous Galerkin method for low-frequency time- harmonic Maxwell equations," Mathematics of Computation, vol. 72, no. 243, pp. 1179-1214, 2003.
[15] Babuska, Ivo, and J. Osborn. "Eigenvalue problems, in "Handbook of numerical analysis", vol. II. " PG Ciarlet and JL Lions Editions, Elsevier science Publishers BV (North-Holland), pp. 645-785, 1991.