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Some Properties I −Cauchy and I −localized Sequence on G −metric Space

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Abstract— *The metric space is typically denoted as $(\boldsymbol{X},\boldsymbol{G})$, which is a generalization of metric spaces. One of the extensively studied properties in $\boldsymbol{G}$ −metric spaces is convergence. In a $\boldsymbol{G}$ −metric space, a sequenc $(\boldsymbol{x_n})$ is said to $\boldsymbol{G}$ −converge to $\boldsymbol{x}$ if $\boldsymbol{\lim G(x,x_n,x_m)=0}$, which means that for every $\boldsymbol{\varepsilon}\in\mathbb{R},\boldsymbol{\varepsilon>0}$, there exists $\boldsymbol{K}\in\mathbb{N}$ such that for all $\boldsymbol{m,n}\in\mathbb{N}$ with $\boldsymbol{m,n\geq K}$, we have $\boldsymbol{G(x,x_n,x_m)<\varepsilon}$. Many mathematicians have discussed the concept of convergence, including the notion of statistical convergence first introduced by Fast (1951). This research aims to investigate the concept of $\boldsymbol{I}$ −convergence, which is a generalization of statistical convergence. Furthermore, the study establishes the connection between $\boldsymbol{I}$ −Cauchy sequences, an extension of $\boldsymbol{I}$ −convergence, and convergence in $\boldsymbol{G}$ −metric spaces. Additionally, the properties of $\boldsymbol{I}$ −Cauchy and $\boldsymbol{I^*}$ −Cauchy sequences, as well as $\boldsymbol{I}$ −localized sequences in $\boldsymbol{G}$ −metric spaces, are examined. The objective of this research is to enhance the understanding of the properties of $\boldsymbol{I}$ −Cauchy sequences within the context of G−metric spaces. The new findings from this study can contribute to the development of the theory of $\boldsymbol{G}$ −metric spaces and expand our understanding of $\boldsymbol{I}$ −convergence in sequences and $\boldsymbol{I}$ −Cauchy sequences. These results may also have potential applications in various fields involving mathematical analysis and modeling.*

Keywords— *$\boldsymbol{G}$ −metric space, $\boldsymbol{I}$ −convergence, $\boldsymbol{I}$ −Cauchy Sequence, $\boldsymbol{I}$ −localized.*

I. INTRODUCTION

In 1951 Fast published the idea of statistical convergence of a sequence of real numbers [1]. A sequence (x_n) is called to converge statistically to $x\in\mathbb{R}$ and denoted by $\lim_{n\to\infty} stat\ x_n = x$, if for each $\varepsilon>0$ the set $A(\varepsilon)=\{n\in\mathbb{N}: G(x_n,x_n,x)\geq\varepsilon\}$ has $\delta\big(A(\varepsilon)\big)$. In 2000, Kostyrko et al. [2] introduced a generalization of statistical convergence term as I −convergence and I^* − convergence using the ideal notation of the set of natural numbers $\mathbb{N}$.

Many other researches on the notion of I −convergence and I^* − convergence of sequence were extensively investigated by [3] they study some important topological properties. Another work has also been done by Banerjee & Rahul in 2016 [4] on double sequences. Nabiev, et al. [5] showed the decomposition theorem of I^* −convergent sequence and introduced the idea of I −Cauchy and I^* − Cauchy sequence. They also proved that I^* −Cauchy sequence is I −Cauchy and if the ideal I satisfies the condition (AP), they are equivalent. In 2010, the idea of I −divergent and I^* −divergent on metric spaces was studied by Das dan Ghosal [6]. Later Banerjee at al. studied the concept on S-metric spaces which are generalization of metric spaces [7]. We will use the following notations.

Definition 1.1. *[8] Let $X\neq\emptyset$ then a family of sets $I\subset 2^X$ is called an ideal if*

a) $A,B\in I$ implies $A\cup B\in I$
b) $A\in I, B\subset A$ imply $B\in I$

If $I\neq\{\emptyset\}$ and $X\notin I$ then the ideal I is called nontrivial.

Definition 1.2. *[9] Let $X\neq\emptyset$. A non-empty family $\mathcal{F}\subset 2^X$ is called a filter if*

a) $\emptyset\notin\mathcal{F}$
b) $A,B\in\mathcal{F}$ implies $A\cap B\in\mathcal{F}$
c) $A\in\mathcal{F}, A\subset B\subset X$ imply $B\in\mathcal{F}$

Lemma 1.1. *[2] Let I be a nontrivial ideal of X. Then the family $\mathcal{F}(I)=\{M\subset X: \exists A\in I: M=X\backslash A\}$ is a filter on X (It is often referred to as the filter related with the ideal I)*

Definition 1.3. *[2] A nontrivial ideal I is said to be admissible if for each $x\in X,\{x\}\in I$*

In 2006, the G −metric space is a generalization of the metric space first introduced by Mustafa dan Sims [10]. Many works on this space have been conducted because there are many mathematical concepts that can be studied, such as those conducted by Gaba in 2017 [11] and Jakfar, et al. [12] they discussed metrics in G −metric space and showed that the metric can be derived from the G −metric in such a way that the convergence.

Definition 1.4. *[10] Let X be a non-avoid set. A Function $G: X\times X\times X\to\mathbb{R}^+$ is called G −metric on X iff for each $x,y,z\in X$ satisfy this following conditions.*

i. $G(x,y,z)=0$ if only if $x=y=z$
ii. $G(x,x,y)>0$ if $x\neq y$ for all $x,y\in X$
iii. $G(x,y,z)=G(x,z,y)=G(y,x,z)=G(y,z,x)=G(z,x,y)=G(z,y,x)$
iv. $G(x,x,y)\leq G(x,y,z)$ for all $x,y,z\in X$ with $y\neq z$
v. $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$

The pair (X,G) is called a G −metric space.

Proposition 1.1. *(in [10]**)** Let (X,G) a G −metric space, then for any $x,y,z\in X$ and $a\in X$ it follows that $G(x,y,z)\leq G(x,x,y)+G(x,x,z)$*

Definition 1.5. *[11] Let (X,G) be a G −metric space and (x_n) be sequence of points of X. The sequence (x_n) is said to be the G − convergent to ke x if $\lim G(x,x_n,x_m)=0$. It's mean for any $\varepsilon\in\mathbb{R},\varepsilon>0$ there exists $K\in\mathbb{N}$ such that $G(x,x_n,x_m)<\varepsilon$, for all $m,n\in\mathbb{N}$ with $m,n\geq K$.*

Definition 1.6. *[10] Let (X,G) be a G −metric space and a sequence (x_n) on X is called G −Cauchy if for any $\varepsilon\in\mathbb{R},\varepsilon>0$ there exists $K\in\mathbb{N}$ such that $G(x_n,x_m,x_l)<\varepsilon$ for all $l,m,n\in\mathbb{N}$ with $l,m,n\geq K$.*

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In recently, Nabieve, et al. [13] firstly introduced the idea of I −localized and I^* −localized sequence on metric spaces and show some of their properties in relation to I −Cauchy concept and Granados, et al. [14] studied I −localized doubles sequence and furthered this idea with the help of triples using ideal sets on metric spaces [15]. In addition, Banerjee, et al. [16] also studied I −localized and I^* −localized sequence and investigated some results related to I −Cauchy sequence in S −metric space.

In this paper, the concept of I and I^* − convergence in G −metric space is defined. In section II, we have investigated the relation between I −Cauchy sequence and G − convergence in G −metric. In section III, we have discussed some properties of the I and I^* −Cauchy sequence in G −metric space. In section IV, we have studied properties of I and I^* −localized sequence related to the concept of I −Cauchy sequence.

II. Preliminary Result

Unless explicitly stated, we assume that $I \subset 2^{\mathbb{N}}$ is a nontrivial ideal of the set of all positive integers $\mathbb{N}$ and that (X, G) is an G −metric space.

Definition 2.1. *A sequence (x_n) on X is said to be I −convergent to $x \in X$ if for every $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N}: G(x_n, x_n, x) \geq \varepsilon\} \in I$.*

Definition 2.2. *An admissible ideal I is said to satisfy the property (AP) if for every countable family $\{A_1, A_2, A_3, \dots.\}$ of mutually disjoint sets of I, there exists a countable family of sets $\{B_1, B_2, B_3, \dots.\}$ such that for each $i \in \mathbb{N}$, $A_i \Delta B_i$ is a finite set and $\bigcup_{i=1}^{\infty} B_i \in I$.*

Definition 2.3. *A sequence (x_n) on X is said to be I^* − convergent to $x \in X$ if there exists a set $M \in \mathcal{F}(I)$,*

$$M = \{m_1, m_2, \dots, m_k, \dots: m_{k-1} < m_k\} \subset \mathbb{N}$$

such that $\lim_{k\to\infty} G(x_{m_k}, x_{m_k}, x) = 0$.

Definition 2.4. *Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence (x_n) on X is called an I −Cauchy sequence in (X, G) if for every$\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $A(\varepsilon) = \{n \in \mathbb{N}: G(x_n, x_n, x_{n_0}) \geq \varepsilon\} \in I$.*

Definition 2.5. *Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence (x_n) on X is called an I^* −Cauchy sequence in (X, G) if there exists a set $M \in \mathcal{F}(I)$, $M = \{m_1, m_2, \dots, m_k, \dots: m_{k-1} < m_k\} \subset \mathbb{N}$ such that the subsequence (x_{m_k}) is an ordinary Cauchy sequence in X i.e., $\lim_{k,r\to\infty} G(x_{m_k}, x_{m_k}, x_{m_r}) = 0$*

Theorem 2.1. *Let I be an ideal admissible on $\mathbb{N}$. If (x_n) in (X, G) is G −convergent to x then (x_n) is I −convergent.*

Proof. Let a sequence (x_n) is G − convergent to x. This means that for every $\varepsilon > 0$, there is a positive integer K such that for any natural number $n \geq K$, $G(x_n, x_n, x) < \varepsilon$ holds. Therefore, for any such number $\varepsilon > 0$ there exist a finite set $A(\varepsilon) = \{n \in \mathbb{N}: G(x_n, x_n, x_{n_0}) \geq \varepsilon\}$ for $n < K$. Hence, since $A(\varepsilon)$ is a finite set, $A(\varepsilon) \in I$ is an admissible ideal. It is proved that a sequence (x_n) G −convergent to x. So for such number $\varepsilon > 0$, $A(\varepsilon) = \{n \in \mathbb{N}: G(x_n, x_n, x_{n_0}) \geq \varepsilon\}$ is an admissible ideal. Thus, (x_n) is I − convergent to x.

Theorem 2.2. *Let I be an ideal admissible on $\mathbb{N}$. If (x_n) in (X, G) is I −Cauchy then (x_n) is G −Cauchy.*

Proof. Let (x_n) be an I −Cauchy sequence in (X, G). Then by definition there exists a positive integer $n_0 = n_0(\varepsilon)$ for every $\varepsilon > 0$ such that a set $A(\varepsilon) = \{n \in \mathbb{N}: G(x_n, x_n, x_{n_0}) \geq \varepsilon\} \in I$. It can be shown that (x_n) is I −Cauchy if for any given $\varepsilon > 0$, there exists $B = B(\varepsilon) \in I$ such that $m, n \notin B$ implies $G(x_m, x_n, x_n) < \varepsilon$. Let us take n_0. Then for every $\varepsilon > 0$, for all $m, n \geq n_0$, we have $G(x_m, x_n, x_{n_0}) < \varepsilon$. Hence, we get that (x_n) be G −Cauchy in (X, G).

III. I −Cauchy and I^* −Cauchy Conditions

Theorem 3.1. *Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal. If (x_n) is an I^* −Cauchy sequence in (X, G) then (x_n) is I −Cauchy.*

Proof. Let (x_n) is an I^* −Cauchy sequence. Then by definition there exists $M = \{m_1, m_2, \dots, m_k, \dots: m_{k-1} < m_k\} \subset \mathbb{N}$, $M \in \mathcal{F}(I)$ such that $G(x_{m_k}, x_{m_k}, x_{m_r}) < \varepsilon$ for all $\varepsilon > 0$ there exists a positive integer $k_0 = k_0(\varepsilon)$ for every $k, r > k_0 = k_0(\varepsilon)$. Choose $n_0 = n_0(\varepsilon) = m_{k_0+1}$, then for the real number $\varepsilon > 0$, we get $G(x_{m_k}, x_{m_k}, x_{n_0}) < \varepsilon$ with $k > k_0$. Let $H = \mathbb{N} \backslash M$ and it is clear that $H \in I$ so $A(\varepsilon) = \{n \in \mathbb{N}: G(x_n, x_n, x_{n_0}) \geq \varepsilon\} \subset H \cup = \{m_1, m_2, \dots, m_k, \dots: m_{k-1} < m_k\} \in I$.

Since for every $\varepsilon > 0$ there can be found a positive integer $n_0 = n_0(\varepsilon)$ such that $A(\varepsilon) \in I$, then (x_n) is I −Cauchy.

Lemma 3.1. (*in [5]*) *If I be an admissible ideal that satisfies the property (AP) then for every countable family $(P_n) \in \mathcal{F}(I)$ for all $n \in \mathbb{N}$ there exists a set $P \in \mathcal{F}(I)$ such that $P \backslash P_n$ is finite for all $n \in \mathbb{N}$.*

Theorem 3.2. *Let I be an admissible ideal that satisfies the property (AP). If (x_n) is I −Cauchy sequence in (X, G) then (x_n) is I^* −Cauchy also.*

Proof. Let (x_n) be I −Cauchy sequence in (X, G). Then by definition, for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that $A(\varepsilon) = \{n \in \mathbb{N}: G(x_n, x_n, x_{n_0}) \geq \varepsilon\} \in I$. Let $P_k = \{n \in \mathbb{N}: G(x_n, x_n, x_{m_k}) < \frac{1}{k}\}$ for $k \in \mathbb{N}$, where $m_k = n_0\left(\frac{1}{k}\right)$. It is clear that $P_k \in \mathcal{F}(I)$ for $k \in \mathbb{N}$. Since I satisfy the property (AP), then then by Lemma 3.1 there exists a set $P \in \mathcal{F}(I)$ such that $P \backslash P_k$ is finite for all $k \in \mathbb{N}$.

Then let $\varepsilon > 0$ and $j \in \mathbb{N}$ dengan $j > \frac{2}{\varepsilon}$. Since $P \backslash P_j$ is finite set if $m, n \in P$, so there exists $k = k(j)$ such that $m, n \in P_j$ for all $m, n > k(j)$. Therefore, the result of it $G(x_n, x_n, x_m) \leq G(x_n, x_n, x_{m_k}) + G(x_m, x_m, x_{m_k}) < \varepsilon$ for $m, n \in k(j)$. Thus, for any $\varepsilon > 0$ there exists $k = k(\varepsilon) \in \mathbb{N}$ such that for $m, n > k(\varepsilon)$ and $m, n \in P \in \mathcal{F}(I)$ implies $G(x_n, x_n, x_{m_k}) < \varepsilon$. This shows that the sequence *is* I^* −Cauchy in (X, G).

Theorem 3.3. *Let (X, G) be a G −metric space space containing at least one accumulation point. If for every sequence (x_n) I −Cauchy is I^* −Cauchy then I satisfies the property (AP).*

Proof. The approach used in [6] is used in the proof of this theorem.

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IV. Basic Properties $I-$localized and I^*-localized

Throughout the previous discussion using the notation I as the admissible ideal of $\mathbb{N}$ and X is $G-$metric space. We now give some definitions and properties of the localized sequence associated with the ideal I in $G-$metric space.

Definition 4.1. *Let (x_n) is a sequence on X. If a positive real number sequence $(a_n = G(x_n, x_n, x))$ converges in $x \in M$ and $M \subset X$ then (x_n) is called sequence in M.*

Definition 4.2. *(i) A sequence (x_n) in X is said to be $I-$localized in subset $M \subset X$ if for each $x \in M$ the positive real sequence $a_n = (G(x_n, x_n, x))_{n\in\mathbb{N}}$ is $I-$ convergent in X. (ii) The maximal subset on (x_n) is $I-$localized, is called the $I-$locator of (x_n) and it's denoted by $loc_I(x_n)$. (iii) A sequence (x_n) is said to be $I-$localized everywhere if the $I-$locator (x_n) is the whole set X and denote as $loc_I(x_n) = X$.*

Lemma 4.1. *The inequality $|G(z,z,y) - G(x,x,y)| \leq G(z,z,x)$ holds good for any $x, y, z \in X$.*

Proof. Since $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$, then $G(x,x,y) \leq G(x,y,z) = G(z,x,y) \leq G(z,z,x) + G(z,z,y)$ by using ii in Definition 1.5. So we have

$$G(x,x,y) \leq G(z,z,x) + G(z,z,y)$$

Again we have

$$G(y,y,z) \leq G(y,z,x) = G(x,y,z) \leq G(x,x,y) + G(x,x,z)$$

Then we obtain

$$-G(z,z,x) \leq G(z,z,y) - G(x,x,y) \leq G(z,z,x)$$

Hence

$$|G(z,z,y) - G(x,x,y)| \leq G(z,z,x) \qquad (4.1)$$

Theorem 4.1. *If (x_n) is an $I-$Cauchy sequence in X then it is $I-$localized everywhere.*

Proof. Let (x_n) is an $I-$Cauchy sequence in X, *then* for every $\varepsilon > 0$ there exists a $n_0 = n_0(\varepsilon)$ such that the set $A(\varepsilon) = \{n \in \mathbb{N}: G(x_n, x_n, x_{n_0}) \geq \varepsilon\} \in I$. Using the Lemma 4.1, we have $|G(x_n, x_n, x) - G(x_{n_0}, x_{n_0}, x)| \leq G(x_n, x_n, x_{n_0})$ so we obtain $\{n \in \mathbb{N}: |G(x_n, x_n, x) - G(x_{n_0}, x_{n_0}, x)| \geq \varepsilon\} \subset \{n \in \mathbb{N}: G(x_n, x_n, x_{n_0}) \geq \varepsilon\} \in I$

This shows that for each $x \in X$ the number sequence $(G(x_{n_0}, x_{n_0}, x))$ is $I-$convergent. Hence the sequence (x_n) is $I-$localized everywhere.

Definition 4.3. *A sequence (x_n) is said to be I^*-localized in X, if for each $x \in X$ the real sequence $(G(x_n, x_n, x))$ is I^*-convergent.*

Theorem 4.2. *Let I be an admissible ideal. If a sequence (x_n) in X is I^*-localized on the subset $M \subset X$, then (x_n) is $I-$localized on the set M and $loc_{I^*}(x_n) \subset loc_I(x_n)$.*

Proof. Let (x_n) be I^*-localized on $M \subset X$, by Definition 4.2, then for each $x \in X$ the number sequence $(G(x_n, x_n, x))$ is I^*-convergent.Therefore, there exist a set $H \in I$ such that for $H^C = \mathbb{N}\backslash H = \{k_1 < k_2 < \cdots < k_j\}$ we have $\lim_{j\to\infty} G(x_j, x_j, x)$ for all $x \in M$. By definition then $G(x_n, x_n, x)$ is an I^*-Cauchy sequence which implies the $G(x_n, x_n, x)$ is an $I-$Cauchy. Hence, for each $x \in M$ the number sequence $(G(x_n, x_n, x))$ $I-$convergent, i. e. (x_n) is $I-$localized on M and consequence $loc_{I^*}(x_n) \subset loc_I(x_n)$.

Proposition 4.1. *Let (X, G) be $G-$metric space, then*

i) If X has no limit point, then I and I^*-localized sequences are the same in X, and $loc_I(x_n) = loc_{I^*}(x_n)$ for any $(x_n) \in X$.

ii) If X has a limit point x, then there is an admissible ideal I for which there exists an $I-$localized (y_n) in X such that (y_n) is not I^*-localized.

It can be proved by standard techniques (see [13]).

Theorem 4.3. *If I satisfies the property (AP) and (x_n) is an $I-$localized on $M \subset X$, then (x_n) is I^*- localized on M.*

Proof. Let I satisfies the property (AP) and (x_n) is an $I-$localized sequence on $E \subset X$. Then by Definition 4.2. the number sequence $(G(x_n, x_n, x))_{n\in\mathbb{N}}$ is $I-$convergent *to* $\rho = \rho(x) \in \mathbb{R}^+$such that for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N}: |G(x_n, x_n, x) - \rho| \geq \varepsilon\} \in I$. Then given $A_1 = \{n \in \mathbb{N}: |G(x_n, x_n, x) - \rho| \geq 1\}$ and for $k \geq 2$ with $k \in \mathbb{N}$, $A_k = \{n \in \mathbb{N}: \frac{1}{k} \leq |G(x_n, x_n, x) - \rho| < \frac{1}{k-1}\}$. It is clear that for each $i, j \in \mathbb{N}$ and $i \neq j$, $A_i \cap A_j = \emptyset$. By the definition of property (AP), there exists a countable family of sets $\{B_1, B_2, B_3, \dots.\}$ such that for $i \in \mathbb{N}$, $A_i \Delta B_i$ is finite set and $\bigcup_{i=1}^{\infty} B_i \in I$.

Then we shall show that sequence (x_n) is I^*- localized. Then it needs to be shown that for each $x \in E$ the number sequence $(G(x_n, x_n, x))_{n\in\mathbb{N}}$ is I^*-convergent to x. Let $\mathbb{N}\backslash B = M = \{m_1, m_2, \dots, m_k: m_{k-1} < m_k\} \mathcal{F}(I)$ such that $\lim_{n\to\infty, n\in M} G(x_n, x_n, x) = \rho$. Let for any $\delta > 0$, chosen a $k \in \mathbb{N}$ such that $\frac{1}{k+1} < \delta$. Then $\{n \in \mathbb{N}: |G(x_n, x_n, x) - \rho| \geq \delta\} \subset \bigcup_{i=1}^{k+1} A_i$. Since $A_i \Delta B_i$ is finite for $i = 1, 2, \dots, k+1$, then we chose $n_0 \in \mathbb{N}$ such that we obtained

$$\left(\bigcup_{i=1}^{k+1} B_i\right) \cap \{n \in \mathbb{N}: n \geq n_0\}$$
$$= \left(\bigcup_{i=1}^{k+1} A_i\right) \cap \{n \in \mathbb{N}: n \geq n_0\}$$

If $n \geq n_0$ and $n \notin B$, then $n \notin \bigcup_{i=1}^{k+1} B_i$ and consecuence $n \notin \bigcup_{i=1}^{k+1} A_i$ and now we have $|G(x_n, x_n, x) - \rho| < \frac{1}{k+1} < \delta$. With that said for $x \in E$, the number sequence $(G(x_n, x_n, x))_{n\in\mathbb{N}}$ is I^*- convergent. Hence (x_n) is I^*- localized.

Theorem 4.4. *If X has a limit point and every $I-$localized sequence implies I^*-localized then I will have the property (AP).*

Proof. Let x is a limit point in X and by definition there exists a sequence (x_n) in X such that $x = \lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ for $a_n = G(x_n, x_n, x)$ with $n \in \mathbb{N}$. Let for $n \in \mathbb{N}$, (A_n) is a mutually disjoint family and $A_n \neq \emptyset$, $A_n \in I$.

Given sequence $(y_n) = (x_j)$ with $n \in A_j$. For each $\delta > 0$, chosen $m \in \mathbb{N}$ such that $a_m < \delta$. Then $A(\delta) = \{n \in \mathbb{N}: G(y_n, y_n, x) \geq \delta\} \subset A_1 \cup A_2 \cup \dots \cup A_m$. So, we obtain $A(\varepsilon) \in I$ and $I - \lim_{n\to\infty}(y_n) = x$. Then we have (y_n) is $I-$localized sequence in X. Since (y_n) is also I^*- localized sequence in X such that $I^* - \lim_{n\to\infty}(y_n) = x$. Then there exists $B \in I$ such that for $M = \mathbb{N}\backslash B = \{m_1, m_2, \dots, m_k: m_{k-1} < m_k\}$ we have $\lim_{k\to\infty} y_{m_k} = x$. Then for each $j \in \mathbb{N}$, $B_j = A_j \cap B$ and

$B_j \in I$. And so $\bigcup_{j=1}^{\infty} B_j = B \cap \bigcup_{j=1}^{\infty} A_j \subset B$. With that said $j \in \mathbb{N}$, $\bigcup_{j=1}^{\infty} B_j \in I$.

Since $\lim_{k\to\infty} y_{m_k} = x$, then A_j has only a finite number of elements that are same as the set of M. Then there exists $k_0 \in \mathbb{N}$ such that $A_j \subset (A_j \cap B) \cup \{m_1, m_2, \dots, m_{k_0}\}$. Hence, $A_j \Delta B_j = A_j \backslash B_j \subset \{m_1, m_2, \dots, m_{k_0}\}$ and It can be concluded that for every $j \in \mathbb{N}$, $A_j \Delta B_j$ is a finite set and it is proven that ideal I satisfies the property (AP).

Definition 4.4. *Let a sequence (x_n) on X is called I −bounded if there exists $x \in X$ such that for any $B \in \mathbb{R}$, $B > 0$ $\{n \in \mathbb{N}: G(x_n, x_n, x) > B\} \in I$.*

Theorem 4.5. *Every I −localized sequence is I −bounded.*

Proof. Let (x_n) be I −localized on $M \subset X$. By the Definition 4.2. for all $x \in M$ the number sequence $(G(x_n, x_n, x))_{n\in\mathbb{N}}$ is I −convergent. Let $(G(x_n, x_n, x))_{n\in\mathbb{N}}$ conveges to $\beta = \beta(x) \in \mathbb{R}$. And given $K > 0$, such that $\{n \in \mathbb{N}: |G(x_n, x_n, x) - \beta| > K\} \in I$. Then

$$\{n \in \mathbb{N}: G(x_n, x_n, x) - \beta > K\} \cup \{n \in \mathbb{N}: G(x_n, x_n, x) - \beta > -K\} \in I.$$

Therefore $\{n \in \mathbb{N}: G(x_n, x_{n_0}, x_{n_0}) > \beta + K\} \in I$ and this shows that (x_n) is I −bounded.

Theorem 4.6. *Let I be an admissible ideal with the property (AP) and $L = loc_I(x_n)$. Let z is a point in X such that for any $\varepsilon > 0$ there exists $x \in L$ satisfying $\{n \in \mathbb{N}: |G(x_n, x_n, x) - G(x_n, x_n, z)| \geq \varepsilon\} \in I$, then $z \in L$*

Proof. The proof of this theorem follows the same general steps as approach as in [16]

V. Conclusion

In this paper, the notion of I and I^* −Cauchy sequence, I and I^* −localized sequence, and the relation between I −Cauchy and G −convergence in G −metric space. It is also known that G −metric space is one of the generalizations of metric space. As further work, it is also desirable to study these properties and also to study other properties on other generalized forms of metric spaces such as M −metric spaces, cone metric spaces, and so on.

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