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Impact of the Properties of the Elliptical Birefringent Optical Fiber on the Nature and Propagation of Wave and Solitary Wave Solutions

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Abstract— *In this work, we evaluate the impact of the characteristic properties of the birefringent optical fiber on the choice of the types of solutions likely to propagate there. We assume that the optical fiber is bathed in an environment where any variation within it is subject to a coefficient of variation and that beyond the coefficients of dispersion, dissipation and nonlinear, the spatial and temporal variations are subject to characteristic coefficients. Subsequently, the solution waveforms that we want to propagate in the fiber or to verify the coupled nonlinear partial differential equation characterizing the dynamics of propagation in this transmission medium are selected. This is done using readjustments or modifications imposed on these characteristic properties, materialized by the coefficients of the terms of the coupled nonlinear partial differential equations. The choice of these solutions is conditioned by the constraint relations which bind the characteristic coefficients of the terms of the equations.*

Keywords— *birefringent optical fiber, solitary wave, characteristic coefficient, i-B functions, propagation; nonlinear, dispersive, dissipative, partial differential equation.*

I. INTRODUCTION

Optical fiber is currently the waveguide that is most at the heart of modern telecommunications. Since the 1920s, as certain reference books on the subject indicate to us [1-4], the date of the first speculations by scientists on this transmission medium, its interest has grown to the point where we no longer speak of telecommunications without alluding to this famous fiber as its name indicates. These fibers are microscopic the size of a strand of hair and most often made from glass (silica) or plastic. Depending on the polarization mode, a basic distinction is made between single-mode fibers and multi-mode fibers, for which the works cited above have largely provided explanations.

The anguish over fiber optics has remained intact and this is reflected in many articles which have been published to this day [5-10]. The aim of these numerous publications is precisely to push the limits of perfection concerning the propagation of the signals in this waveguide.

Science is ultimately a discipline that allows us to push the limits of understanding the phenomena of the universe or simply to discover certain hidden facets of the universe, we subscribe to this logic to be interested in the propagation of waves in birefringent optical fibers. This phenomenon arises from the fact that the fiber behaves like it had two refractive indices and at the same time favoring the propagation of two signals, even if it can be noted that these two signals move at different speeds, one which goes faster while the other goes slowly.

But not to stay with the highly scientific definition, we can simply define the birefringent optical fiber as a fiber which simultaneously propagates two different signals or two different waves; even as nowadays there are fibers which propagate several signals at the same time, the starting point remains the birefringent fiber.

A signal which propagates in the optical fiber, as demonstrated by numerous studies, has to cope with numerous phenomena such as dispersion; dissipation, non-linearity, cross-phase modulation, selft-phase modulation, etc. which are most often at the origin of instabilities or even of the total disappearance of the signal emitted in the waveguide [11-13]. The observation we make is that these phenomena are closely related to the nature of the waveguide, that is to say depend on its characteristic properties. To this end, we want to demonstrate that the waveguide in general and the birefringent optical fiber in particular sufficiently impact through its properties the wave that propagates inside. In other words, each characteristic property of the fiber is more favorable to the propagation of a certain type of signal and that a better propagation of a designated signal can be done through the modification of the initial properties of the propagation medium or simply the adaptation of the properties so as to favor the movement of the signal in the support [14,15].

The work that we try to present in this manuscript, demonstrates analytically that a good propagation of the waves in the fiber is linked to a judicious choice of the characteristic properties of the fiber. What is the procedure? To achieve this, we assume that the birefringent optical fiber is in a medium where each effect has a coefficient, that is to say in addition to the coefficients of dispersion, dissipation and nonlinearity that are taken into account the most; often in the modeling, we will assume that there are also coefficients of spatial and even temporal variation.

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This being the case, we solve the coupled nonlinear partial differential equations which govern the propagation in the elliptical birefringent optical fiber, by means of an appropriate choice of the form of solution whose analytical sequence is embodied by the iB-function. The choice of the iB-function here is not a matter of chance; it is because this form of solution embodies on it several forms of functions which can simply lead to different solutions, in particular the traveling wave solutions, solitary wave solutions and others [16-20]. Beyond the choice of the solution function, we obtain through the various calculations, the constraint equations linking the characteristic coefficients of the terms of the equations, which allow choosing the appropriate solutions. These constraint equations in some cases propose the modification of the structure of the equation and naturally of the fibers in order to obtain solutions.

If many works concerning solitary and traveling waves have already been published in optical fibers and other waveguides [21-43], very few tackle it from our angle in this article.

This work is broadly organized as follows: Section 2 describes and presents the coupled nonlinear partial differential equations that govern wave propagation in birefringent optical fiber, Section 3, draws up the range equations of coefficients to be determined, The pairs (n, m) and (n', m') for which there is a grouping of the terms of the range equations and the fields of possibilities of the solutions are established in section 4. The possible solutions and the impact of the properties of the transmission medium are inventoried in section 5. Some trigonometric solutions are proposed in section 6; section 7 is devoted to the propagation of some obtained solutions and a conclusion closes the analyzes.

II. Equations That Govern Wave Propagation In Birefringent Optical Fibers

The equations that model the dynamics of propagation in birefringent optical fiber vary depending on whether the fiber is linear, circular or elliptical. Thus, depending on the angle θ of incidence of injection of the signal into the fiber, the fiber is linear birefringent when the angle is $\theta = 0$, circular when the angle is $\theta = \pi/2$ and elliptical when the angle is any. Regarding this work, we are interested in the case of the elliptical birefringent optical fiber. Thus, if we assume that the two polarization axes are oriented by the Ox and Oy directions, the equations which model the dynamics of signal propagation are given by,

$$\frac{\partial A_x}{\partial z} + \beta_{1x}\frac{\partial A_x}{\partial t} + i\frac{\beta_2}{2}\frac{\partial^2 A_x}{\partial \tau^2} + \frac{\alpha}{2}A_x - i\gamma\left(\left|A_x\right|^2 + B\left|A_y\right|^2\right)A_x = 0, \quad (1)$$

and

$$\frac{\partial A_y}{\partial z} + \beta_{1y}\frac{\partial A_y}{\partial t} + i\frac{\beta_2}{2}\frac{\partial^2 A_y}{\partial \tau^2} + \frac{\alpha}{2}A_y - i\gamma\left(\left|A_y\right|^2 + B\left|A_x\right|^2\right)A_y = 0, \quad (2)$$

where A_x and A_y are the amplitudes of the signals in the two axes, β_{1x}, β_{1y} are the coefficients of dispersion ; α the coefficient of dissipation, γ and B the coefficients of nonlinearity. To remain in the philosophy of our study, we attribute to terms of (1) and (2) arbitrary coefficients $n_i\left(i = 1,2,3,...,11\right)$ and these equations take general forms

$$n_0\frac{\partial U_1}{\partial z} + n_1\frac{\partial U_1}{\partial t} + in_2\frac{\partial^2 U_1}{\partial \tau^2} + n_3U_1 - i\left(n_4\left|U_1\right|^2 + n_5\left|U_2\right|^2\right)U_1 = 0, \quad (3)$$

and

$$n_6\frac{\partial U_2}{\partial z} + n_7\frac{\partial U_2}{\partial t} + in_8\frac{\partial^2 U_2}{\partial \tau^2} + n_9U_2 - i\left(n_{10}\left|U_2\right|^2 + n_{11}\left|U_1\right|^2\right)U_2 = 0, \quad (4)$$

where $n_i\left(i = 1,2,...,11\right)$ are the amplitudes of the signals which propagate in the optical fiber. Equations (3) and (4) are equations for which we actually want to construct solutions. Due to the possibilities of solutions offered by the iB-functions, we choose to construct the solutions of these equations in the forms

$$U_1\left(z,t\right) = aJ_{n,m}\left(\alpha z - \alpha_0 t\right)\exp\left[-i\left(kz - \omega t\right)\right], \quad (5)$$

and

$$U_2\left(z,t\right) = bJ_{n',m'}\left(\alpha z - \alpha_0 t\right)\exp\left[-i\left(kz - \omega t\right)\right], \quad (6)$$

where a, b, n, m, n', m', α, α_0, k and ω are the constants to be determined. $J_{n,m}\left(\alpha t - \alpha_0 t\right)$, and $J_{n',m'}\left(\alpha z - \alpha_0 t\right)$, are iB-functions which can relate to hyperbolic and trigonometric functions according to the choice of its parameters[16-20].

In order to facilitate the calculations we set the change of variables $U = aJ_{n,m}\left(\alpha z - \alpha_0 t\right)$ and $U = bJ_{n',m'}\left(\alpha z - \alpha_0 t\right)$ and equations (5) and (6) are written

$$U_1\left(z,t\right) = U\left(z,t\right)\exp\left[-i\left(kz - \omega t\right)\right], \quad (7)$$

and

$$U_2\left(z,t\right) = V\left(z,t\right)\exp\left[-i\left(kz - \omega t\right)\right]. \quad (8)$$

The evaluation of the different terms of the (7) and (8) allows to transform them as follows

$$n_0\frac{\partial U}{\partial z} + \left(n_1 - 2n_2\omega\right)\frac{\partial U}{\partial t} + in_2\frac{\partial^2 U}{\partial t^2} + \left[n_3 + i\left(n_1\omega - n_0k - n_2\omega^2\right)\right]U$$
$$-i\left(n_4\left|U\right|^2 + n_5\left|V\right|^2\right)U = 0, \quad (9)$$

and

$$n_6\frac{\partial V}{\partial z} + \left(n_7 - 2n_8\omega\right)\frac{\partial V}{\partial t} + in_8\frac{\partial^2 V}{\partial t^2} + \left[n_9 + i\left(n_7\omega - n_6k - n_8\omega^2\right)\right]V$$
$$-i\left(n_{10}\left|V\right|^2 + n_{11}\left|U\right|^2\right)V = 0. \quad (10)$$

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By making the change of variable $\xi = \alpha z - \alpha_0 t$, (9) and (10) become

$$\left[n_0\alpha - n_1\alpha_0 + 2n_2\alpha_0\omega\right]\frac{\partial U}{\partial \xi} + in_2\alpha_0^2\frac{\partial^2 U}{\partial \xi^2} + \left[n_3 + i\left(n_1\omega - n_0k - n_2\omega^2\right)\right]U - i\left[n_4|U|^2 + n_5|V|^2\right]U = 0, \quad (11)$$

and

$$\left[n_6\alpha - n_7\alpha_0 + 2n_8\alpha_0\omega\right]\frac{\partial V}{\partial \xi} + in_8\alpha_0^2\frac{\partial^2 U}{\partial \xi^2} + \left[n_9 + i\left(n_7\omega - n_6k - n_8\omega^2\right)\right]V - i\left[n_{10}|V|^2 + n_{11}|U|^2\right]V = 0. \quad (12)$$

After all the simplifying transformations, the problem is reduced to solving (11) and (12).

III. COEFFICIENT RANGE EQUATIONS

The transformations performed in the previous section require to look for the solutions of (11) and (12) in the forms

$$U(\xi) = aJ_{n,m}(\xi), \quad (13)$$

and

$$V(\xi) = aJ_{n',m'}(\xi). \quad (14)$$

Thus, the insertion of ansatz (13) and (14) in (11) and (12) leads respectively to the equations

$$\begin{aligned}
&am\left[n_0\alpha - n_1\alpha_0 + 2n_2\alpha_0\omega\right]J_{n-1,m-1} \\
&-an\left[n_0\alpha - n_1\alpha_0 + 2n_2\alpha_0\omega\right]J_{n+1,m+1} \\
&+in_2\alpha_0^2 am(n-1)J_{n-2,m-2} \\
&-in_2\alpha_0^2\left[am(n-1) + an(m+1)\right]J_{n,m} \\
&+in_2\alpha_0^2 an(n+1)J_{n+2,m+2} \\
&+a\left[n_3 + i\left(n_1\omega - n_0k - n_2\omega^2\right)\right]J_{n,m} \\
&-in_4a|a|^2 J_{3n,3m} - in_5a|b|^2 J_{2n'+n,2m'+m} = 0,
\end{aligned} \quad (15)$$

and

$$\begin{aligned}
&bm'\left[n_6\alpha - n_7\alpha_0 + 2n_8\alpha_0\omega\right]J_{n'-1,m'-1} \\
&-bn'\left[n_6\alpha - n_7\alpha_0 + 2n_8\alpha_0\omega\right]J_{n'+1,m'+1} \\
&+in_8\alpha_0^2 am'(n'-1)J_{n'-2,m'-2} \\
&-in_8\alpha_0^2\left[m'(n'-1) + bn'(m'+1)\right]J_{n',m'} \\
&+in_8\alpha_0^2 bn'(n'+1)J_{n'+2,m'+2} \\
&+b\left[n_9 + i\left(n_7\omega - n_6k - n_8\omega^2\right)\right]J_{n',m'} \\
&-in_{10}b|b|^2 J_{3n',3m'} - in_{11}b|a|^2 J_{2n+n',2m+m'} = 0.
\end{aligned} \quad (16)$$

Equations (15) and (16) are main equations which will be at the center of the discussion for the search for solutions. They are called the coefficient range equations because they are the basis for determining the constants relating to the forms of expected solutions.

IV. PAIRS (n, m) AND (n', m') FOR WHICH THERE IS A GROUPING OF THE TERMS OF (16) AND (17) AS WELL AS THE FIELD OF POSSIBILITIES OF SOLUTIONS

The search for pairs for which there is a grouping of terms in (15) and (16) requires to solve 12 systems of equations involving n, m, n' and m', and at the end of which we obtain the following values

$$n, m \in \{-1, -1/2, 0, 1/2, 1\}, \quad (17)$$

and

$$n', m' \in \{-1, -1/2, 0, 1/2, 1\}. \quad (18)$$

In this search for pairs favoring the grouping of terms, we have in each case two non-explicit relations such as $n = n'$ and $m = m'$. We notice that the probabilities of obtaining the pairs ($-1,-1$), ($-1/2, 1/2$), ($0,0$), ($1/2, 1/2$) and ($1,1$) are equal such that

$$P(-1,-1) = P(-1/2,-1/2) = P(0,0) = P(1/2,1/2) = P(1,1) = 1/6$$

There are no dominant pairs and under these conditions, the search field for solutions is vast and it is necessary to combine several pairs if not all the pairs resulting from the two fields (n, m) and (n', m') to search for non-trivial solutions.

The field of search for solutions is vast because there are no dominant pairs, that is to say ones possessing high probabilities that they will be used to constitute the restricted field of possibilities of solutions.

The fields of the possibilities of solutions, that is to say the tables regrouping the values of the pairs for which the solutions will be sought are given below.

Table 1: The field of possible solutions associated with the pairs (n, m)

(n, m)	-1	$-1/2$	0	$1/2$	1
-1	$(-1,-1)$	$(-1,-1/2)$	$(-1,0)$	$(-1,1/2)$	$(-1,1)$
$-1/2$	$(-1/2,-1)$	$(-1/2,-1/2)$	$(-1/2,0)$	$(-1/2,1/2)$	$(-1/2,1)$
0	$(0,-1)$	$(0,-1/2)$	$(0,0)$	$(0,1/2)$	$(0,1)$
$1/2$	$(1/2,-1)$	$(1/2,-1/2)$	$(1/2,0)$	$(1/2,1/2)$	$(1/2,1)$
1	$(1,-1)$	$(1,-1/2)$	$(1,0)$	$(1,1/2)$	$(1,1)$

Table 2: The field of possible solutions associated with the pairs (n', m')

(n', m')	-1	$-1/2$	0	$1/2$	1
-1	$(-1,-1)$	$(-1,-1/2)$	$(-1,0)$	$(-1,1/2)$	$(-1,1)$
$-1/2$	$(-1/2,-1)$	$(-1/2,-1/2)$	$(-1/2,0)$	$(-1/2,1/2)$	$(-1/2,1)$
0	$(0,-1)$	$(0,-1/2)$	$(0,0)$	$(0,1/2)$	$(0,1)$
$1/2$	$(1/2,-1)$	$(1/2,-1/2)$	$(1/2,0)$	$(1/2,1/2)$	$(1/2,1)$
1	$(1,-1)$	$(1,-1/2)$	$(1,0)$	$(1,1/2)$	$(1,1)$

V. SOLUTIONS AND IMPACT ON THE TRANSMISSION MEDIUM

First, we focus the search for solutions on the pairs ($-1,-1$), ($-1/2,-1/2$), ($1/2,1/2$) and ($1,1$). Since these are the solutions excluding the center pair ($0,0$), we simplify

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the range equations (14) and (15) respectively by $J_{n,m}$ and $J_{n',m'}$ just for the purpose of making calculations easier and thereby obtain the following simplified range equations

$$\begin{aligned}
&-in_2\alpha_0^2\left[am(n-1)+an(m+1)\right]\\
&+a\left[n_3+i\left(n_1\omega-n_0k-n_2\omega^2\right)\right]\\
&+am\left[n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right]J_{-1,-1}\\
&-an\left[n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right]J_{1,1}\\
&+in_2\alpha_0^2am(n-1)J_{-2,-2}\\
&+in_2\alpha_0^2an(n+1)J_{2,2}-in_4a|a|^2J_{2n,2m}\\
&-in_5a|b|^2J_{2n',2m'}=0,
\end{aligned}\tag{19}$$

and

$$\begin{aligned}
&-in_8\alpha_0^2\left[m'(n'-1)+bn'(m'+1)\right]\\
&+b\left[n_9+i\left(n_7\omega-n_6k-n_8\omega^2\right)\right]\\
&+bm'\left[n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right]J_{-1,-1}\\
&-bn'\left[n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right]J_{1,1}\\
&+in_8\alpha_0^2am'(n'-1)J_{-2,-2}+in_8\alpha_0^2bn'(n'+1)J_{2,2}\\
&-in_{10}b|b|^2J_{2n',2m'}-in_{11}b|a|^2J_{2n,2m}=0.
\end{aligned}\tag{20}$$

The various cases enumerated above are examined below:

- case $(n,m)=(-1/2,-1/2)$ and $(n',m')=(-1/2,-1/2)$

For values of the pairs (n,m) and (n',m') above, (18) and (19) give respectively

$$\begin{aligned}
&a\left[-i\frac{n_2\alpha_0^2}{2}+n_3+i\left(n_1\omega-n_0k-n_2\omega^2\right)\right]\\
&-\left[\begin{array}{l}\frac{1}{2}a\left(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right)\\+ia|a|^2n_4+in_5a|b|^2\end{array}\right]J_{-1,-1}\\
&+\frac{1}{2}a\left(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right)J_{1,1}\\
&+\frac{3}{4}in_2\alpha_0^2aJ_{-2,-2}-\frac{in_2}{4}\alpha_0^2aJ_{2,2}=0,
\end{aligned}\tag{21}$$

and

$$\begin{aligned}
&b\left[-i\frac{n_8\alpha_0^2}{2}+n_9+i\left(n_1\omega-n_6k-n_8\omega^2\right)\right]\\
&-\left[\begin{array}{l}\frac{1}{2}a\left(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right)\\+ib|b|^2n_{10}+in_5b|a|^2\end{array}\right]J_{-1,-1}\\
&+\frac{1}{2}b\left(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right)J_{1,1}\\
&+\frac{3}{4}in_8\alpha_0^2bJ_{-2,-2}-\frac{in_8}{4}\alpha_0^2bJ_{2,2}=0.
\end{aligned}\tag{22}$$

Since we are looking for the non-trivial solutions ($a\neq 0, b\neq 0$), (21) and (22) hold if and only if we have the following equations

$$n_2=n_8=0,\tag{23}$$

$$n_3+i\left(n_1\omega-n_0k\right)=0,\tag{24}$$

$$\frac{1}{2}\left(n_0\alpha-n_1\alpha_0\right)+i\left(|a|^2n_4+n_5|b|^2\right)=0,\tag{25}$$

$$n_9+i\left(n_7\omega-n_6k\right)=0,\tag{26}$$

$$\frac{1}{2}\left(n_6\alpha-n_7\alpha_0\right)+i\left(|a|^2n_{10}+n_{11}|b|^2\right)=0,\tag{27}$$

and

$$\frac{1}{2}\left(n_6\alpha-n_7\alpha_0\right)=0.\tag{28}$$

With regard to (23) to (28), (24) and (26) give rise to three possibilities of analysis, namely the case where n_3 and n_9 are pure imaginaries, the case where equations (24) and (26) are complex with the imaginary parts and the real parts which are zero and the case where n_0, n_1, n_6 and n_7 are pure imaginary ($n_0\rightarrow in_0$, $n_1\rightarrow in_1$, $n_6\rightarrow in_6$, $n_7\rightarrow in_7$ / $i^2=-1$).

- In the case where n_3 and n_9 are pure imaginary, then the constraint relations for which the solutions are possible as well as the values of a and b are

$$\frac{n_3}{n_9}=\frac{n_0k-n_1\omega}{n_6k-n_7\omega},\tag{29}$$

$$\frac{n_0}{n_6}=\frac{n_1}{n_7},\tag{30}$$

$$\frac{n_4}{n_{11}}=\frac{n_5}{n_{10}},\tag{31}$$

and

$$|b|^2=-\frac{n_4}{n_5}|a|^2 \ or\ |b|^2=-\frac{n_{11}}{n_{10}}|a|^2.\tag{32}$$

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We obtain from relation (32), the magnitudes of a and b such that

$$|b|=\sqrt{-\frac{n_4}{n_5}}|a|,\ n_4n_5\prec 0\ or\ |b|=\sqrt{-\frac{n_{11}}{n_{10}}}|a|,\ n_{11}n_{10}\prec 0. \quad (33)$$

To give solutions in this case, we will also distinguish two outcomes, the case where a and b are real and the case where a and b are complex even if the real case is included in the complex case.

– First case: a and b are real: the solutions are given by

$$U(\xi)=aJ_{-\frac{1}{2},-\frac{1}{2}}(\xi)\Rightarrow U_1(z,t)=aJ_{-\frac{1}{2},-\frac{1}{2}}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right], \quad (34)$$

and

$$V(\xi)=\pm a\sqrt{\frac{-n_4}{n_5}}J_{-\frac{1}{2},-\frac{1}{2}}(\xi) \quad (35)$$

$$\Rightarrow U_2(z,t)=\pm a\sqrt{\frac{-n_4}{n_5}}J_{-\frac{1}{2},-\frac{1}{2}}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right],$$

– Second case: a and b are complex

In this sub-case, there exist real number β such that $a=|a|\exp i\beta$ and $b=|a|\sqrt{\frac{-n_4}{n_5}}\exp i\beta$ and the solutions are given by

$$U(\xi)=|a|J_{-\frac{1}{2},-\frac{1}{2}}(\xi)\exp i\beta$$

$$\Rightarrow U_1(z,t)=|a|J_{-\frac{1}{2},-\frac{1}{2}}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\beta)\right], \quad (36)$$

and

$$V(\xi)=|a|\sqrt{\frac{-n_4}{n_5}}J_{-\frac{1}{2},-\frac{1}{2}}(\xi)\exp i\beta$$

$$\Rightarrow U_2(z,t)=|a|\sqrt{\frac{-n_4}{n_5}}J_{-\frac{1}{2},-\frac{1}{2}}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\beta)\right]. \quad (37)$$

The solutions obtained here, are obtained by means of the fact that $n_2=n_8=0$. The physical interpretation that holds corresponds to the case where the birefringent optical fiber is very weakly dispersive. But the signals that propagate there continue to influence each other.

The system of coupled nonlinear partial differential equations which describes the evolution of the waves within it on the mathematical level reduces to

$$n_0\frac{\partial U_1}{\partial z}+n_1\frac{\partial U_1}{\partial t}+n_3U_1-i\left(n_4|U_1|^2+n_5|U_2|^2\right)U_1=0, \quad (38)$$

and

$$n_6\frac{\partial U_2}{\partial z}+n_7\frac{\partial U_2}{\partial t}+n_9U_2-i\left(n_{10}|U_2|^2+n_{11}|U_1|^2\right)U_2=0. \quad (39)$$

- In the case where n_3 and n_9 are complex, then the constraint relations for which the solutions are possible as well as the values of a and b remain the same except that henceforth we will have in addition $n_3=n_9=0$ and (29) which change into

$$\frac{n_0}{n_1}=\frac{n_6}{n_7}=\frac{\omega}{k}. \quad (40)$$

The solutions of the equations remain those obtained in the equations (34), ..., (37). Except that in addition to the fact that we already have $n_2=n_8=0$, we add $n_3=n_9=0$. The physical interpretation returns to the case where the fiber is very weakly dispersive and very weakly dissipative. Overall, therefore, the solutions obtained in this study portion are likely to be propagated in very weakly dispersive elliptical birefringent optical fibers or in elliptical birefringent optical fibers which are both very weakly dispersive and very weakly dissipative.

In this case where the birefringent fiber is very weakly dispersive and very weakly dissipative, the equations which model the dynamics of evolution of the wave within it are reduced to

$$n_0\frac{\partial U_1}{\partial z}+n_1\frac{\partial U_1}{\partial t}-i\left(n_4|U_1|^2+n_5|U_2|^2\right)U_1=0, \quad (41)$$

and

$$n_6\frac{\partial U_2}{\partial z}+n_7\frac{\partial U_2}{\partial t}-i\left(n_{10}|U_2|^2+n_{11}|U_1|^2\right)U_2=0. \quad (42)$$

- Case where $n_0\to in_0,\ n_1\to in_1,\ n_6\to in_6,\ n_7\to in_7\ /\ i^2=-1$

The resolution of equations (24) and (26) permit to obtain

$$|a|=\left[\frac{(n_{10}n_1-n_5n_7)\alpha_0+(n_5n_6-n_{10}n_0)\alpha}{2(n_{10}n_4-n_5n_{11})}\right]^{\frac{1}{2}}, \quad (43)$$

and

$$|b|=\left[\frac{n_5(n_7n_4-n_1n_{11})\alpha_0+(n_0n_5n_{11}-n_0^2n_4+n_5n_6n_4-n_{10}n_0n_4)\alpha}{2n_5(n_{10}n_4-n_5n_{11})}\right]^{\frac{1}{2}}. \quad (44)$$

Then there exists a real θ such that we have

$$a=\left[\frac{(n_{10}n_1-n_5n_7)\alpha_0+(n_5n_6-n_{10}n_0)\alpha}{2(n_{10}n_4-n_5n_{11})}\right]^{\frac{1}{2}}\exp i\theta, \quad (45)$$

and

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$$b=\left[\frac{n_5\left(n_7n_4-n_1n_{11}\right)\alpha_0+\left(n_0n_5n_{11}-n_0^2n_4+n_5n_6n_4-n_{10}n_0n_4\right)\alpha}{2n_5\left(n_{10}n_4-n_5n_{11}\right)}\right]^{\frac{1}{2}}\exp i\theta.$$

(46)

provided that the quantities under radical are all positive. The solutions in this case are

$$U_1(z,t)=\left[\frac{\left(n_{10}n_1-n_5n_7\right)\alpha_0+\left(n_5n_6-n_{10}n_0\right)\alpha}{2\left(n_{10}n_4-n_5n_{11}\right)}\right]^{\frac{1}{2}}J_{-\frac{1}{2},-\frac{1}{2}}\left(\alpha z-\alpha_0t\right)\exp\left[-i\left(kz-\omega t-\theta\right)\right], \quad (47)$$

and

$$U_2(z,t)=\left[\frac{n_5\left(n_7n_4-n_1n_{11}\right)\alpha_0+\left(n_0n_5n_{11}-n_0^2n_4+n_5n_6n_4-n_{10}n_0n_4\right)\alpha}{2n_5\left(n_{10}n_4-n_5n_{11}\right)}\right]^{\frac{1}{2}}J_{-\frac{1}{2},-\frac{1}{2}}\left(\alpha z-\alpha_0t\right)\exp\left[-i\left(kz-\omega t-\theta\right)\right]. \quad (48)$$

As in the cases treated above, these solutions are also valid both for very weakly dispersive elliptical birefringent optical fibers as well as for very weakly dispersive and very weakly dissipative fibers.

- case $(n,m)=(1/2,1/2)$ and $(n',m')=(1/2,1/2)$

For values of the pairs (n,m) and (n',m') above, (19) and (20) give respectively

$$a\left[-i\frac{n_2\alpha_0^2}{2}+n_3+i\left(n_1\omega-n_0k-n_2\omega^2\right)\right]+\frac{a}{2}\left(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right)J_{-1,-1}-\left[\frac{1}{2}a\left(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right)+ia|a|^2n_4+in_5a|b|^2\right]J_{1,1}+\frac{in_2a\alpha_0^2}{4}J_{-2,-2}+\frac{3in_2\alpha_0^2a}{4}J_{2,2}=0, \quad (49)$$

and

$$b\left[-i\frac{n_8\alpha_0^2}{2}+n_9+i\left(n_7\omega-n_6k-n_8\omega^2\right)\right]+\frac{a}{2}\left(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right)J_{-1,-1}-\left[\frac{1}{2}b\left(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right)+ib|b|^2n_{10}+in_{11}b|a|^2\right]J_{1,1}+\frac{in_8b\alpha_0^2}{4}J_{-2,-2}+\frac{3in_8\alpha_0^2b}{4}J_{2,2}=0. \quad (50)$$

Since we are looking for the non-trivial solutions ($a\neq 0, b\neq 0$), we observe that (43) and (44) hold in the case where we obtain exactly (23) to (28) and the similar analysis lead to the solutions:

- First case: a and b are real: the solutions are given by

$$U(\xi)=aJ_{\frac{1}{2},\frac{1}{2}}(\xi) \Rightarrow U_1(z,t)=aJ_{\frac{1}{2},\frac{1}{2}}\left(\alpha z-\alpha_0t\right)\exp\left[-i\left(kz-\omega t\right)\right], \quad (51)$$

and

$$V(\xi)=\pm a\sqrt{\frac{-n_4}{n_5}}J_{\frac{1}{2},\frac{1}{2}}(\xi) \Rightarrow U_2(z,t)=\pm a\sqrt{\frac{-n_4}{n_5}}J_{\frac{1}{2},\frac{1}{2}}\left(\alpha z-\alpha_0t\right)\exp\left[-i\left(kz-\omega t\right)\right]. \quad (52)$$

- Second case: a and b are complex

In this sub-case, there exist real number θ such that $a=|a|\exp i\theta$ and $b=|a|\sqrt{\frac{-n_4}{n_5}}\exp i\theta$ and the solutions are given by

$$U(\xi)=|a|J_{\frac{1}{2},\frac{1}{2}}(\xi)\exp i\theta \Rightarrow U_1(z,t)=|a|J_{\frac{1}{2},\frac{1}{2}}\left(\alpha z-\alpha_0t\right)\exp\left[-i\left(kz-\omega t-\theta\right)\right], \quad (53)$$

and

$$V(\xi)=|a|\sqrt{\frac{-n_4}{n_5}}J_{-\frac{1}{2},-\frac{1}{2}}(\xi)\exp i\theta \Rightarrow U_2(z,t)=|a|\sqrt{\frac{-n_4}{n_5}}J_{\frac{1}{2},\frac{1}{2}}\left(\alpha z-\alpha_0t\right)\exp\left[-i\left(kz-\omega t-\theta\right)\right]. \quad (54)$$

As in the case for $(n,m)=(-1/2,-1/2)$ and $(n',m')=(-1/2,-1/2)$ the solutions obtained here, are obtained by means of the fact that $n_2=n_8=0$. The physical interpretation that holds corresponds to the case where the birefringent optical fiber is very weakly dispersive. But the signals that propagate there continue to influence each other. The system of coupled nonlinear partial differential equations which describes the evolution of the waves within it is mathematically given by (38) and (39).

- In the case where n_3 and n_9 are complex, then the constraint relations for which the solutions are possible as well as the values of a and b remain the same .

The solutions of equations remain those obtained in (34) to (37). Except that in addition to the fact that we already have $n_2=n_8=0$, we add $n_3=n_9=0$. The physical interpretation returns to the case where the fiber is very weakly dispersive and very weakly dissipative. Overall, therefore, the solutions obtained in this study portion are likely to be propagated in very weakly dispersive elliptical

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birefringent optical fibers or in elliptical birefringent optical fibers which are both very weakly dispersive and very weakly dissipative.
In this case where the birefringent fiber is very weakly dispersive and very weakly dissipative, the equations which model the dynamics of evolution of the wave within it are also given by (41) and (42).

- Case where

$$n_0 \to in_0,\, n_1 \to in_1,\, n_6 \to in_6,\, n_7 \to in_7 \,/\, i^2 = -1$$

The resolution of (25) and (27) permit to obtain similar solutions as above ;

$$|a| = \left[\frac{(n_{10}n_1 - n_5n_7)\alpha_0 + (n_5n_6 - n_{10}n_0)\alpha}{2(n_{10}n_4 - n_5n_{11})}\right]^{\frac{1}{2}}, \qquad (55)$$

and

$$|b| = \left[\frac{n_5(n_7n_4 - n_1n_{11})\alpha_0 + (n_0n_5n_{11} - n_0^2n_4 + n_5n_6n_4 - n_{10}n_0n_4)\alpha}{2n_5(n_{10}n_4 - n_5n_{11})}\right]^{\frac{1}{2}}. \qquad (56)$$

Then there exists a real θ such that we have

$$a = \left[\frac{(n_{10}n_1 - n_5n_7)\alpha_0 + (n_5n_6 - n_{10}n_0)\alpha}{2(n_{10}n_4 - n_5n_{11})}\right]^{\frac{1}{2}} \exp i\theta, \qquad (57)$$

and

$$b = \left[\frac{n_5(n_7n_4 - n_1n_{11})\alpha_0 + (n_0n_5n_{11} - n_0^2n_4 + n_5n_6n_4 - n_{10}n_0n_4)\alpha}{2n_5(n_{10}n_4 - n_5n_{11})}\right]^{\frac{1}{2}} \exp i\gamma. \qquad (58)$$

The solutions in this case are

$$U_1(z,t) = \left[\frac{(n_{10}n_1 - n_5n_7)\alpha_0 + (n_5n_6 - n_{10}n_0)\alpha}{2(n_{10}n_4 - n_5n_{11})}\right]^{\frac{1}{2}} \times J_{\frac{1}{2},\frac{1}{2}}(\alpha z - \alpha_0 t)\exp\left[-i(kz - \omega t - \theta)\right], \qquad (59)$$

and

$$U_2(z,t) = \left[\frac{n_5(n_7n_4 - n_1n_{11})\alpha_0 + (n_0n_5n_{11} - n_0^2n_4 + n_5n_6n_4 - n_{10}n_0n_4)\alpha}{2n_5(n_{10}n_4 - n_5n_{11})}\right]^{\frac{1}{2}} \times J_{\frac{1}{2},\frac{1}{2}}(\alpha z - \alpha_0 t)\exp\left[-i(kz - \omega t - \gamma)\right]. \qquad (60)$$

As in the cases treated above, these solutions are also valid both for very weakly dispersive elliptical birefringent optical fibers as well as for very weakly dispersive and very weakly dissipative fibers.

❖ case $(n,m) = (-1/2,-1/2)$ and $(n',m') = (1/2,1/2)$

Equations (19) and (20) give respectively

$$\begin{aligned} &a\left[-i\frac{n_2\alpha_0^2}{2} + n_3 + i(n_1\omega - n_0k - n_2\omega^2)\right] \\ &-\left[\frac{a}{2}(n_0\alpha - n_1\alpha_0 + 2n_2\alpha_0\omega) + ia|a|^2 n_4\right]J_{-1,-1} \\ &+\left[\frac{1}{2}a(n_0\alpha - n_1\alpha_0 + 2n_2\alpha_0\omega) - in_5a|b|^2\right]J_{1,1} \\ &+\frac{3in_2a\alpha_0^2}{4}J_{-2,-2} - \frac{in_2\alpha_0^2a}{4}J_{2,2} = 0, \end{aligned} \qquad (61)$$

and

$$\begin{aligned} &b\left[-i\frac{n_8\alpha_0^2}{2} + n_9 + i(n_7\omega - n_6k - n_8\omega^2)\right] \\ &+\left[\frac{b}{2}(n_6\alpha - n_7\alpha_0 + 2n_8\alpha_0\omega) - ib|a|^2 n_{11}\right]J_{-1,-1} \\ &-\left[\frac{b}{2}(n_6\alpha - n_7\alpha_0 + 2n_8\alpha_0\omega) + ib|b|^2 n_{10}\right]J_{1,1} \\ &-\frac{in_8b\alpha_0^2}{4}J_{-2,-2} + \frac{3in_8\alpha_0^2b}{4}J_{2,2} = 0. \end{aligned} \qquad (62)$$

Equations (61) and (62) hold if and only if we have constraint equations that follow

$$\frac{n_3}{n_9} = \frac{n_1\omega - n_0k}{n_7\omega - n_6k}, \qquad (63)$$

$$\frac{|b|^2}{|a|^2} = -\frac{n_4}{n_5} = -\frac{n_{11}}{n_{10}}, \qquad (64)$$

and

$$\frac{n_4}{n_5} = \frac{n_{11}}{n_{10}}. \qquad (65)$$

It overflows from the relation (64) that

$$|b| = \sqrt{-\frac{n_4}{n_5}}|b|,\; n_4n_5 \prec 0. \qquad (66)$$

To remain only in two possible cases, the solutions are as follows:

- if a and b are real then the solutions are given by

$$\begin{aligned} &U(\xi) = aJ_{-\frac{1}{2},-\frac{1}{2}}(\xi) \\ &\Rightarrow U_1(z,t) = J_{-\frac{1}{2},-\frac{1}{2}}(\alpha z - \alpha_0 t)\exp\left[-i(kz - \omega t - \theta)\right], \end{aligned} \qquad (67)$$

and

$$\begin{aligned} &V(\xi) = \pm\sqrt{\frac{-n_4}{n_5}}aJ_{\frac{1}{2},\frac{1}{2}}(\xi) \\ &\Rightarrow U_2(z,t) = \pm\sqrt{\frac{-n_4}{n_5}}aJ_{\frac{1}{2},\frac{1}{2}}(\alpha z - \alpha_0 t)\exp\left[-i(kz - \omega t)\right]. \end{aligned} \qquad (68)$$

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- if a and b are complex then there exist a real θ such that the solutions are given by

$$U(\xi)=|a|J_{-\frac{1}{2},-\frac{1}{2}}(\xi)\exp i\theta$$
$$\Rightarrow U_1(z,t)=|a|J_{-\frac{1}{2},-\frac{1}{2}}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\theta)\right], \quad (69)$$

and

$$V(\xi)=|a|\sqrt{\frac{-n_4}{n_5}}J_{\frac{1}{2},\frac{1}{2}}(\xi)\exp i\theta$$
$$\Rightarrow U_2(z,t)=|a|\sqrt{\frac{-n_4}{n_5}}J_{\frac{1}{2},\frac{1}{2}}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\theta)\right]. \quad (70)$$

These solutions also work in very weakly dispersive elliptical birefringent optical fibers as well as in very weakly dispersive and very weakly dissipative elliptical birefringent optical fibers.

We must also notice that these solutions obtained so far can interchange forms in the case where we choose to express a as a function of b ($|a|=\sqrt{\frac{-n_5}{n_4}}|b|$, $n_4 n_5 \prec 0$).

- case $(n,m)=(-1,-1)$ and $(n',m')=(-1,-1)$

Equations (19) and (20) give respectively

$$\begin{aligned}&a\left[n_3+i\left(n_1\omega-n_0k-n_2\omega^2-2n_2\alpha_0^2\right)\right]\\&-a\left(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right)J_{-1,-1}\\&+a\left(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right)J_{1,1}\\&+i\left[2an_2\alpha_0^2-an_5|b|^2-an_4|a|^2\right]J_{-2,-2}=0,\end{aligned} \quad (71)$$

and

$$\begin{aligned}&b\left[n_9+i\left(n_7\omega-n_6k-n_8\omega^2-2n_8\alpha_0^2\right)\right]\\&-a\left(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right)J_{-1,-1}\\&+a\left(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right)J_{1,1}\\&+i\left[2an_8\alpha_0^2-bn_{11}|a|^2-bn_{10}|b|^2\right]J_{-2,-2}=0.\end{aligned} \quad (72)$$

Equations (71) and (72) are satisfied if and only if for values of a and b other than zero, we have

$$n_3+i\left(n_1\omega-n_0k-n_2\omega^2-2n_2\alpha_0^2\right)=0, \quad (73)$$
$$n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega=0, \quad (74)$$
$$2n_2\alpha^2-n_5|b|^2-|a|^2n_4=0, \quad (75)$$
$$n_9+i\left(n_7\omega-n_6k-n_8\omega^2-2n_8\alpha_0^2\right)=0, \quad (76)$$
$$n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega=0, \quad (77)$$

and

$$2n_8\alpha^2-n_{11}|a|^2-|b|^2n_{10}=0. \quad (78)$$

To solve the above equations, we will consider two possibilities: the case where n_3, $n_1\omega-n_0k-n_2\omega^2-2n_2\alpha_0^2$, n_9 and $n_7\omega-n_6k-n_8\omega^2-2n_8\alpha_0^2$ are real and the case where these quantities are complex.

- Case where the coefficients are real

Solving (73) and (76) imposes on us the constraints $n_3=0$, $n_1\omega-n_0k-n_2\omega^2-2n_2\alpha_0^2=0$, $n_9=0$ and $n_7\omega-n_6k-n_8\omega^2-2n_8\alpha_0^2=0$ such that the other constraints given by (74) and (77) remain valid. These conditions being met, the problem is reduced to solving (75) and (78). Thus, (75) and (78) can still be written

$$n_4|a|^2+n_5|b|^2=2n_2\alpha_0^2, \quad (79)$$

and

$$n_{11}|a|^2+n_{10}|b|^2=2n_8\alpha^2. \quad (80)$$

Solving (78) and (79) leads to

$$|a|^2=\frac{2\left(n_5n_8-n_{10}n_2\right)\alpha_0^2}{n_5n_{11}-n_4n_{10}}, \quad (81)$$

and

$$|b|^2=\frac{2\left(n_2n_{11}-n_4n_8\right)\alpha_0^2}{n_5n_{11}-n_4n_{10}}. \quad (82)$$

To determine the expressions of a and b, we also assume two possibilities:

- If a and b are real, then we will have

$$a=\alpha_0\left(\frac{2\left(n_5n_8-n_{10}n_2\right)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}},\ \left(n_5n_8-n_{10}n_2\right)\left(n_5n_{11}-n_4n_{10}\right)\succ 0 \quad (83)$$

and

$$b=\alpha_0\left(\frac{2\left(n_2n_{11}-n_4n_8\right)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}},\ \left(n_2n_{11}-n_4n_8\right)\left(n_5n_{11}-n_4n_{10}\right)\succ 0. \quad (84)$$

The solutions are thus given by

$$U(\xi)=\alpha_0\left(\frac{2\left(n_5n_8-n_{10}n_2\right)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}J_{-1,-1}(\xi)$$
$$\Rightarrow U_1(z,t)=\alpha_0\left(\frac{2\left(n_5n_8-n_{10}n_2\right)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}} \quad (85)$$
$$\times J_{-1,-1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right]$$

and

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$$V(\xi)=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}J_{-1,-1}(\xi)$$

$$\Rightarrow U_2(z,t)=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}} \quad (86)$$

$$\times J_{-1,-1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right].$$

– If a and b are complex, there exist the reals β and θ such that

$$a=\alpha_0\left(\frac{2(n_5n_8-n_{10}n_2)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}\exp i\beta, \quad (87)$$

and

$$b=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}\exp i\theta. \quad (88)$$

The solutions in this case are given by

$$U(\xi)=\alpha_0\left(\frac{2(n_5n_8-n_{10}n_2)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}J_{-1,-1}(\xi)\exp i\beta$$

$$\Rightarrow U_1(z,t)=\alpha_0\left(\frac{2(n_5n_8-n_{10}n_2)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}} \quad (89)$$

$$\times J_{-1,-1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\beta)\right],$$

and

$$V(\xi)=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}J_{-1,-1}(\xi)\exp i\theta$$

$$\Rightarrow U_2(z,t)=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}} \quad (90)$$

$$\times J_{-1,-1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\theta)\right].$$

For $n_3=n_9=0$, the dissipation terms disappear in the initial equations; which translates on the physical level that we are in the cases of very weak dissipations. On the mathematical level, the equations which would admit by means of the various constraint relations, for exact solutions the above solutions are given by

$$n_0\frac{\partial U_1}{\partial z}+n_1\frac{\partial U_1}{\partial t}+in_2\frac{\partial^2 U_1}{\partial \tau^2}-i\left(n_4|U_1|^2+n_5|U_2|^2\right)U_1=0, \quad (91)$$

and

$$n_6\frac{\partial U_2}{\partial z}+n_7\frac{\partial U_2}{\partial t}+in_8\frac{\partial^2 U_2}{\partial \tau^2}-i\left(n_{10}|U_2|^2+n_{11}|U_1|^2\right)U_2=0. \quad (92)$$

- Case where the constants n_3 and n_9 are complex

Under these conditions, we can write $n_3=-i\left(n_1\omega-n_0k-n_2\omega^2-2n_2\alpha_0^2\right)$ and $n_9=-i\left(n_7\omega-n_6k-n_8\omega^2-2n_8\alpha_0^2\right)$. The other constraint relations remaining unchanged and valid, (3) and (4) without undergoing any structural modification admit the same solutions as in the case where $n_3=n_9=0$. On the physical level and without carrying out any checks, we think that this case can be compared to cases of strong dissipations.

❖ case $(n,m)=(1,1)$ and $(n',m')=(1,1)$

Equations (18) and (19) give respectively

$$a\left[n_3+i\left(n_1\omega-n_0k-n_2\omega^2-2n_2\alpha_0^2\right)\right]+a\left(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right)J_{-1,-1}$$
$$-a\left(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega\right)J_{1,1}+i\left(2n_2\alpha_0^2a-a|a|^2n_4-an_5|b|^2\right)J_{2,2}=0, \quad (93)$$

and

$$b\left[n_9+i\left(n_7\omega-n_6k-n_8\omega^2-2n_8\alpha_0^2\right)\right]+b\left(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right)J_{-1,-1}$$
$$-b\left(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega\right)J_{1,1}+i\left(2n_8\alpha_0^2b-b|b|^2n_{10}-n_5b|a|^2\right)J_{2,2}=0. \quad (94)$$

Equations (93) and (94) are verified if we have relations almost identical to the relations obtained in the case $(n,m)=(-1,-1)$ and $(n',m')=(-1,-1)$. For an identical analysis, we obtain the following solutions:

- Case where $n_3=n_9=0$ and $n_3\neq 0, n_9\neq 0$ with a and b real, the solutions are

$$U(\xi)=\alpha_0\left(\frac{2(n_5n_8-n_{10}n_2)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}J_{1,1}(\xi)$$

$$\Rightarrow U_1(z,t)=\alpha_0\left(\frac{2(n_5n_8-n_{10}n_2)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}} \quad (95)$$

$$\times J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right],$$

and

$$V(\xi)=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}J_{1,1}(\xi)$$

$$\Rightarrow U_2(z,t)=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}} \quad (96)$$

$$\times J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right].$$

- Case where $n_3=n_9$ and $n_3\neq 0, n_9\neq 0$ with a and b complex, the solutions are

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$$U(\xi)=\alpha_0\left(\frac{2(n_5n_8-n_{10}n_2)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}J_{1,1}(\xi)\exp i\beta$$

$$\Rightarrow U_1(z,t)=\alpha_0\left(\frac{2(n_5n_8-n_{10}n_2)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}} \quad (97)$$

$$\times J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\beta)\right],$$

and

$$V(\xi)=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}J_{1,1}(\xi)\exp i\theta$$

$$\Rightarrow U_2(z,t)=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}} \quad (98)$$

$$\times J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\theta)\right].$$

We also remain in the very weakly dispersive elliptical birefringent optical fibers with modification of the structure of the initial equations, and in the case of highly dispersive birefringent optical fibers without modification of the structure of the initial equations, with regard to this other series of solutions.

- case $(n,m)=(-1,-1)$ and $(n',m')=(1,1)$

Equations (19) and (20) give respectively

$$\begin{aligned}&a\left[n_3+i\left(n_1\omega-n_0k-n_2\omega^2-2n_2\alpha_0^2\right)\right]\\&-a(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega)J_{-1,-1}\\&+a(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega)J_{1,1}\\&+i\left(2n_2\alpha_0^2a-a|a|^2n_4\right)J_{2,2}-ian_5|b|^2J_{2,2}=0,\end{aligned} \quad (99)$$

and

$$\begin{aligned}&b\left[n_9+i\left(n_7\omega-n_6k-n_8\omega^2-2n_8\alpha_0^2\right)\right]\\&+b(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega)J_{-1,-1}\\&-b(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega)J_{1,1}\\&+i\left(2n_8\alpha_0^2b-b|b|^2n_{10}\right)J_{2,2}-in_{11}b|a|^2J_{-2,-2}=0.\end{aligned} \quad (100)$$

Equations (99) and (100) are verified if and only if we have for $a\neq 0$ and $b\neq 0$ the relations (73), (74), (76) and (77) that still appear in this analytical framework ; except the fact that the relations which mark the difference in the analyzes are given below

$$n_5=n_{11}=0, \quad (101)$$

$$|a|^2=2n_2\alpha_0^2, \quad (102)$$

and

$$|b|^2=2n_8\alpha_0^2. \quad (103)$$

The observation of the obtained equations places us in fact in the case where $n_5=n_{11}=0$, which would simply mean that we are in a scenario where the nonlinear interaction of each wave or signal in the fiber is negligible or simply has no effects. This would also mean that each signal evolves on its own without influence from the other. Equations (3) and (4) which govern the dynamics in this case are

$$n_0\frac{\partial U_1}{\partial z}+n_1\frac{\partial U_1}{\partial t}+in_2\frac{\partial^2U_1}{\partial\tau^2}+n_3U_1-in_4|U_1|^2U_1=0, \quad (104)$$

and

$$n_6\frac{\partial U_2}{\partial z}+n_7\frac{\partial U_2}{\partial t}+in_8\frac{\partial^2U_2}{\partial\tau^2}+n_9U_2-in_{10}|U_2|^2U_2=0. \quad (105)$$

We notice that (104) and (105) are similar and each describe the dynamics of propagation of a signal in a single-mode optical fiber.

- Case where $n_3=n_9=0$ and $n_3\neq 0, n_9\neq 0$ with a and b real, the solutions for $n_2\succ 0, n_8\succ 0$, are given by

$$\begin{aligned}&U(\xi)=\pm\alpha_0\sqrt{2n_2}J_{-1,-1}(\xi)\\&\Rightarrow U_1(z,t)=\pm\alpha_0\sqrt{2n_2}J_{-1,-1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right],\end{aligned} \quad (106)$$

and

$$\begin{aligned}&V(\xi)=\pm\alpha_0\sqrt{2n_8}J_{1,1}(\xi)\\&\Rightarrow U_2(z,t)=\pm\alpha_0\sqrt{2n_8}J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right].\end{aligned} \quad (107)$$

- Case where $n_3=n_9$ and $n_3\neq 0, n_9\neq 0$ with a and b complex, the solutions for $n_2\succ 0, n_8\succ 0$, are given by

$$\begin{aligned}&U(\xi)=\alpha_0\sqrt{2n_2}J_{-1,-1}(\xi)\exp i\beta\\&\Rightarrow U_1(z,t)=\alpha_0\sqrt{2n_2}J_{-1,-1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\beta)\right],\end{aligned} \quad (108)$$

and

$$\begin{aligned}&V(\xi)=\alpha_0\sqrt{2n_8}J_{1,1}(\xi)\exp i\theta\\&\Rightarrow U_2(z,t)=\alpha_0\sqrt{2n_8}J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\theta)\right].\end{aligned} \quad (109)$$

In the case where $n_3=n_9=0$, we always stay in single-mode optical fibers except that apart from the fact that there are no longer any non-linear interactions between the signals which propagate, there are very low dissipations. The nonlinear partial differential equations which govern the wave propagation in this case reduce to

$$n_0\frac{\partial U_1}{\partial z}+n_1\frac{\partial U_1}{\partial t}+in_2\frac{\partial^2U_1}{\partial\tau^2}{}_1-in_4|U_1|^2U_1=0, \quad (110)$$

and

$$n_6\frac{\partial U_2}{\partial z}+n_7\frac{\partial U_2}{\partial t}+in_8\frac{\partial^2U_2}{\partial\tau^2}-in_{10}|U_2|^2U_2=0. \quad (111)$$

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- case $(n,m)=(1,1)$ and $(n',m')=(-1,-1)$

An approach similar to that followed in the case $(n,m)=(-1,-1)$ and $(n',m')=(1,1)$ treated above, allows to see that for the case $(n,m)=(1,1)$ and $(n',m')=(-1,-1)$, there is a decoupling of the nonlinear partial differential equations and we are in fact immersed in the single-mode optical fiber. the solutions thus obtained are as follows:

- Case where $n_3=n_9=0$ and $n_3\neq 0, n_9\neq 0$ with a and b real, the solutions for $n_2\succ 0, n_8\succ 0$, are given by

$$U(\xi)=\pm\alpha_0\sqrt{2n_2}J_{1,1}(\xi)$$
$$\Rightarrow U_1(z,t)=\pm\alpha_0\sqrt{2n_2}J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right], \quad (112)$$

and

$$V(\xi)=\pm\alpha_0\sqrt{2n_8}J_{-1,-1}(\xi)$$
$$\Rightarrow U_2(z,t)=\pm\alpha_0\sqrt{2n_8}J_{-1,-1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right]. \quad (113)$$

- Case where $n_3=n_9$ and $n_3\neq 0, n_9\neq 0$ with a and b complex, the solutions for $n_2\succ 0, n_8\succ 0$, are given by

$$U(\xi)=\alpha_0\sqrt{2n_2}J_{1,1}(\xi)\exp i\beta$$
$$\Rightarrow U_1(z,t)=\alpha_0\sqrt{2n_2}J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\beta)\right], \quad (114)$$

and

$$V(\xi)=\alpha_0\sqrt{2n_8}J_{-1,-1}(\xi)\exp i\theta$$
$$\Rightarrow U_2(z,t)=\alpha_0\sqrt{2n_8}J_{-1,-1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\theta)\right]. \quad (115)$$

Solutions (112) to (115) also verify the modified partial differential equations (104) and (105) as well as (110) and (111).

- case $(n,m)=(1,1)$ and $(n',m')=(1,0)$

Equations (18) and (19) give respectively

$$\begin{aligned}&a(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega)\\&+a\left[n_3+i\left(n_1\omega-n_0k-n_2\omega^2-2n_2\alpha_0^2\right)\right]J_{1,1}\\&-a(n_0\alpha-n_1\alpha_0+2n_2\alpha_0\omega)J_{2,2}\\&+i\left(2n_2\alpha_0^2a-a|a|^2n_4\right)J_{3,3}-in_5a|b|^2J_{3,1}=0,\end{aligned} \quad (116)$$

and

$$\begin{aligned}&-b(n_6\alpha-n_7\alpha_0+2n_8\alpha_0\omega)J_{2,1}\\&+b\left[n_9+i\left(n_7\omega-n_6k-n_8\omega^2-n_8\alpha_0^2\right)\right]J_{1,0}\\&+i\left(2n_8\alpha_0^2b-n_{11}b|a|^2\right)J_{3,2}-in_{10}b|b|^2J_{3,0}=0.\end{aligned} \quad (117)$$

Equations (116) and (117) are verified if and only if we have for $a\neq 0$ and $b\neq 0$ the relations (73) and (77) that still appear in this analytical framework; except the fact that the relations which mark the difference in the analyzes are given below

$$n_5=n_{10}=0, \quad (118)$$

$$n_4|a|^2=2n_2\alpha_0^2, \quad (119)$$

$$n_9+i\left(n_7\omega-n_6k-n_8\omega^2-n_8\alpha_0^2\right)=0, \quad (120)$$

and

$$n_{11}|a|^2=2n_8\alpha_0^2. \quad (121)$$

The exploitation of (119) and (121) makes it possible to write

$$|a|^2=\frac{2n_2\alpha_0^2}{n_4} \text{ with } \frac{n_4}{n_{11}}=\frac{n_2}{n_8}$$
$$\Rightarrow |a|=\alpha_0\sqrt{\frac{2n_2}{n_4}}, n_2n_4\succ 0. \quad (122)$$

We notice here that $n_5=n_{10}=0$, in this situation the signal of amplitude U_1 does not receive the nonlinear influence of the second signal of amplitude U_2. Only the second signal receives the influence of the first signal. So we can simply say that the signal U_1 is insensitive to the effects of the signal U_2.

The structure of the system is modified, so (3) and (4) are modified as

$$n_0\frac{\partial U_1}{\partial z}+n_1\frac{\partial U_1}{\partial t}+in_2\frac{\partial^2 U_1}{\partial\tau^2}+n_3U_1-in_4|U_1|^2U_1=0, \quad (123)$$

and

$$n_6\frac{\partial U_2}{\partial z}+n_7\frac{\partial U_2}{\partial t}+in_8\frac{\partial^2 U_2}{\partial\tau^2}+n_9U_2-in_{11}|U_1|^2U_2=0. \quad (124)$$

The solutions are such as

- Case where $n_3=n_9=0$ and $n_3\neq 0, n_9\neq 0$ with a and b real, the solutions for $n_2n_4\succ 0,$ are given by

$$U(\xi)=\pm\alpha_0\sqrt{\frac{2n_2}{n_4}}J_{1,1}(\xi)$$
$$\Rightarrow U_1(z,t)=\pm\alpha_0\sqrt{\frac{2n_2}{n_4}}J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right], \quad (125)$$

and

$$V(\xi)=bJ_{1,0}(\xi)$$
$$\Rightarrow U_2(z,t)=bJ_{1,0}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right]. \quad (126)$$

- Case where $n_3=n_9$ and $n_3\neq 0, n_9\neq 0$ with a and b complex, the solutions for $n_2n_4\succ 0,$ are given by

$$U(\xi)=\pm\alpha_0\sqrt{\frac{2n_2}{n_4}}J_{1,1}(\xi)\exp i\theta$$
$$\Rightarrow U_1(z,t)=\pm\alpha_0\sqrt{\frac{2n_2}{n_4}}J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\theta)\right], \quad (127)$$

and

$$V(\xi)=bJ_{1,0}(\xi)$$
$$\Rightarrow U_2(z,t)=bJ_{1,0}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right]. \quad (128)$$

In the case where in addition to the fact that $n_5=n_{10}=0$, we have $n_3=n_9=0$; the studied structure is further modified by an absence of dissipation or a very low proportion of dissipation. The equations describing the dynamics become

$$n_0\frac{\partial U_1}{\partial z}+n_1\frac{\partial U_1}{\partial t}+in_2\frac{\partial^2 U_1}{\partial\tau^2}-in_4|U_1|^2U_1=0, \quad (129)$$

and

$$n_6\frac{\partial U_2}{\partial z}+n_7\frac{\partial U_2}{\partial t}+in_8\frac{\partial^2 U_2}{\partial\tau^2}-in_{11}|U_1|^2U_2=0. \quad (130)$$

❖ case $(n,m)=(1,0)$ and $(n',m')=(1,1)$

With regard to the symmetry of (3) and (4) that we solve, we can easily deduce the solutions of the case $(n,m)=(1,0)$ and $(n',m')=(1,1)$ from the solutions obtained in the case $(n,m)=(1,1)$ and $(n',m')=(1,0)$. We find ourselves here in the situation where it is rather the signal of amplitude U_2 which is insensitive to the effects of the signal of amplitude U_1. There is always modification of the studied system and the equations which embody the dynamics under these conditions are given by

$$n_0\frac{\partial U_1}{\partial z}+n_1\frac{\partial U_1}{\partial t}+in_2\frac{\partial^2 U_1}{\partial\tau^2}+n_3U_1-in_5|U_2|^2U_1=0, \quad (131)$$

and

$$n_6\frac{\partial U_2}{\partial z}+n_7\frac{\partial U_2}{\partial t}+in_8\frac{\partial^2 U_2}{\partial\tau^2}+n_9U_2-in_{10}|U_2|^2U_2=0. \quad (132)$$

Equations (131) and (132) can be further reduced if we take into account the very low dissipation ($n_3=n_9=0$) and we consequently obtain

$$n_0\frac{\partial U_1}{\partial z}+n_1\frac{\partial U_1}{\partial t}+in_2\frac{\partial^2 U_1}{\partial\tau^2}-in_5|U_2|^2U_1=0, \quad (133)$$

and

$$n_6\frac{\partial U_2}{\partial z}+n_7\frac{\partial U_2}{\partial t}+in_8\frac{\partial^2 U_2}{\partial\tau^2}+n_9U_2-in_{10}|U_2|^2U_2=0. \quad (134)$$

As in the case $(n,m)=(1,1)$ and $(n',m')=(1,0)$, the solutions will be given by

- Case where $n_3=n_9=0$ and $n_3\neq 0, n_9\neq 0$ with a and b real, the solutions for $n_8n_{10}\succ 0,$ are given by

$$U(\xi)=aJ_{1,0}(\xi)$$
$$\Rightarrow U_1(z,t)=aJ_{1,0}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right]. \quad (135)$$

and

$$V(\xi)=\pm\alpha_0\sqrt{\frac{2n_8}{n_{10}}}J_{1,1}(\xi)$$
$$\Rightarrow U_2(z,t)=\pm\alpha_0\sqrt{\frac{2n_8}{n_{10}}}J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right], \quad (136)$$

- Case where $n_3=n_9$ and $n_3\neq 0, n_9\neq 0$ with a and b complex, the solutions for $n_8n_{10}\succ 0,$ are given by

$$U(\xi)=aJ_{1,0}(\xi)$$
$$\Rightarrow U_1(z,t)=aJ_{1,0}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t)\right], \quad (137)$$

and

$$V(\xi)=\pm\alpha_0\sqrt{\frac{2n_8}{n_{10}}}J_{1,1}(\xi)\exp i\theta$$
$$\Rightarrow U_2(z,t)=\pm\alpha_0\sqrt{\frac{2n_8}{n_{10}}}J_{1,1}(\alpha z-\alpha_0 t)\exp\left[-i(kz-\omega t-\theta)\right]. \quad (138)$$

VI. Other Forms of Solutions: Trigonometric Solutions

Another advantage of the iB-functions used in this work is that they allow easy transition from implicit hyperbolic forms to trigonometric forms. Thus, by making the correspondences $\alpha\to i\alpha$ and $\alpha_0\to i\alpha_0$ in the different solutions obtained, we can have the pairs of corresponding trigonometric solutions. The solutions determined being numerous, we will limit to a few examples

- For (38) and (39), we have

$$U_1(z,t)=\left[\frac{(n_{10}n_1-n_5n_7)\alpha_0+(n_5n_6-n_{10}n_0)\alpha}{2(n_{10}n_4-n_5n_{11})}\right]^{\frac{1}{2}}\cot an^{\frac{1}{2}}(\alpha z-\alpha_0 t)$$
$$\times\exp\left[-i\left(kz-\omega t-\theta+\frac{\pi}{4}+p\pi\right)\right], p=0,1 \quad (139)$$

and

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$$U_2(z,t)=\left[\frac{n_5(n_7n_4-n_1n_{11})\alpha_0+(n_0n_5n_{11}-n_0^2n_4+n_5n_6n_4-n_{10}n_0n_4)\alpha}{2n_5(n_{10}n_4-n_5n_{11})}\right]^{\frac{1}{2}}$$
$$\times\cot an^{\frac{1}{2}}(\alpha z-\alpha_0 t)\exp\left[-i\left(kz-\omega t-\theta+\frac{\pi}{4}+p\pi\right)\right],\ p=0,1.$$
(140)

- For (91) and (92) we have

$$U_1(z,t)=\alpha_0\left(\frac{2(n_5n_8-n_{10}n_2)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}\cot an(\alpha z-\alpha_0 t) \quad (141)$$
$$\times\exp[-i(kz-\omega t-\beta)],$$

and

$$U_2(z,t)=\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}\cot an(\alpha z-\alpha_0 t) \quad (142)$$
$$\times\exp[-i(kz-\omega t-\theta)],$$

- For (104) and (105), we have

$$U_1(z,t)=-\alpha_0\left(\frac{2(n_5n_8-n_{10}n_2)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}\tan(\alpha z-\alpha_0 t) \quad (143)$$
$$\times\exp[-i(kz-\omega t-\beta)],$$

and

$$U_2(z,t)=-\alpha_0\left(\frac{2(n_2n_{11}-n_4n_8)}{n_5n_{11}-n_4n_{10}}\right)^{\frac{1}{2}}\tan(\alpha z-\alpha_0 t)$$
$$\times\exp[-i(kz-\omega t-\theta)]. \quad (144)$$

- For (110) and (111), we have

$$U_1(z,t)=\alpha_0\sqrt{2n_2}\cot an(\alpha z-\alpha_0 t)\exp[-i(kz-\omega t-\beta)], \quad (145)$$

and

$$U_2(z,t)=-\alpha_0\sqrt{2n_8}\tan(\alpha z-\alpha_0 t)\exp[-i(kz-\omega t-\theta)]. \quad (146)$$

- For (123) and (124)

$$U_1(z,t)=\pm\alpha_0\sqrt{\frac{2n_2}{n_4}}\tan(\alpha z-\alpha_0 t)\exp[-i(kz-\omega t)], \quad (147)$$

and

$$U_2(z,t)=b\cos ec(\alpha z-\alpha_0 t)\exp[-i(kz-\omega t)]. \quad (148)$$

The solutions presented above come from the values of the pairs proposed by the fields of possibilities of solutions. We do not pretend to have isolated all the possible solutions. To be sure, it is necessary to check all the pairs of the field of possibilities of solutions.

VII. Study of Propagation of Some Solutions

For each modification made to the initial birefringent optical fiber, we will normally have to do a propagation study to assess the practical feasibility of the solutions obtained. But in view of the already very important volume of work, we have chosen to dwell on a few solutions. What is very important to note here is that the solutions to be propagated in each case respond to particular modifications of the starting birefringent optical fiber. Thus, we used the split-step method to discretize the coupled nonlinear partial differential equations corresponding to the modifications made on the birefringent fiber.

First case: $(n,m)=(1/2,1/2)$ and $(n',m')=(1/2,1/2)$

a) n_3 and n_9 are pure imaginary.

The nonlinear partial differential equations (37) and (38) are discretized so that the envelopes $U_1(z,t)$ and $U_2(z,t)$ are given by (50) and (51) in the case where a and b are real, and (52) and (53) in the case where a and b are complex. The profiles obtained are as follows

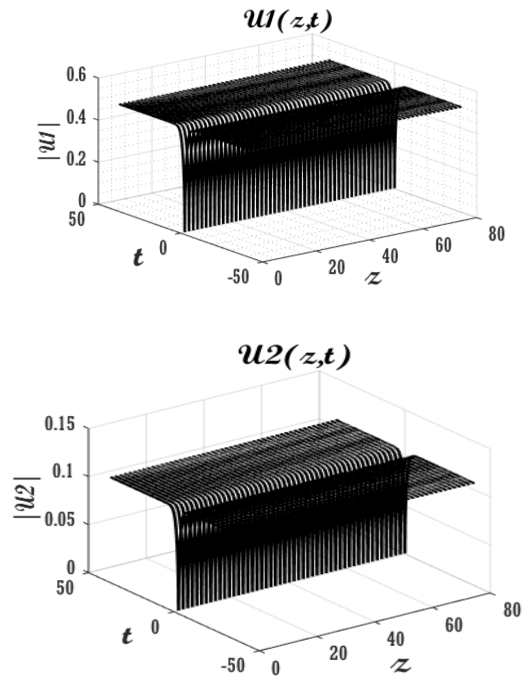
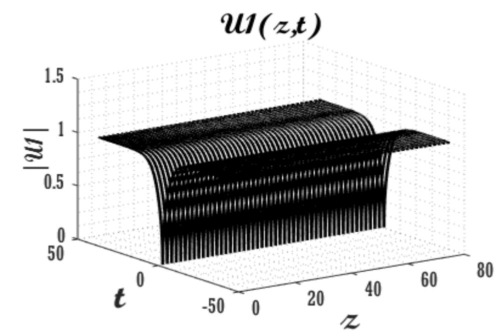


Fig. 1: The profiles are obtained for: $n_0=15$, $n_1=0.0007$, $n_4=0.005$, $n_5=-0.11$, $n_6=1$, $n_{10}=0.0015$ $\alpha_0=0.5$, $k_0=0.1$, $\omega_0=0.2$ and $a=0.5$.



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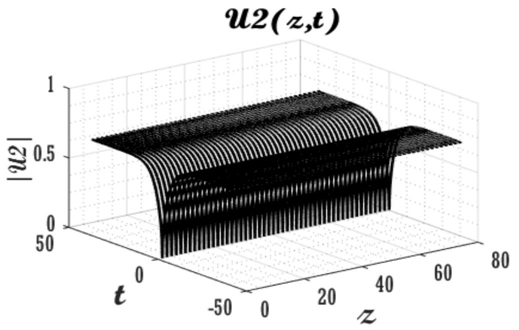

Fig. 2: The profiles are obtained for: $n_0 = 12$, $n_1 = 0.001$, $n_4 = 0.05$, $n_5 = -0.11$, $n_6 = 10$, $n_{10} = 0.12$ $\alpha_0 = 0.5$, $k_0 = 1.2$, $\omega_0 = 0.15$ and $a = 1$.

b) n_3 and n_9 are real parts and equal to zero

The nonlinear partial differential equations (40) and (41) are discretized so that the envelopes $U_1(\mathrm{z,t})$ and $U_2(\mathrm{z,t})$ are given by the relations (50) and (51) in the case where a and b are real, and relations (52) and (53) in the case where a and b are complex. The profiles obtained are:

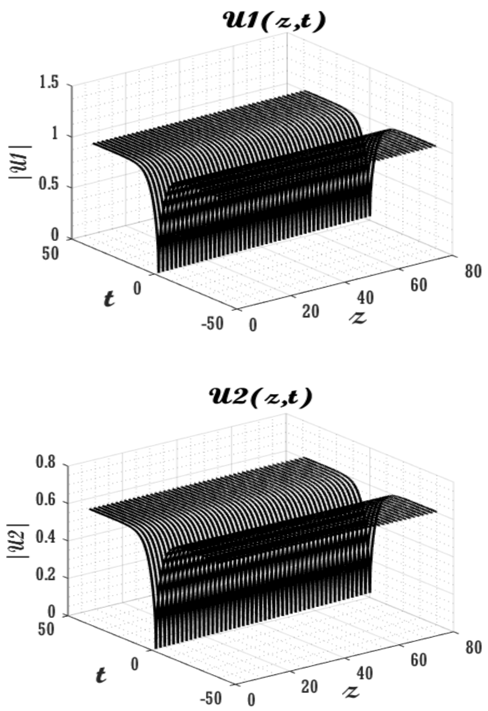

Fig.3: The profiles are obtained for: $n_0 = 20$, $n_1 = 0.01$, $n_4 = 0.04$, $n_5 = -0.11$, $n_6 = 10$, $n_{10} = 0.12$ $\alpha_0 = 0.5$, $k_0 = 1.2$, $\omega_0 = 0.15$ and $a = 1$.

c) Case where $n_0 \leftarrow in_0, n_1 \leftarrow in_1$, $n_6 \leftarrow in_6$, $n_7 \leftarrow in_7$

The profiles obtained are:

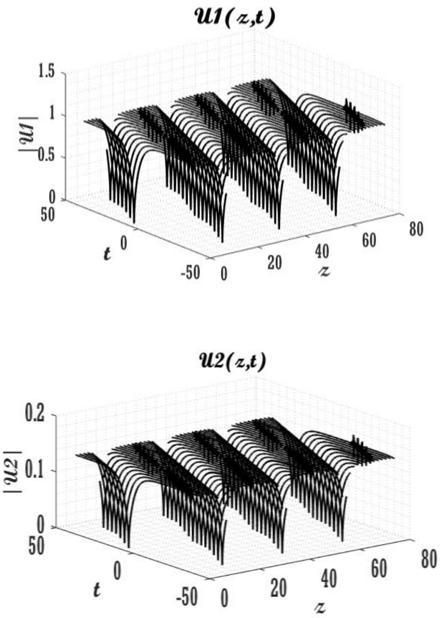

Fig. 4: The profiles are obtained for: $n_0 = 0.1$, $n_4 = 0.04$, $n_5 = -0.21$, $n_6 = 1.5$, $n_{10} = 0.55$ $\alpha_0 = 0.15$, $\alpha = 0.5$ $k_0 = 1.2$, $\omega_0 = 0.75$ and $a = 1$.

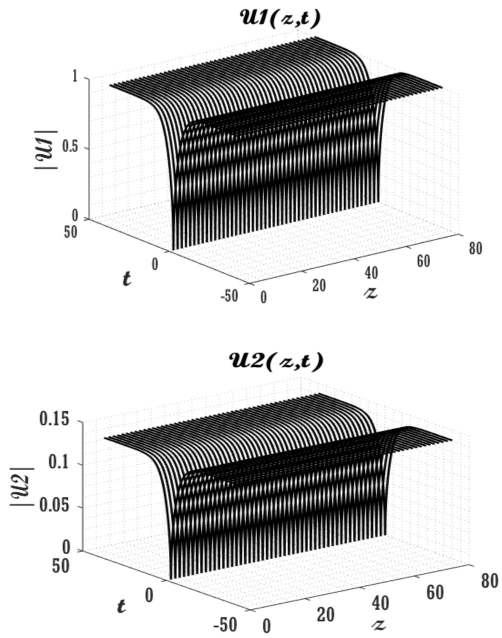

Fig. 5: The profiles are obtained for: $n_0 = 0.1$, $n_4 = 0.04$, $n_5 = -0.21$, $n_6 = 1.5$, $n_{10} = 0.55$ $\alpha_0 = 0.15$, $\alpha = 15$ $k_0 = 1.2$, $\omega_0 = 0.75$ and $a = 1$.

Second case: $(n,m) = (1,1)$ and $(n',m') = (1,1)$,

a) Case where $n_3 = 0, n_9 = 0$

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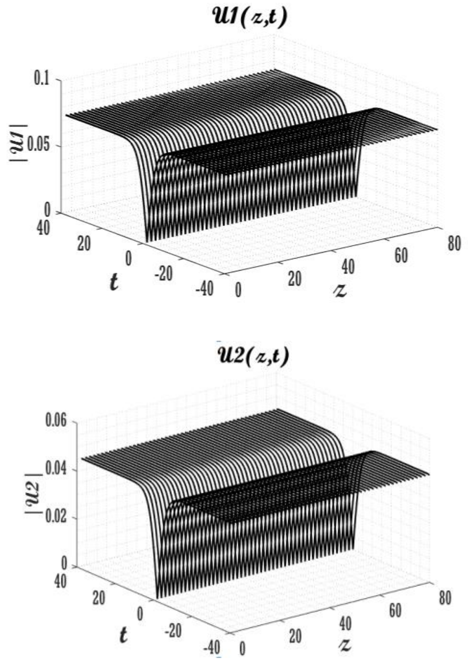

Fig.6: The profiles are obtained for: $n_0 = -35$, $n_1 = 0.003$, $n_2 = 0.002$ $n_4 = 0.005$, $n_5 = 0.11$, $n_6 = 1$, $n_7 = 0.0001$, $n_8 = 0.0025$, $n_{10} = 0.015$, $n_{11} = 0.052$, $\alpha_0 = 0.25$, $\theta = \pi$.

b) Case where $n_3 = n_9$ and $n_3 \neq 0, n_9 \neq 0$

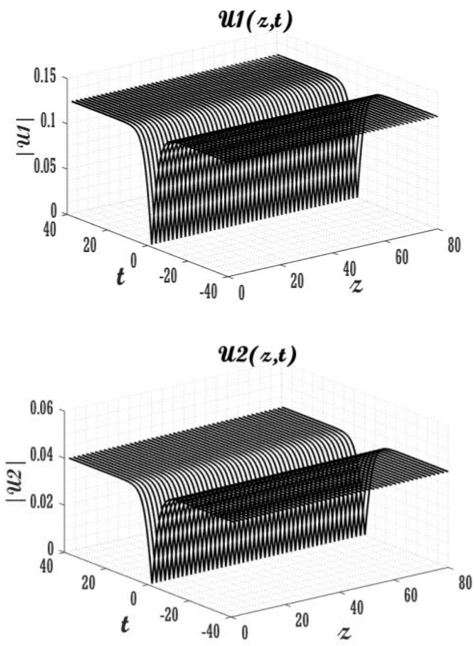

Fig.7: The profiles are obtained for: $n_0 = -29$, $n_1 = 0.003$, $n_2 = 0.002$ $n_4 = 0.005$, $n_5 = 0.11$, $n_6 = 1$, $n_7 = 0.0001$, $n_8 = 0.0025$, $n_{10} = 0.015$, $n_{11} = 0.052$, $\alpha_0 = 0.25$, $\theta = \pi$.

Third case: $(n, m) = (1,1)$ and $(n', m') = (1,0)$, $n_3 \neq 0, n_9 \neq 0$

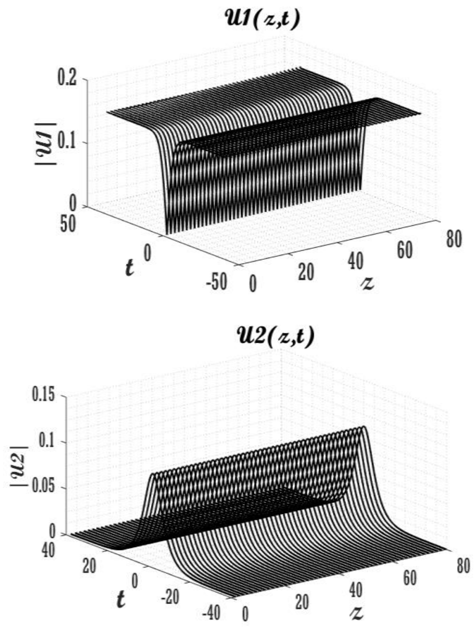

Fig. 8: The profiles are obtained for: $n_0 = 500$, $n_1 = 0.5$, $n_2 = 0.1$ $n_4 = 0.5$, $n_6 = 30.6$, $n_7 = 0.0001$, $n_8 = 0.0025$, $\alpha_0 = 0.25$, $\theta = \pi$, $b = 0.1$, $k = 0.00001$

Fourth case: $(n, m) = (1,0)$ and $(n', m') = (1,1)$, $n_3 \neq 0, n_9 \neq 0$

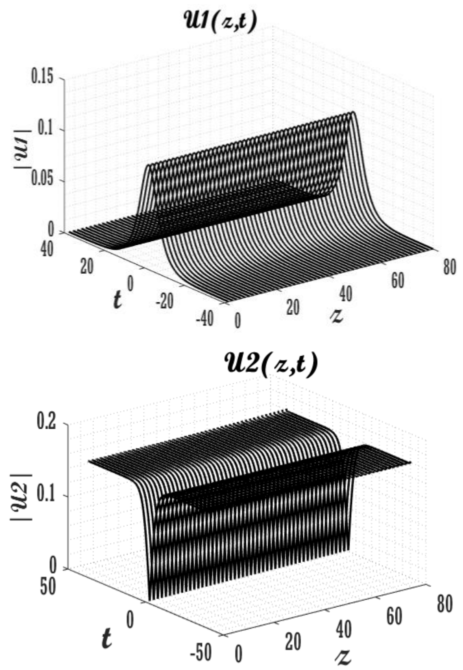

Fig.9: The profiles are obtained for: $n_0 = 100$, $n_1 = 0.5$, $n_2 = 0.1$ $n_5 = 0.5$, $n_6 = 30.6$,

$$n_7 = 0.0001,\ n_8 = 0.0025,\ \alpha_0 = 0.25,\ \theta = \pi,$$
$$a = 0.1,\ k = 0.00001$$

What is very important to remember is that Figs.1 to 9 show the prototypes of solitary waves which propagate in the birefringent optical fiber according to the modifications made.

Each Fig. obtained corresponds to a precise modification made on the birefringent optical fiber under study. These modifications made on the physical structure studied are the suppression of the dispersion of the dissipation and even of the nonlinearity. This transmission medium being birefringent, we also played on the mutual influence of the two waves propagating in the medium. Figs. above are a few chosen to illustrate the impact of the properties of the fiber on the nature of the waves.

VIII. Conclusion

This work aimed to study the impact of the properties of the elliptical birefringent optical fiber on the waves likely to propagate therein. But all was not only to study the impact of the properties, but also to determine the prototype wave solutions. To achieve our objective, we have carefully chosen a solution function which must embody the waveforms that we want them to be solutions of the coupled nonlinear partial differential equations which govern the dynamics of wave propagation in the considered waveguide.

iB-functions were chosen because of their flexibility, because not only are they easy to handle, but they offer a wider field of search for solutions.

The calculations made it possible to determine the fields of the possibilities of solutions, that is to say a field in which we must find solutions. Certainly, all the pairs in the field have not led to non-trivial solutions, but we are sure that the majority of solutions, when they exist, are found there. The problem is just to have the patience to verify the existence of solutions for all the pairs of fields. Thus, for what concerns this work, implicit hyperbolic and trigonometric solutions have been determined for certain pairs of fields. But not all pairs because some pairs lead to trivial solutions and even to impossibilities. It should be pointed out that on the 25 pairs of the table constituting the field of the possibilities of solutions, we have combined about ten pairs and the cases of the non-trivial solutions obtained are those exposed in the body of this article.

To return to the expected results, we can confirm that the objectives have been achieved, because we have had cases of analyzes where the solutions were only possible by removing the dissipation coefficients (case of the very weakly dissipative elliptical birefringent optical fiber); either by eliminating the dispersion coefficients (case of very weakly dispersive optical fiber); or by eliminating both the dissipation and dispersion coefficients (case of the very weakly dissipative and dispersive birefringent optical fiber). In some cases, the mutual nonlinear influence has been broken so that each wave behaves as if it is alone in its fiber (case of single mode fiber). Another very important scenario is the situation where a signal is non-reversibly subjected to the non-linear influence of the other (case where one of the signals is insensitive to the non-linear effects of the other). As part of our analyzes, we have also noticed that the coefficients assigned to the spatial and temporal variations do not impact the solutions in their entirety, this can justify the fact that in the majority of the equations which model the dynamics of the waves, these variations are assigned coefficients one.

The different solutions and nonlinear partial differential equations obtained describing the different modifications and different dosages of their coefficients (properties), sufficiently prove the impact of the properties on the quality of the signal to propagate in the fiber. This study, which has been analytically successful, deserves a real extension in the Engineering of manufacturing of optical fiber transmission media. The most direct proposition is that of going particularly in the direction of the manufacturing optical fiber, while having a look at the type of signal that can best accommodate it. The principle being to play on the properties of the fiber or of the waveguide so as to adapt it to the requirements of the signal that we want to propagate there.

The work that we have started here is a process that we will extend to several other waveguides in order to draw all the conclusions that will be necessary in the future.

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