

# Modelling Surfactant-Fuel Thin Film Profile

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**Abstract**— Tankers carrying large amounts of explosive fuels are involved in transportation accidents; these incidents can cause catastrophic consequences. In the present, we specifically develop the governing equations to the mathematical model for the drainage of liquid in a thin film spread over a fuel spillage surface. Discretize the governing equations' numerical solution with the accompanying initial/boundary conditions on a staggered grid. We determine the rate of thinning and lifetime of films by tracking the velocity, drainage, and flux change of each connected region. In this research, the depletion of surfactant-fuel thin-film foam was attained by varying Reynolds number,  $Re$  to show that given different interface inertial forces to viscous forces, the surfactant-fuel thin-film foam shows different velocity, drainage, and flux profiles. The study simulation shows that for  $Re=0.01, 0.1$ , and  $1$  the velocity profile stabilizes after approximately  $0.5m$ , and the flow continues. Surfactant-fuel thin-film foam with  $Re=1$  has the maximum and minimum velocities are less than that of  $Re=0.01$  and  $Re=0.1$ .  $Re=10$ , the velocity profile stabilizes after approximately  $0.8$ .  $Re$  greater than  $1$  shows the surfactant-fuel thin-film is depleted compared to when  $Re$  is less than  $1$ . We conclude that to suppress the gas vapors and control the fire incidences,  $Re \neq \gg 1$ . We recommended that there is a need to consider the effect of variation in temperature, velocity, besides Reynolds number, in determining the lifetime of a surfactant-fuel thin foam. There is a need to have an experimental analysis of the effect of varying other parameters such as the shear rate on the stability of the equilibrium interface or boundary between the fuel and soluble surfactant.

**Keywords**— Drainage profile: Flux Profile: Surfactant-Fuel Thin Film: Velocity Profile: Reynolds Number.

## I. INTRODUCTION

Suppression of evaporation of hydrocarbons liquids and fuels by aqueous films is of practical importance in firefighting. It may even be of potential value in preventing pollution in the event of large-scale spills involving the liquids. Developed in the 1960s, aqueous film-forming foam (AFFF) relies on a mixture of surfactant and solvents to rapidly suppress fuel spill vapors and fires [34, 21]. AFFF formulations have been highly effective and have achieved worldwide use [30]. Low concentrations of fluorosurfactant additives have found numerous usage in modern high-performance as fire-fighting foams against the traditional fire extinguishers. Fluoro-surfactants gave these foams the ability to form a thin film, spreading films on the surface of burning liquids. The films provided significant resistance to diffusion of flammable vapors [27]. Synthetic surfactants are widely used in household cleaning products (detergents), cosmetics, and personal care products (shampoo and toothpaste). Foam presence is not guaranteed beneficial. For instance, in the brewing industry, foam reduces vessel capacity hence lessening finished product foaming potential [3]. The application of foam on fuel can be used to prevent various accidents and incidents caused by petroleum fuel leakage, accidental burning and explosions. These are common in transportation and public safety, and have brought a bit of tremendous loss of life and property.

The surfactant can rapidly suppress surface tension, which is essential for thin-film spreading and necessary for vapor suppression [14]. Additionally, a quick reduction of the surface tension can generate high-quality foams. Foam drainage is an essential element in the formation and early development of

thin-film [36]. It is also essential given the widespread assumption that the aqueous film forms the main barrier of fuel spill vapor suppression of AFFF observed in numerous large scale tests [36, 13].

Thin liquid films play a central role in many real-life applications. The interface between the liquid and the surrounding fluid (usually a gaseous phase) is a free and deformable boundary. Thin liquid films can display a variety of dynamics and interfacial instabilities. A clear understanding of the film drainage phenomena in fuel spill vapor suppression is still lacking. Models for thin liquid film drainage have been proposed by numerous researchers dating back to 1886, when Reynolds [28] solved for the drainage velocity of the fluid between two plane parallel disks. Flow in thin liquid films is critically dependent on the tangential extensibility of the two interfaces in intimate contact with the thinning film. If neither of the interfaces will support tangential stress, thinning is generally very rapid such as we have mobile film surfaces in comparison to the situation where either one or both the surfaces is tangentially in-extensible or immobile [11]. The hydrodynamics of thin films with rigid interfaces was addressed by Reynolds [28].

Exerowa [10] extensively discusses the rate of thinning of thin-film. Exerowa and Sheludko and develops the most common method for studying the drainage of individual foam films [10, 31]. Previously, thinning rates of thin-film have been calculated in foams with Newtonian liquid phases. In the case of plane-parallel film, with tangentially immobile surfaces, one obtains the well-known Reynolds equation for the velocity of thinning [28]. For two identical surfaces, the equation was first derived by Radoev *et al.* [26]. An important practical parameter

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is the film lifetime  $T$ , defined as when the film will thin from the initial thickness to the critical thickness of rupture. At very low surfactant concentrations, the film drains more quickly because of the high mobility of the interface. In aqueous-like liquids, surfactants can be added to lower the gas-liquid surface tension and decrease the thinning rates by orders of magnitude [37, 7].

In this work, we try to estimate the distance before depletion of surfactant-fuel thin-film foam. We varied  $Re$  to show that given different interface Reynolds numbers (inertial forces to viscous forces), the surfactant-fuel thin-film foam will show different velocity profiles. We show through simulations that varying  $Re$  leads to stabilization of velocity profile of thin-film due to surfactant fuel foam as the flow continues. We establish that to suppress the gas vapors and control the fire incidences, the velocity of the thin-film due to surfactant fuel foam must be stabilized.

The differences in thin film thinning dynamics remain unexplored among firefighting foams. Measurements of aqueous film under vapor conditions are needed to quantify its contribution to vapor suppression. The proposed study focuses on;

- i. To develop the governing equations to the mathematical model for the drainage of liquid in a thin film spread over a fuel spill surface.
- ii. To discretize and numerically solve the governing equations with the accompanying initial/boundary conditions on a staggered grid.
- iii. To determine velocity profile of thin-film under varied Reynolds number.

## II. FLUID DYNAMICS AND THIN FILM EQUATIONS

The set-up of the foam-fuel system is shown in the sketch in Figure 1. We assume that the two fluids are immiscible and

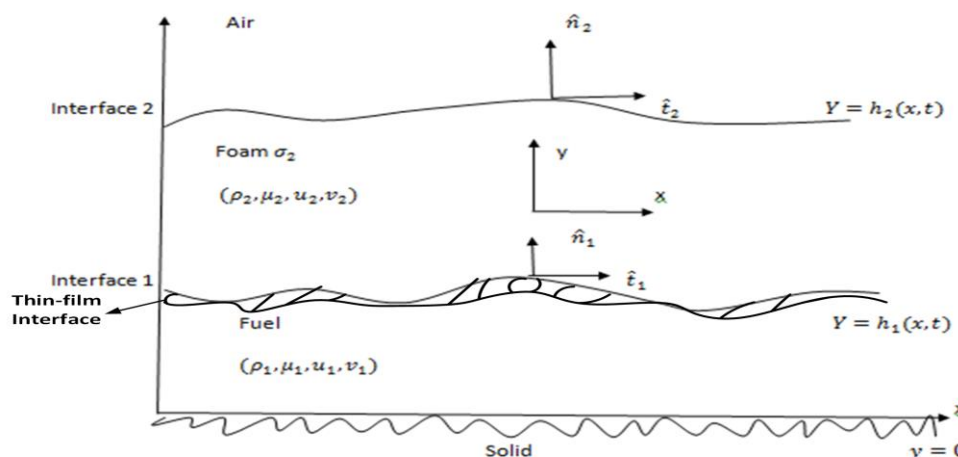


Fig. 1. Free surface foam flow over fuel surface showing thin-film interface which is the region of interest.  
Example of a figure caption

### 2) Thin film equations

In the following, we consider a thin parametrizable film with a centerline  $H(x, t)$  and thickness  $h(x, t)$ . The interfaces are situated at  $H(x, t) \pm \frac{1}{2}h(x, t)$ . We will present the basic

equations describing the behavior of a Newtonian liquid and a surfactant.

### A. Problem Formulation

#### 1) Governing equations

Foam spread can be modeled as a shallow free-surface flow. Consider the free surface foam over fuel in figure 1. We need the governing equations for the dynamics of the field and suitable boundary conditions to derive the thin film evolution equations.

We consider an idealized system consisting of foam and fuel, both with uniform initial thickness, and are brought into direct contact at  $t = 0$ . The fuel layer corresponds to  $h_1$ , and the foam layer corresponds to  $h_2$ . We use subscripts 1 and 2 to denote the properties of the fuel and foam layers, respectively. The two-phase flows can be treated quite similarly to single-phase flows. The fact that the two unique fluids are present can be handled either by using two separate sets of Navier-Stokes, one for each fluid or more conveniently using the same set of equations for fluids but with variable density and viscosity fields, that is

$$\partial_x u_k + \partial_y v_k = 0 \quad (1)$$

$$\rho_k \left( \frac{\partial u_k}{\partial t} + u_k \frac{\partial u_k}{\partial x} + v_k \frac{\partial u_k}{\partial y} \right) = -\frac{\partial p_k}{\partial x} + \mu_k (u_{k,xx} + u_{k,yy}) + \rho_k g \quad (2)$$

$$\rho_k \left( \frac{\partial v_k}{\partial t} + v_k \frac{\partial v_k}{\partial x} + u_k \frac{\partial v_k}{\partial y} \right) = -\frac{\partial p_k}{\partial y} + \mu_k (v_{k,xx} + v_{k,yy}) + \rho_k g \quad (3)$$

The subscript  $k$  takes the values 1 and 2 representing the fuel and foam layers, respectively.

equations describing the behavior of a Newtonian liquid and a surfactant.

#### 3) Newtonian fluid

We are investigating the ability of foam to arrest explosions due to fuel spill vapor; hence the liquid under consideration is

a thin film that drains from foam once it is applied to a fuel spill surface. We consider incompressible Newtonian fluids. We employ a lubrication approximation of the governing Navier-Stokes equations since the height of a thin film is small compared to its width [1, 22, 15]. This leads to a system of coupled nonlinear partial differential equations, which can also be modeled as a shallow free-surface flow. The Navier-Stokes equations in 2 - D have the form;

$$u_x + v_y = 0 \tag{4}$$

$$\rho(u_t + uu_x + vv_y) = -P_x + \mu(u_{xx} + v_{yy}) + \rho g_1 \tag{5}$$

$$\rho(v_t + uv_x + vv_y) = -P_y + \mu(v_{xx} + u_{yy}) + \rho g_2 \tag{6}$$

The indices denote the derivatives. (4) represents conservation of mass and (5)-(6) represent conservation of momentum in  $x$  - and  $y$  - direction, respectively. With liquid velocity given by  $u = (u, v)$ , the liquid density, viscosity and pressure given by  $\rho, \mu$ , and  $P$ . The left-hand sides of the momentum equations denote the inertial forces balanced on the right-hand side by the pressure gradient and viscous forces. We consider the gravitational force  $\rho g$ , which is a body force and acts on the whole fluids.

#### 4) Definition of interface parameters

We define conditions at the free interfaces  $h^\pm(x, t) = H(x, t) \pm \frac{1}{2}h(x, t)$ , however the boundary conditions are more complicated than the no slip conditions and involves force balance and kinematics. We utilize the unit tangent and unit normal to the surfaces defined by:

$$\hat{n}^\pm = \frac{(-\partial_x h^\pm, 1)}{(1+(\partial_x h^\pm)^2)^{\frac{1}{2}}}, \quad \hat{t} = \frac{(1, \partial_x h^\pm)}{(1+(\partial_x h^\pm)^2)^{\frac{1}{2}}} \tag{7}$$

Finally, the mean curvature of the interface is given by

$$\pm \kappa^\pm = -\nabla_1 \cdot \hat{n} = \frac{\partial_{xx}^2 h^\pm}{(1+(\partial_x h^\pm)^2)^{\frac{3}{2}}} \tag{8}$$

In all of the above expressions, "+" belongs to the film foam interface,  $h^+$  and "-" to the fuel film interface  $h^-$ .

#### 5) Interface conditions

We have the following conditions for the evolution of the interfaces;

$$v_1 = h_{1,t} + u_1 h_{1,x} \quad \text{at} \quad y = H(x, t) - \frac{1}{2}h(x, t) \tag{9}$$

$$v_2 = h_{2,t} + u_2 h_{2,x} \quad \text{at} \quad y = H(x, t) + \frac{1}{2}h(x, t) \tag{10}$$

Equations (9)-(10) are the kinematic boundary conditions (in the absence of interfacial mass transfer) that balances the normal component of the liquid velocity at the interface with the speed of the interface. Additionally, there are conditions for the equilibrium of the normal and tangential forces:

$$\sigma k^\pm = \hat{n}^\pm \cdot \tau \cdot \hat{n}^\pm \tag{11}$$

$$\hat{t}^\pm \cdot \nabla \sigma = \hat{t}^\pm \cdot \tau \cdot \hat{t}^\pm, \tag{12}$$

where  $\tau$  is the stress tensor of the liquid,  $\hat{n}$  is the unit outward vector normal to the surface,  $\hat{t}$  is the unit vector tangential to the interface,  $\kappa$  is the mean curvature of the interface. We denote the liquid stress tensor by  $\tau$ , so for a Newtonian liquid we have

$$\tau = \begin{bmatrix} -P + 2\mu u_x & \mu(u_y + v_x) \\ \mu(u_y + v_x) & -P + 2\mu v_y \end{bmatrix} \tag{13}$$

If  $\sigma^\pm$  is the surface tension of the fuel-film and film-foam interfaces, then substituting the stress tensors in (11) and (12) we find

$$\pm \sigma^\pm \kappa^\pm = -P + \mu \frac{(u_x(H_x \pm \frac{1}{2}h_x)^2 - (H_x \pm \frac{1}{2}h_x)v_x - 2u_y(H_x \pm \frac{1}{2}h_x) + 2v_y)}{1+(H_x \pm \frac{1}{2}h_x)^2} \tag{14}$$

$$\pm \sigma_x^\pm = \mu \frac{((u_y + v_x)(1 - (H_x \pm \frac{1}{2}h_x)^2) + 2(H_x \pm \frac{1}{2}h_x)(v_y - u_x))}{\sqrt{1+(H_x \pm \frac{1}{2}h_x)^2}} \tag{15}$$

We shall refer to (14) as "normal force balance" and (15) as the "tangential force balance" at each surface. We must also specify the boundary conditions at the ends of the film. This depends on the situation under considerations. The gradient of surface tension in the tangential force balance is commonly known as Marangoni stress. We also require a third condition to locate the interface. Assuming that there is no evaporation, we employ the kinematic condition, and the motion of the interface (9) and (10) becomes

$$u = H_t \pm \frac{1}{2}h_t + u(H_x \pm \frac{1}{2}h_x) \tag{16}$$

We have an additional unknown quantity here, the surface tension  $\sigma$ , which is related to the surfactant concentration

#### B. Non-dimensionalization

We assume that the dimension of this film under consideration in  $x$  - direction is of the magnitude  $L$  and that its typical thickness is  $h = \epsilon L \ll L$ , where  $\epsilon \ll 1$  is a small parameter. Furthermore, we assume that the curvature of the centerline of the film is small such that  $H \ll L$ . Finally, we expect that the surface tension varies around a constant value  $\gamma$  in the magnitude  $\nabla \gamma \ll \gamma$ . Based on these assumptions, we introduce the following dimensionless variables.

$$x = Lx', \quad y = \epsilon Ly', \quad t = \frac{L}{U} t' \quad u = Uu', \quad v = \epsilon Uv', \quad p = \frac{\epsilon U}{L} p' \tag{17}$$

$$h = \epsilon Lh', \quad H \pm \frac{1}{2}h = \epsilon L(H' \pm \frac{1}{2}h'), \quad \sigma^\pm = \gamma + \Delta \gamma \sigma'^{\pm} \tag{17}$$

Moreover, we introduce the following similarity parameters;

i. The capillary number  $Ca = \frac{\mu U}{\gamma}$  which is the ratio of viscous and capillary forces.

ii. The Marangoni number  $Ma = \frac{\Delta \gamma}{\mu U}$ , which describes the relation between Marangoni and viscous forces.

iii. The Froude number  $Fr = \frac{U^2}{L}$ , which has the dimension of an acceleration.

iv. The dimensionless ratio  $\frac{g}{Fr}$  characterizes the relation between gravitational and inertial forces, where  $g$  is the absolute value of the gravitational acceleration.

v. The dimensionless Reynolds number  $Re = \frac{\rho U L}{\mu}$ , which characterizes the relation between inertial and viscous forces. We decompose the surface tension into a constant component  $\gamma$  and a variable  $\sigma^\pm \Delta \gamma$ , where  $\Delta \gamma$  is the material property.

We substitute these into (4)-(6), to obtain (dropping primes);

$$u_x + v_y = 0,$$

$$\epsilon^2 Re(u_t + uu_x + vv_y) = u_{yy} + \epsilon^2 \left( -P_x + u_{xx} + \frac{g \cdot Re}{Fr} e_{gx} \right).$$

$$\epsilon^2 Re(v_t + uv_x + vv_y) = v_{yy} - P_y + \epsilon^2 \left( v_{xx} + \frac{g \cdot Re}{Fr} e_{gy} \right).$$

The values  $e_{gx}$  and  $e_{gy}$  are the coefficients of the unit vector in the direction of the gravity, that is,  $g = ge_g$

The motion of the interfaces (9)-(10) becomes

$$u = H_t \pm \frac{1}{2} h_t + u \left( H_x \pm \frac{1}{2} h_x \right) \quad (18)$$

We assumed the foam layer is stationary meaning fuel spill on a plateau. The force balances in normal and tangential directions (11)-(12) are;

$$\pm \left( \frac{\epsilon}{Ca} + \epsilon Ma \sigma^\pm \right) \frac{(H_{xx} \pm \frac{1}{2} h_{xx})}{1 + \epsilon^2 (H_x \pm \frac{1}{2} h_x)^2} = -P + \frac{2\epsilon^2 u_x (H_x \pm \frac{1}{2} h_x)^2 - 2\epsilon^2 v_x (H_x \pm \frac{1}{2} h_x) - 2u_y (H_x \pm \frac{1}{2} h_x) + 2v_y}{1 + \epsilon^2 (H_x \pm \frac{1}{2} h_x)^2} \quad (19)$$

In the normal direction and,

$$\pm \epsilon Ma \sigma_x^\pm = \frac{(u_y + \epsilon^2 v_x)(1 - \epsilon^2 (H_x \pm \frac{1}{2} h_x)^2 + 2\epsilon^2 (H_x \pm \frac{1}{2} h_x)(v_y - u_x))}{\sqrt{1 + \epsilon^2 (H_x \pm \frac{1}{2} h_x)^2}}, \quad (20)$$

tangential direction. The fluid mechanics problem is closed by imposing a kinematic condition on the interfaces.

$$u = H_t \pm \frac{1}{2} h_t + u \left( H_x \pm \frac{1}{2} h_x \right) \quad (21)$$

A simplified one-dimensional model for the evolution of the film can be derived by expanding the dependent variables as asymptotic series in powers of the small parameter  $\epsilon$ . Finally, we need the solution of the governing equations (4)-(6) as perturbation series in powers of the small parameter  $\epsilon$ .

### C. Reduction of Thin-Film Equations

We expand the dimensionless equations in terms of the small parameter  $\epsilon$ . At the moment, we make the following assumptions.

1.  $\frac{\epsilon^2 g \cdot Re}{Fr} \ll 1$
2.  $\epsilon Ma \ll 1$
3.  $\epsilon^2 Re \ll 1$

In the proceeding section, we considered the changes we have to make if the above assumptions do not hold. We make ansatz<sup>23</sup>  $\phi = \phi_0 + \epsilon^2 \phi_1 + \dots$  where  $\phi$  stands for any of the unknowns. Then the leading-order problem reduces to:

$$u_{0x} + v_{0y} = 0 \quad (22)$$

$$u_{0yy} = 0 \quad (23)$$

$$v_{0yy} = P_{0y} \quad (24)$$

With boundary conditions on  $y = H_0 + \frac{1}{2} h_0$ ; we have

$$v_0 = H_{0t} + u_0 H_{0x} \pm \frac{1}{2} h_{0t} \pm \frac{1}{2} u_0 h_{0x} \quad (25)$$

$$\pm \frac{\epsilon}{Ca} \left( H_{0xx} + \frac{1}{2} h_{0xx} \right) = -P_0 + 2v_{0y} \quad (26)$$

$$u_{0y} = 0 \quad (27)$$

These equations can be simplified further. Integrating (23) and applying (27) gives

$$u_0 = u_0(x, t), \quad (28)$$

that is, the longitudinal velocity, which is constant across the film. Thus, such flows are often termed "extensional". Next we integrate (22) over  $\left[ H_0 - \frac{1}{2} h_0, y \right]$  and  $\left[ y, H_0 + \frac{1}{2} h_0 \right]$ , respectively. This yields with (25) the following equations:

$$0 = u_{0x} \left( y - H_0 + \frac{1}{2} h_0 \right) + v_0(y) + \frac{h_{0t}}{2} + u_0 \frac{h_{0x}}{2} - H_{0t} - u_0 H_{0x}$$

$$0 = u_{0x} \left( H_0 + \frac{1}{2} h_0 - y \right) - v_0(y) + \frac{h_{0t}}{2} + u_0 \frac{h_{0x}}{2} + H_{0t} + u_0 H_{0x}$$

Adding these gives mass conservation:

$$0 = h_{0t} + (u_0 h_0)_x \quad (29)$$

Subtracting leads to an expression for  $v_0$ :

$$v_0 = y u_{0x} + H_{0t} + (u_0 H_0)_x \quad (30)$$

From (24) and (25), the equation of the center line is found to be

$$H_{0xx} = 0 \quad (31)$$

This relationship forces the center line to be straight and hence, without loss of generality,  $H_0 = 0$ . We include inertial effect across the film will result in an equation for  $H_0$  which must be solved alongside those for  $u$  and  $h$ .

We now also obtain the leading order pressure,  $P_0$ . Substituting for  $v_0$  into (24), integrating and applying (26) gives [37]

$$P_0 = -2u_{0x} - \frac{\epsilon}{2Ca} h_{0xx} \quad (32)$$

The pressure is therefore generated by a combination of external (viscous) and capillary effects. If  $Ca \ll \epsilon$ , the scaling for  $P_0$  is no longer valid. In this case, the pressure gradient may enter into the Navier-Stoke equations in leading order and we obtain a lubrication-type equation. We now have (29), (30) and (32) for the four unknowns  $h_0, u_0, v_0, P_0$ . We need one more equation in order to close the system. This will be taken from the next order  $\epsilon^2$ ; the relevant parts of the  $\phi(\epsilon^2)$  problem is [37],

$$u_{1x} + v_{1y} = 0 \quad (33)$$

$$Re(u_{0t} + u_0 u_{0x} + v_0 u_{0y}) = -P_{0x} + u_{0xx} + u_{1yy} + \frac{gRs}{Fr} e_{gx} \quad (34)$$

The related boundary condition on  $y = \pm \frac{1}{2} h_0$ ,

$$\pm \frac{Ma}{\epsilon} \sigma_x^\pm = u_{1y} + v_{0x} \pm 2h_{0x}(v_{0y} - u_{0x}).$$

Integrating (34) across the film together with this boundary condition leads to

$$h_0 \left[ P_{0x} - u_{0xx} - \frac{gRe}{Fr} e_{gx} + Re(u_{0t} + u_0 u_{0x} + v_0 u_{0y}) \right] = \frac{Ma}{\epsilon} (\sigma^+ + \sigma^-)_x - v_{0x}|_{\frac{h_0}{2}} + v_{0x}|_{-\frac{h_0}{2}} + 2h_{0x}(2u_{0x} + v_{0y}) \quad (35)$$

Under the assumption of symmetry, that is,  $\sigma_x^+ = \sigma_x^-$ , together with (30) and (32), we obtain from (29) and (35) a

system of two ODE's for the two unknowns in  $h$  and  $u$  (dropping the subscripts)

$$0 = h_t + (uh)_x \tag{36}$$

$$0 = \frac{Ma}{\epsilon} (2\sigma_x) + \frac{\epsilon}{2Ca} hh_{xxx} - Reh(u_t + uu_x) + h \frac{gRs}{Fr} e_{g_x} + 4(hu_x)_x \tag{37}$$

We note that the surface tension  $\sigma$  still appears in these equations. We will relate this quantity to the surfactant concentration in the following sections. If the surfactant is not present, the surface tension is constant, and the corresponding terms drop out.

#### D. Special Case

We have assumed that  $\epsilon^2 Re \ll 1$ ,  $\frac{\epsilon^2 g.Re}{Fr} \ll 1$  and  $\epsilon Ma \ll 1$ . In this section, we consider some special cases in which these conditions do not hold.

##### 1) Dominant boundary forces

We consider the case when  $\epsilon Ma$  is of order one or higher. Since we assume  $\Delta y \ll y$  this means  $Ca \ll \epsilon$  As we can see from (32), we have to rescale the pressure and will use the scaling  $P = \frac{\mu U}{(\epsilon^2 L)} p'$ . In this case, the dimensionless Navier-Stoke equations in leading order becomes;

$$u_{0x} + u_{0y} = 0 \tag{38}$$

$$P_{0x} = u_{0yy} \tag{39}$$

$$P_{0y} = 0. \tag{40}$$

The boundary conditions change accordingly (we assume  $\sigma_y^\pm = 0$ ).

$$2v_0 = \pm h_{0t} \pm u_0 h_{0x} \tag{41}$$

$$\frac{\epsilon^3}{2Ca} h_{0xx} = -P_0 \tag{42}$$

$$\pm \epsilon Ma \sigma_x^\pm = u_{0y}. \tag{43}$$

(40) yields constant pressure across the film, hence (42) gives

$$P_0 = \frac{\epsilon^3}{2Ca} h_{0xx}.$$

Integrating (39) across the film using (43) gives, together with the expression for the pressure:

$$\frac{\epsilon^3}{2Ca} h_{0xxx} + Ma(\sigma_0^+ + \sigma_0^-)_x = 0$$

Then integrating (39) twice with (43) yields

$$u_0 = \bar{u} + \frac{\epsilon Ma}{2} (\sigma_0^+ - \sigma_0^-)_x y + \frac{\epsilon^3}{4Ca} h_{0xxx} \left( \frac{h_0^2}{12} - y^2 \right),$$

where

$$\bar{u} = \frac{u_0 \left( \frac{h_0}{2} \right) + u_0 \left( \frac{-h_0}{2} \right)}{2} + \frac{\epsilon^3}{4Ca} h_{0xxx} \left( \frac{h_0^2}{6} \right)$$

Note that the Marangoni term cancels due to symmetry. We observe that we no longer have constant velocity  $u_0$  across the films; but that we obtain a parabolic velocity profile as in lubrication theory. Finally, integrating (38) across the film together with (41), we obtain mass conservation:

$$h_{0t} + (h_0 \bar{u})_x = 0.$$

The final system for  $h$  and  $\bar{u}$  is then given by (using symmetry and leaving the subscripts):

$$\begin{aligned} 0 &= h_t + (\bar{u}h)_x \\ 0 &= \frac{2Ma}{\epsilon} \sigma_x + \frac{\epsilon}{2Ca} hh_{xxx}, \end{aligned}$$

Apart from the fact that the tangential velocity is no longer constant, across the film, this is exactly the same model as we obtained before in (36)-(37) for the case that inertial and viscosity can be neglected; that is, the film is dominated by surface forces.

##### 2) Model of a fast film

We will now consider the case in which the liquid drains out of the film very fast; that is, the velocity scaling is so large that the condition  $\epsilon^2 Re \ll 1$  from the previous section no longer holds. Assuming that inertia forces enter in leading order, we obtain the following system.

$$u_{0x} + v_{0y} = 0$$

$$\epsilon^2 Re (u_{0t} + u_0 u_{0x} + v_0 u_{0y}) = u_{0yy}$$

$$\epsilon^2 Re (v_{0t} + u_0 v_{0x} + v_0 v_{0y}) = v_{0yy} - P_{0y}$$

With boundary conditions:

$$v_0 = \pm \frac{1}{2} h_{0t} \pm \frac{1}{2} u_0 h_{0x}$$

$$\frac{\epsilon}{2Ca} h_{0xx} = -P_0 \pm u_{0x} h_{0x} + 2v_{0y}$$

$$\pm \epsilon Ma \sigma_{0x}^\pm = u_{0y}$$

There are two possible scenarios;

- Capillary and Marangoni forces are negligible, and the tangential velocity  $u$  is constant across the film. Then we obtain the following hyperbolic system:  $h_t + (uh)_x = 0, u_t + uu_x = 0$ . That is, the flow is completely inertia-dominated.
- Capillary and Marangoni forces enter at leading order; the  $u$  is not constant across the film. In this case, we are not able to simplify the system further. If the velocity scaling is very fast, capillary and Marangoni forces only appear in leading order if  $\epsilon$  is large. Hence, the model describes a relatively thick film for which the flow film approximation does not hold and the full problem has to be solved.

#### E. Effects of pressure Drop Across the Film

This section shall show how the thin film equations of Section D(1) may be modified to take account of pressure drop across the film. We shall set  $Ma \sim \mathcal{O}(\epsilon)$ ,  $Ca \sim \mathcal{O}(\epsilon)$  in all the calculations that follows, that is, we work in the distinguished limit viscous, capillary, and Marangoni forces already balance. If we wish to incorporate a difference in pressure across the film, as in the case of a film between two bubbles of different sizes, or in a glass bottle manufacture, we must rewrite boundary conditions as

$$\begin{aligned} \pm \frac{\epsilon}{Ca} \left( H_{xx} \pm \frac{1}{2} h_{xx} \right) &= P_\pm - P + \frac{2\epsilon^2 u_x (h_x \pm \frac{1}{2} h_x)^2}{1 + \epsilon^2 (h_x \pm \frac{1}{2} h_x)^2} - \\ &\frac{2\epsilon^2 v_x (H_x \pm \frac{1}{2} h_x) - 2u_y (H_x \pm \frac{1}{2} h_x) + 2v_y}{1 + \epsilon^2 (H_x \pm \frac{1}{2} h_x)^2}, \end{aligned} \tag{44}$$

where  $P_\pm$  represents the nondimensional pressure above and below the film. If we allow the pressure difference,  $\Delta P = P_+ - P_-$ , to enter the problem at leading order, we modify (26) to read  $\pm \frac{\epsilon}{Ca} \left( H_{0xx} \pm \frac{1}{2} h_{0xx} \right) = P_\pm - P_0 + 2v_{0y}$  on  $y = H_0 \pm \frac{1}{2} h_0$

$$\tag{45}$$

Integrating the normal momentum balance and applying the above boundary conditions results in the following equation for the center line

$$-\Delta P + \frac{2\epsilon}{Ca} H_{0xx} = 0 \tag{46}$$

That is, the center line is a parabola. The leading order pressure is modified to read

$$P_0 = \frac{(P_+ + P_-)}{2} - \frac{\epsilon}{2Ca} H_{0xx} - 2u_{0x} \quad (47)$$

The equation for conservation of mass and the longitudinal force balance remains the same. Therefore, the effect of including a pressure drop across the film is to force the centerline to have non-zero curvature.

## F. Surfactant

### 1) Physical model

As discussed in the previous sections, the presence of a surfactant reduces the surface tension of an interface, and the surface tension becomes a function of the interfacial concentration,  $C$ , that is,  $\sigma = \sigma(C)$ . All surfactants tend to be partially soluble in the bulk liquid. In general, there will be a relationship between the surfactant adsorbed at the surface and the concentration in bulk [32, 20]. There is a need to model a foam stabilized by the effect of a surfactant on the surface tension. We denote its bulk concentration by  $C(x, y, t)$  and surface surfactant concentration on the top and bottom surfaces by  $\Gamma^\pm(x, t)$  respectively. We assume that the bulk concentration is governed by convection and diffusion. Thus, we obtain the following equation for  $C(x, y, t)$ : (37)

$$C_t + uC_x + vC_y = D_s(C_{xx} + C_{yy}) \quad (48)$$

The diffusivity  $D_s$  is a material parameter that we assume to be independent of space and time.

### 2) Conditions at the free interfaces

Surfactants tend to assemble at the surface of the liquid. Therefore, it is not sufficient to consider the concentration at the surface as the trace of the bulk concentration, but a new quantity is introduced, the surface concentration  $\Gamma$ . We assume that  $\Gamma$  is governed by convection, diffusion and a flux of surfactant from the bulk onto the surface, [4] thus it is described by

$$\Gamma_t + \nabla_{\Gamma} \cdot (u_{\Gamma} \Gamma) = \nabla_{\Gamma} \cdot (D_{\Gamma} \nabla_{\Gamma} \Gamma) + j \quad (49)$$

$$\Gamma_t + \frac{\partial(u_s \Gamma)}{\partial s} - D_s \frac{(\partial^2 \Gamma)}{(\partial s^2)} = j$$

The index  $\Gamma$  stands for the surface, that is, the directions spanned by the tangential vector. The material parameter  $D_{\Gamma}$  is the surface diffusivity assumed to be constant, and  $j$  is the flux of surfactant from the bulk. We have to close the system by adding some more equations for the newly introduced unknowns  $\Gamma$  and  $j$  as well as for the surface tension  $\sigma$ . We need a relation between  $\Gamma$  and  $C$  which is introduced by a constitutive equation for the flux  $j$ ,

$$j = j(C, \Gamma).$$

There are several such models in chemical literature, and we apply one of the most common ones, the Langmuir-Hinshelwood equation [4, 2].

$$j = k_1(C(\Gamma_{\infty} - \Gamma) - k_2\Gamma) \quad (50)$$

In (49), sets the rate of adsorption of surfactant at the surface to be proportional to the subsurface concentration,  $C(x, H, t)$  and to the amount of space available at the surface. The material parameter  $k_1$  and  $k_2$  also determine the relative magnitudes of adsorption and desorption. It is often assumed [2] that the adsorption process happens on a much faster time scale than the

other effects. In this case, (50) reduces to a relation for the thermodynamic equilibrium called the Langmuir isotherm.

$$\Gamma = \frac{(\Gamma_{\infty} C)}{(k_2 + C)} \quad (51)$$

A relation between the flux  $j$  and the bulk concentration  $C$  can be divided under the assumption that the flux onto the surface in the bulk is controlled by diffusion and therefore. This is given by

$$j = -D_s \frac{\partial C}{\partial n},$$

so that,

$$j = \frac{D_s}{\sqrt{1 + (H_x \pm \frac{1}{2} h_x^{\pm})^2}} \left( C_y - \left( H_x \pm \frac{1}{2} h_x^{\pm} \right) C_x \right) \quad (52)$$

$$\mp \frac{D_s}{\sqrt{1 + (H_x \pm \frac{1}{2} h_x)^2}} \left( C_y - \left( H_x \pm \frac{1}{2} h_x \right) C_x \right)$$

In contrast to equation (50), describes the behavior in bulk and not at the interface. However, due to continuity reasons, the two expressions are equal at the interface. Therefore, we can eliminate  $j$  and obtain two equations by equation (52) and (49) on the other hand and (52) and (50) on the other hand. Using (7) we get

$$k_1 = (C(\Gamma_{\infty} - \Gamma) - k_2\Gamma) = \frac{D_s}{\sqrt{1 + (H_x \pm \frac{1}{2} h_x)^2}} \left( C_y - \left( H_x \pm \frac{1}{2} h_x \right) C_x \right) \quad (53)$$

### 3) Effect of surfactant on the surface tension

At free surface, a surfactant is able to expel its hydrophobic tail from the solution and this reduces the surface energy of the system. For this purpose, it is necessary to impose the resulting constitutive relation between the surface tension and the surface concentration. Therefore, we apply Frumkin equation [2].

$$\sigma^* - \sigma = -R\theta\Gamma_{\infty} \ln \left( 1 - \frac{\Gamma}{\Gamma_{\infty}} \right), \quad (54)$$

(54) models relation for a wide range of surfactants. The term  $\sigma^* - \sigma$  is sometimes called the "surface pressure". If the surfactant concentration is above the critical micelle concentration (see section 1.1.3), we must include diffusion of micelles and interplay between the bulk and the micellar concentrations; we shall not model such since it is beyond the remit of this work.  $R$  denotes the gas constant,  $\theta$  the temperature, and  $\sigma^*$  the surface tension of the pure liquid without surfactant. Note that this equation has a singularity for  $\Gamma = \Gamma_{\infty}$  and therefore becomes invalid in this limit. However, we will only consider relatively small concentrations in which the model is a good approximation of the real behavior. In thermodynamic equilibrium, we can plug (51) into (54), and we obtain after differentiation.

$$\sigma_x = -R\theta\Gamma_{\infty} \frac{C_x}{(k_2 + C)} \quad (55)$$

$$\sigma_y = -R\theta\Gamma_{\infty} \frac{C_y}{(k_2 + C)} \quad (56)$$

For the thin-film model (36)-(37), only the partial derivatives of the surface tension are needed. For small concentrations  $C \ll k_2$ , we can simplify (51) and (55)-(56) even further to obtain a linear relation.

$$\Gamma = \frac{(\Gamma_{\infty} C)}{k_2}$$

$$\sigma_x = -\frac{(R\theta\Gamma_\infty)}{k_2} C_x$$

$$\sigma_y = -\frac{(R\theta\Gamma_\infty)}{k_2} C_y$$

4) Non-dimensionalization

Additionally, to the dimensionless variables introduced earlier, we non-dimensionalize  $C$  and  $\Gamma$  by

$$C = C^* C'$$

$$\Gamma^\pm = \Gamma^* \Gamma^\pm$$

We also introduce some similarity parameters

i. The Peclet number  $Pe = \frac{UL}{D_s}$  which characterizes the relation of convection and diffusion in the bulk.

ii. The replenishment number  $S = \frac{(D_s C^*)}{(U \Gamma^*)}$ , which is the relation of diffusion from the bulk unto the surface and convection at the surface.

iii. Moreover, we introduce  $\Lambda = \frac{\Gamma^*}{\Gamma_\infty}$  and  $\Pi = \frac{C^*}{k_2}$  which describe the order of magnitude of the concentrations compared to the saturation concentrations

With these, the convection-diffusion (48) for the bulk concentration of the surfactant becomes (dropping primes):

$$\epsilon^2 Pe (C_t + u C_x + v C_y) = \epsilon^2 C_{xx} + C_{yy} \quad (57)$$

At the interfaces  $y = H \pm \frac{1}{2}h$

$$\Gamma_t^\pm \sqrt{1 + \epsilon^2 ((H_x \pm \frac{1}{2}h_x)^2)} + (U_\Gamma \Gamma^\pm)_x = \pm \frac{S}{\epsilon} (C_y + \epsilon^2 (H_x \pm \frac{1}{2}h_x) C_x) \quad (58)$$

Equation (53) becomes

$$\frac{D_s}{\epsilon k_1 L \Gamma^* \sqrt{1 + \epsilon^2 ((H_x \pm \frac{1}{2}h_x)^2)}} (C_y + \epsilon^2 1 + \epsilon^2 ((H_x \pm \frac{1}{2}h_x)^2) C_x) = \frac{C}{\Lambda} - C\Gamma - \frac{\Gamma}{\Pi} \quad (59)$$

Finally, we have the Frumkin (54) for the relation of surface tension and surfactant concentration;

$$\frac{(\sigma^* - \gamma)}{(R\theta\Gamma_\infty)} - \frac{\nabla\gamma}{(R\theta\Gamma_\infty)} \sigma^\pm = -\ln\left(1 - \frac{(\Gamma^* \Gamma^\pm)}{\Gamma_\infty}\right) \quad (60)$$

$$\Gamma^\pm = \frac{\frac{(\Gamma_\infty \Pi) C}{\Gamma}}{1 + \Pi C} \quad (61)$$

$$\Gamma^\pm = \frac{\left(\frac{\Pi C}{\Lambda}\right)}{(1 + \Pi C)}$$

In the presence of a surfactant,

$$\sigma = \frac{R\Gamma_\infty}{\nabla\gamma}$$

We obtain

$$0 = h_t + (uh)_x \quad (62)$$

$$0 = \frac{\epsilon}{2Ca} h h_{xxx} + \frac{2Ma}{\epsilon} \sigma_x - Reh(u_t + uu_x) + h \frac{gRe}{Fr} e g_x + 4(hu_x)_x \quad (63)$$

$$(hC_x)_x - Pe h(C_t + uC_x) - \frac{2}{\epsilon S} (C_t + (u_s C)_x) = 0 \quad (64)$$

Where

$$\sigma_x = -\Gamma_x = C_x \quad (65)$$

### III. MODELLING VELOCITY, DRAINAGE AND VOLUMETRIC FLUX PROFILE OF THIN-FILM

We have derived a system of equations for the description of a thin film between two free surfaces in the previous chapter.

We have considered all the phenomena that we assume to play an important role in the thinning process. In particular, these are gravity, inertia, viscosity, capillarity and marangoni forces due to the presence of surfactants. Moreover, we have derived equations modeling pure liquids and surfactant. The approach to drainage in foam lamellae and isolated films are central to the process of the thinning rate and lifetime of aqueous thin film spread over a fuel spill on a plateau to mitigate fuel vapors. We deal with the aqueous film arising in application of aqueous film forming foam, AFFF, which contains small amounts of fluorocarbon that enable a higher-density aqueous film to form on top of the lower-density hydrocarbon fuel-spill surface by lowering the surface tension of the film. It is the film that suppresses the transport of the fuel vapor from the pool surface to the surroundings [33]. In order to gain knowledge about its decay rate, we study the thinning of a single foam film. Therefore, we have to study the environment of the given process and the general behavior of foam in the process. In the following, we assume on this basis that we can consider a thin film starting as a horizontal uniformly spread liquid on the surface of fuel and thinning due to drainage. The computation starts at the time when the film is thin enough such that the thin film approximation can be applied.

#### A. Assumptions

The following assumptions are made about the model:

- i. The film is long and thin so that a typical length scale,  $L$  is much larger than a typical thickness.
- ii. The liquid density  $\rho$  is constant.
- iii. The shear viscosity  $\mu$  is constant.
- iv. The surface tension  $\gamma$  to be equal on the two interfaces.
- v. The liquid velocity is characterized by a typical value  $U$ .
- vi. Exterior forces acting on the thin film due to the fuel and foam flow are negligible
- vii. The geometry of the computational domain is considered to be constant in time

#### B. Initial and Boundary Conditions

As we have already mentioned in the previous section, no boundary conditions at the ends of the thin film have been considered. We are dealing with surfactant stabilized thin film. Therefore, we discuss conditions at the boundaries of the computational domain. Moreover, the challenge of finding initial values for the thin film problem is addressed.

##### 1) 6.2.1 One-Dimensional Problem

We consider a case of a two-dimensional foam. Apart from the initial condition, we need conditions at the boundary  $x = x_L = 0$  and  $x = x_R = 1$ .

**Conditions at  $x = x_L = 0$**  The film thickness,  $h$ , the velocity  $u$  and the concentration  $C$ , respectively, we have

$$h_x = 0 \quad (66)$$

$$u = 0 \quad (67)$$

$$C_x = 0 \quad (68)$$

**Conditions at the interface  $x = x_R = 1$**

Recall that in the thin film approximation, for a viscous dominated film, the momentum equation reduces to  $4(hu_x)_x = 0$  in 1D. The first boundary condition for the film thickness  $h$ , at the right boundary is therefore

$$h_x = 0 \tag{69}$$

A possibility to define the condition for  $h$ :

Use the reduced momentum equation and set

$$4(hu_x)_x = 0 \tag{70}$$

If  $h_s$  can be obtained, the first approach is to be preferred. However, in general we do not have this information such that we will use the second alternative, although it is a less exact approximation. We consider the velocity  $u$ , for which one more condition at  $x = x_R = 1$  is needed. There are several possibilities for this and in the following, we will discuss their advantages and disadvantages

- i. Another possible condition is to set  $u(1) = \frac{q}{h}$  but as in the first case, this is only a good approximation close to the interface  $x = x_R = 1$ .
- ii. A second approach is to set constant flux at  $x = x_R = 1$ , that is

$$(uh)_x = 0 \tag{71}$$

Plugging this into the mass equation, we observe that this results in  $h_t(1) = 0$ , such that the thickness of the thin film at the boundary remains constant. If the behavior of the thickness of the interface is known this condition can be improved to

$$(uh)_x = -h_t(1) \tag{72}$$

at the boundary  $x_R = 1$ . In what follows, we will use the second approach.

Finally, the concentration  $C$ , is considered. Then we prescribe a homogeneous Neumann condition,

$$C_x = 0. \tag{73}$$

### C. Parameter Settings

In order to solve the film-thinning problem for surfactant-stabilized film, we need to have some information about the physical parameters appearing in our model. These are approximated and outlined in Table I.

TABLE I. Parameter for the aqueous Surfactant

Parameter	Name	Value range	Unit
$D_s$	Diffusivity	$10^{-10} - 10^{-5}$	$m^2/s$
$D_\Gamma$	Surface diffusivity	$10^{-10} - 10^{-5}$	$m^2/s$
$C^*$	Typical bulk concentration	$\approx 10^{-4}$	$mol/m^3$
$\Gamma_\infty$	Surface saturation concentration	$\leq 10^{-6} - \leq 10^{-5}$	$mol/m^3$
$\Gamma^*$	Typical surface concentration	$\leq 10^{-6} - \leq 10^{-5}$	$mol/m^3$
$\chi_2$	Langmuir parameter	$10^{-2} - 1$	$mol/m^3$
$R$	Universal gas constant	$\approx 8.3$	$J/molK$
$\theta$	Temperature	$\approx 300$	$K$

### D. Existence and Uniqueness of the Linearized Problem

We consider a one-dimension problem in the presence of a surface active agent, in which the constant parameters are set to one and make the assumption that capillary effect can be neglected. Moreover, we do not regard any boundary conditions but assume a Riemann problem on  $\mathbb{R}$ .

**Riemann problem:** Find  $h, u, C: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , such that

$$0 = h_t + (uh)_x,$$

$$0 = u_t + uu_x - u_{xx} - \frac{h_x}{h}u_x + \frac{c_x}{h} - 1,$$

$$0 = C_t + uC_x + \frac{u_x C}{1+h} - \frac{hC_{xx}}{1+h} - \frac{h_x C_x}{1+h}$$

$$c_x = -\sigma_x$$

On  $[0, T] \times \mathbb{R}$  and

$$h = h_0,$$

$$u = u_0,$$

$$C = C_0$$

On  $\{0\} \times \mathbb{R}$ .

Equivalently, we can write the system in conservative form as

$$U_t + F(U)_x = G(U) \tag{74}$$

With

$$U = \begin{bmatrix} h \\ u \\ C \end{bmatrix}, \quad F = \begin{bmatrix} uh \\ \frac{u^2}{2} + \frac{c}{h} \\ uC \\ \frac{1}{1+h} + uC \end{bmatrix}$$

denoting the derivatives with subscripts Eq.(74) can be written in the form

$$U_t + H(U)(U)_x = G(U) \tag{75}$$

We linearize this problem with respect to the initial values  $h_0, u_0$  and  $C_0$ , yielding the **Linearized problem:** Find  $h, u, C; [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$0 = h_t + (hu_0)_x + (h_0u)_x - (h_0u_0)_x, \tag{76}$$

$$0 = u_t + (u_0u)_x - u_{xx} - \frac{h_{0x}}{h_0}u_x - \frac{u_{0x}}{h_0}h_x + \frac{u_{0x}h_{0x}}{h_0^2}h + \frac{c_x}{h_0} - \frac{c_{0x}}{h_0} - 1 - u_0u_{0x} + \frac{c_{0x}}{h_0} \tag{77}$$

$$0 = C_t + u_0C_x + C_{0x}u + \frac{C_0}{1+h_0}u_x + \frac{u_{0x}}{1+h_0}C - \frac{u_{0x}C_0}{(1+h_0)^2}h - \frac{h_0}{1+h_0}C_{xx} - \frac{C_{0xx}}{1+h_0} + \frac{h_0C_{0xx}}{(1+h_0)^2}h - \frac{h_{0x}}{1+h_0}C_x - \frac{C_{0x}}{1+h_0}h_x + \frac{h_{0x}C_{0x}}{(1+h_0)^2} - u_0C_{0x} - \frac{u_{0x}C_0}{(1+h_0)^2} + \frac{h_0C_{0xx}}{(1+h_0)^2} + \frac{h_{0x}C_{0x}}{(1+h_0)^2} \tag{78}$$

On  $[0, T] \times \mathbb{R}$  and

$$h = h_0,$$

$$u = u_0,$$

$$C = C_0$$

On  $\{0\} \times \mathbb{R}$ . For the integral form of the linearized problem, the mathematical definitions of the respective spaces and norms are needed.

#### Definition 1

Let

$$H^m(\mathbb{R})\{u \in L^2(\mathbb{R}): D^2u \in L^2(\mathbb{R}), |\alpha| \leq m\}$$

Be a Sobolev space of order  $m$ . In particular  $H^0(\mathbb{R}) = L^2(\mathbb{R})$ . Define the spaces  $V, H$  by

$$V := H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$$

$$H := L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$$

With  $(\cdot, \cdot)_{H^m}$  and  $(\cdot, \cdot)_{L^2}$  as the standard norms defined on the spaces  $H^m$  and  $L^2$ . Note that  $V$  and  $H$  are Hilbert spaces with respect to  $(\cdot, \cdot)$  and  $(\cdot, \cdot)$ , respectively.

**Definition 2** Let  $V, H$  be a pair of real separable Hilbert spaces with corresponding scalar products  $(\cdot, \cdot)$  and  $(\cdot, \cdot)$ , and norms  $\|\cdot\|$  and  $|\cdot|$ .



Let  $T \in \mathbb{R} \cup \{\infty\}$ ,  $B$  be a Banach space.  $L^2(B) := L^2(0, T; B)$  denotes the space of functions  $t \rightarrow f(t)$ :  $(0, T) \rightarrow B$  such that

- i.  $f$  is measurable for  $dt$ ,
- ii.  $\|f\|_{L^2(B)} = \left(\int_0^t \|f(t)\|_B^2 dt\right)^{\frac{1}{2}} < \infty$

Moreover, we define the space

$$W(V) := W(0, T; V, V') := \{u : u \in L^2(V), u' \in L^2(V')\}$$

Where  $V'$  denotes the dual space of  $V$

Using Definition 1, Integral formulation requires less derivability conditions on the solutions, which allows to obtain discontinuous solutions. These discontinuous solutions do not verify the partial differential equation at every point because the derivatives are not defined at the discontinuities, and must meet a 'jump condition' along them, which is obtained from the integral form (see Lemma 1, Rankine-Hugoniot condition).

**Problem 1**

**(Integral Form):** Find

$$U = \begin{bmatrix} h \\ u \\ C \end{bmatrix} : [0, T] \times \mathbb{R} \rightarrow V$$

fulfilling

$$U(t = 0) = U_0 \in V \tag{79}$$

Such that

$$\frac{d}{dt} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(x, t) dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial}{\partial x} F(U(x, t)) dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} G dx \tag{80}$$

Since the integrand (80) is continuous and it holds for every segment  $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ , then it follows that the integrand must vanish in order to have the partial differential equation given by (74). All the integration are performed over  $\mathbb{R}$ .

**Theorem 1** Discontinuous Solution to (76), and (77) Let  $u_0 \in C'(\mathbb{R})$ . Then the Cauchy problem has a unique solution  $u \in [(0, T) \times \mathbb{R}]$ .

*Proof.* Note that classical solutions are weak solutions and weak solutions that lie in  $C^1(\mathbb{R} \times [0, T])$  satisfy (74), (79) in the classical sense. An important class of the solutions are piecewise classical solutions with discontinuities separating the smooth regions. The following lemma gives a necessary and sufficient condition imposed on these discontinuities such that the solution is a weak solution [12, 16].

**Lemma 2** Rankine-Hugoniot jump condition Assume that  $\mathbb{R} \times [0, T]$  is separated by a smooth curve  $x = x(t)$  into two parts  $\Omega_1$  and  $\Omega_2$ . Suppose that  $u$  is a classical solution of (74) on either side of the curve  $x = x(t)$  and that  $u$  and its derivatives extend continuously to the curve from the left to the right. Furthermore, assume  $u$  is a  $C^1$ -function on  $\Omega_1$  and  $\Omega_2$ , respectively. We therefore obtain the necessary condition

$$(u_R(t) - u_L(t))\dot{x}(t) = f(u_R(t)) - f(u_L(t))$$

for  $u$  to be a solution of the integrated version of the conservation law. It is often written

$$[f(u)] = \dot{x}[u]$$

By allowing shocks [17] we can solve the IVP uniquely. We first look at constant states. The simplest problem to solve is the transition from one constant speed, say  $u_R$  to another speed  $u_L$ . The deceleration requirement is that  $u_R > u_L$ ,  $h_R > h_L$ ,  $c_R > c_L$ .

The shock speed  $s = \dot{x}(t) = \frac{F_R - F_L}{u_R - u_L}$ , is the slope of the line segment connecting two points on the flux curve.

Using definition we can admit discontinuous solutions using the integral form of the conservation laws, that is

$$\frac{d}{dx} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(x, t) dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial}{\partial x} (F(U(x, t))) dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} G dx \tag{81}$$

where

$$F(U) = \begin{bmatrix} u_0 h + h_0 u \\ u_0 u + \frac{c}{1 + h_0} \\ \frac{u c_0}{1 + h_0} + u_0 c \end{bmatrix}, \quad G = \begin{bmatrix} (h_0 u_0)_x \\ 1 + u_0 u_0 x - \frac{C_{0x}}{h_0} \\ u_0 C_{0x} + \frac{u_{0x} C_0}{(1 + h_0)^2} - \frac{h_{0x} C_{0x}}{(1 + h_0)^2} \end{bmatrix}$$

**Theorem 3** Let  $h_0, u_0, C_0$  and their spatial derivatives be bounded by some  $\tilde{M}$  and let  $h_0(x) \geq H_0 > 0$  for all  $x \in \mathbb{R}$ . Then there exists a unique solution  $U$  of problem 1 and the solution fulfills

$$U \in W(0, T; V, V').$$

*Proof.* We prove Theorem 3 by applying results from [8, p.~513], following two conditions:

**Condition 1** For each  $t \in [0, T]$  we are given a continuous function  $U$  on  $V$  with the following property: For  $u \in V$ , the map  $t \rightarrow U(t, u)$  is measurable and there exists a constant  $M = M(t) > 0$  such that

$$|U(t, u)| \leq M \|U\| \quad \forall u \in V$$

**Condition 2** The initial conditions and sources satisfy

$$u_0 \in H^1(0, 1), \quad G \in L^2(V').$$

Using these, the following theorem can be stated:

**Theorem 4** Consider the following problem:

Find  $u$  satisfying

$$u \in W(V)$$

and

$$\frac{d}{dt} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(x, t) dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial}{\partial x} f(U(x, t)) dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} G dx$$

Moreover  $u(0) = u_0$ . Suppose the spaces  $V, H$  are given,  $U(t; X(t; 0, y))$  satisfies Condition 1, and  $u_0, G$  satisfy Condition 2. Then the problem has a unique solution  $u$  fulfilling  $W(0, T; V, V')$ .

For the proof of Theorem 1, we refer to [9].

**Proof Existence.** Let  $u_0(x) \in C_0^1(\mathbb{R})$  and  $f \in C^2(\mathbb{R})$  with  $u_0 \geq 0$  be given. Since  $u$  is defined as the composition of the  $C^1$  maps  $u_0$  and  $(t, x) \rightarrow X(0; t, x)$ . Then there exists a maximal time  $T > 0$  such that (75), (79) has a unique solution  $U \in C^1([0, T] \times \mathbb{R})$ . The corresponding characteristics are straight lines. If  $H(u_0)$  for the system (75), (79) is increasing, the maximal existence time is infinite,  $T = \infty$ . The formula (79) defines a  $C^1$  function. Since  $u$  is defined as the composition of the  $C^1$  maps  $u_0$  and  $(t, x) \rightarrow X(0; t, x)$ . This function obviously satisfies  $u|_{t=0} = u_0$ .

### Uniqueness

Uniqueness follows from the uniqueness of characteristics (the PDE is reduced to an ODE along the characteristic), as follows. Since the map  $t \rightarrow X(t; 0, y)$  is  $C^1$  and  $u \in C^1$ , so is the map  $t \rightarrow u(t, X(t; 0, y))$ . Applying the chain Rule one easily sees that

$$\frac{d}{dt}u(t, X(t; 0, y)) = 0.$$

Hence  $t \rightarrow u(t, X(t; 0, y))$  is constant on  $[0T]$  and  $u(t, X(t; 0, y)) = u(0, X(0; 0, y)) = u_0(y)$ .

Letting  $x = X(t; 0, y)$  we have  $x = X(0; t, y)$  so that  $u(t, x) = u_0(X(0; t, x))$ ,  $\forall (t, x) \in [0, T] \times \mathbb{R}$ .

For the proof of Theorem 2, we refer to [8]

**Condition 1.** We have to show that the function  $U$  is bounded, i.e. for each  $U \in V$ ,  $t \rightarrow U(t, u)$  is measurable and there exists a constant  $M = M(t) > 0$  such that

$$|U(t, u)| \leq M \|U\|$$

Since we assumed that  $h_0, h_{0x}, u_0, u_{0x}, c_0, c_0x \leq \tilde{M}$  are all bounded, we can drag them in front of the integrals in (80). Further  $h_0^{-1}$  is bounded by  $H_0^{-1}$ . We denote the common bound of  $\tilde{M}$  and  $H_0^{-1}$  by  $M_0$  and obtain from (80):

$$|U(U)| \leq M_0 \int |h_x| + M_0 \int |u_x| + M_0 \int |u_x| + M_0 \int |u_{xx}| + M_0^2 \int |u_x| + M_0^2 \int |h_x| + M_0^4 \int |h| + M_0 \int |c_x| + M_0^3 \int |h| + M_0 \int |c_x| + M_0 \int |u| + M_0^2 \int |u_x c| + M_0^2 \int |c| + M_0^4 \int |h| + M_0^2 \int |c_{xx}| + M_0^4 \int |c_x| + M_0^2 \int |h_x| + M_0^4 \int |h| + M_0^4 \int |h_x| + M_0^4 \int |h_x| \leq M \cdot \|U\|$$

Such that  $|\cdot|$  denotes the absolute value not the norms in  $H$ .

**add Condition 2.** The condition on  $U_0$  follows immediately from the assumptions of the problem since  $V \subset H$ .

This concludes the proof of Theorem 3. We have shown that there exists a unique solution  $U$  of problem 2 which fulfills  $U \in W(0, \Delta T; V, V')$ .

### E. A Finite Volume Approach

The staggered finite volume method is advantageous for the discretization of conservation laws such as the thin film equations due to its direct connection to the physical flow properties. In this section, we consider the one-dimensional case with capillary effect negligible. The flux evaluation is the most important step in this solution [18]. The method can be formulated analogously for the two-dimensional problem. We consider the 1-D case as follows: Let  $\Omega = [0,1]$ . Find  $U = (h, u, c)^T: [0, T] \times \Omega \rightarrow \mathbb{R}$ , such that

$$0 = h_t + (uh)_x \tag{82}$$

$$0 = \frac{2Ma}{\epsilon} c_x + Reh(u_t + uu_x) - h \frac{Re.g}{Fr} e_{gx} - 4(hu_x)_x \tag{83}$$

$$0 = \frac{2}{\epsilon s} (C_t + (uC)_x) + Peh(C_t + uC_x) - (hC_x)_x \tag{84}$$

In  $[0, T] \times \Omega$  and

$$u = u_0$$

On  $\{0\} \times \Omega$ . Moreover, the boundary conditions

$$0 = h_x \tag{85}$$

$$0 = c_x \tag{86}$$

at  $x = 0$ , and

$$0 = c_x \tag{87}$$

$$0 = (hu)_x \tag{88}$$

$$u = \frac{q}{h} \tag{89}$$

At  $x = 1$  must be fulfilled. In order to formulate the integral form of problem, we have defined a suitable space.

**Definition 3:** Let  $\Omega = [0,1]$ . Let the sobolev space  $H^m(\Omega)$  be given analogous to Section 6.4 by

$$H^m(\Omega) = \{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega); |\alpha| \leq m\}$$

We can then define the spaces

$$V_h(\Omega) = \{u \in H^2(\Omega); u_x(0) = 0\},$$

$$V_u(\Omega) = \{u \in H^1(\Omega); u(0) = 0\}$$

and

$$V_c(\Omega) = H^1(\Omega).$$

Moreover, we let

$$V := V_h(\Omega) \times V_u(\Omega) \times V_c(\Omega)$$

Thus, we can formulate the

**Piecewise problem** Find  $U: [0, T] \rightarrow V$ , such that

$$\frac{d}{dt}(U) + F(U)_x = 0 \tag{90}$$

where

$$U(0) = U_0 \in V$$

The corresponding finite-dimensional approximation reads

**Finite volume problem** Find  $U_N: [0, T] \rightarrow V_N$ , such that

$$\frac{d}{dt}(U_N) + F(U_N)_x = 0 \tag{91}$$

where

$$U_N(0) = U_{N_0} \in V_N$$

$V_N$  denotes the finite-dimensional approximation space to  $V$ . There are multiple possibilities to define such a space that differ in their approximation qualities. For further information, we refer to [5]. The following is an exemplary approximation space  $V_N$  suitable for our problem.

**Remark (91)** is a finite-dimensional system of ordinary differential equations which can be solved using a standard solver.

We describe the staggered scheme in the one-dimensional case. We consider the time interval  $(0, T)$  and the space domain  $\Omega := (0, 1)$ . The time interval is divided into  $N_t$  time steps of length  $\Delta t$  and for all  $n \in \{0, 1, \dots, N_t\}$ ,  $t^n = n \Delta t$ . The domain  $\Omega$  is divided into  $N_x$  cells of length  $\Delta x$ . The left end, the center and the right end of the  $j$ -th cell are denoted by  $x_{j-\frac{1}{2}}, x_j$  and  $x_{j+\frac{1}{2}}$ , respectively. This method is based on writing the governing equations in integral form over an elementary control volume or cell, hence the general term Finite Volume (FV) Method.

The approximation of  $h$  at point  $x_j$  and at time  $t^n$  is denoted by  $h_j^n$ . The velocity  $u$  is discretized at the interfaces between the cells. The approximation of  $u$  at point  $x_{j+\frac{1}{2}}$  and at time  $t^n$  is denoted  $u_{j+\frac{1}{2}}^n$ . Fluxes, on the other hand are evaluated at the interfaces between  $(x_{j-\frac{1}{2}}$  and  $x_{j+\frac{1}{2}})$ . From the derivation in Chapter 3, thin film drainage model (82)-(84) are balance laws in the form (91).

For the  $j$ th cell, the integration of (91) with respect to  $x$  from control surface  $x_{j-\frac{1}{2}}$  to control surface  $x_{j+\frac{1}{2}}$  yields:

$$\frac{d}{dt} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u dx + F_{x_{j+\frac{1}{2}}} - F_{x_{j-\frac{1}{2}}} = 0 \tag{92}$$

Recalling that the flow variables ( $h$  &  $u$ ) are averaged over the cell, the application of Green's theorem to (92), gives

$$u_j^{-n+1} = u_j^{-n} - \frac{\Delta t}{\Delta x} \left[ F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \right] \quad (93)$$

Where the superscript  $n$  and  $n + 1$  reflect the  $t$  and  $t + \Delta t$  time levels respectively. In (93), the determination of  $u$  at the new time step  $n + 1$  requires the computation of the numerical flux at the cell interfaces at the old time  $n$  and the evaluation of the source term. The source terms are introduced into the solution through a second-order time splitting. The evaluation of the flux term is presented in the next section.

1) Flux Computation

In the Godunov approach the numerical flux  $F_{j+1/2}$ , is determined by solving a local Riemann problem, section 4.5.4 of the Handbook on Numerical Analysis Hyperbolic Problems) at each cell interface.

$$PDE: U_t + (F(U))_x = 0 \quad t \in [t_n, t_{n+1}] \quad (94)$$

$$ICs: u(x, t_n) = \begin{cases} u_L, & \text{if } x < x_{j+\frac{1}{2}}, \\ u_R, & \text{if } x > x_{j+\frac{1}{2}}. \end{cases}$$

Where the left and right states are given, respectively, by  $u_L = u_j^{-n}$  and  $u_R = u_{j+1}^{-n}$

To compute the flux  $F_{j+\frac{1}{2}}$  we need to know the solution  $u$  along the interface  $x = x_{j+\frac{1}{2}}, t_n \leq t \leq t_n + \Delta t$ . We introduce the shock speed at the cell interface;

$$\dot{s} = \frac{f(u_R) - f(u_L)}{u_R - u_L}$$

According to the Rankine-Hugoniot jump condition under the condition that  $f(u)$  is convex, using the method of characteristics: we have, for  $t_n \leq t \leq t_n + \Delta t$ ,

$$u(x_{j+\frac{1}{2}}, t) =$$

$$\begin{cases} u_L & \text{if } f'(u_L) > 0 \text{ and } f'(u_R) > 0 \\ u_R & \text{if } f'(u_L) < 0 \text{ and } f'(u_R) < 0 \\ u_L & \text{if } f'(u_L) \geq 0 \text{ and } f'(u_R) \leq 0 \text{ and } \dot{s} > 0 \\ u_R & \text{if } f'(u_L) \geq 0 \text{ and } f'(u_R) \leq 0 \text{ and } \dot{s} < 0 \end{cases} \quad (95)$$

Note that the four cases correspond either to a shock where, according to Lax's criterion  $f'(u_L) \geq \dot{s} \geq f'(u_R)$  so that  $u_{j+1/2} = u_L$  if  $u_{j+1/2} = u_R$  if  $\dot{s} > 0$ .

The celebrated Godunov's method, [18] is now obtained by simply using the solution to the Riemann problem (95) at each interface  $x_{j+\frac{1}{2}}$  to compute the numerical fluxes

$F_{j-\frac{1}{2}}, F_{j+\frac{1}{2}}$  in (94), yielding

$$F(x_{j+\frac{1}{2}}, t) =$$

$$\begin{cases} f(u_L) & \text{if } f'(u_L) > 0 \text{ and } f'(u_R) > 0 \\ f(u_R) & \text{if } f'(u_L) < 0 \text{ and } f'(u_R) < 0 \\ f(u_L) & \text{if } f'(u_L) \geq 0 \text{ and } f'(u_R) \leq 0 \text{ and } \dot{s} > 0 \\ f(u_R) & \text{if } f'(u_L) \geq 0 \text{ and } f'(u_R) \leq 0 \text{ and } \dot{s} < 0 \end{cases} \quad (96)$$

2) Stability of the Godunov method

The numerical results indicated that stability for the upwind scheme is subtle. It is unconditionally unstable as the scheme (101), instead. the stability depends on parameters  $\Delta x$  and  $\Delta t$ . Numerical results indicate the crucial role played by the ratio  $\parallel$

$\frac{\Delta x}{\Delta t} \parallel$ . Recall that the upwind method is stable under the CFL condition. Therefore, provided the CFL condition

$$\max_j |f'(u_j^n)| \frac{\Delta x}{\Delta t} \leq \frac{1}{2} \quad (97)$$

Is satisfied, Godunov's method is stable [18]. We assume that the scheme used is stable.

3) Consistency

A finite volume scheme (101) with numerical flux function

$$F_{j+\frac{1}{2}}^n = f(u_{j-n+1}^n, \dots, u_{j+n}^n) \quad (98)$$

is consistent if

$$F(u, \dots, u) = f(u) \quad \forall u \in \mathbb{R} \quad (99)$$

4) Convergence

The numerical results shows that the scheme with the flux function

$$F_{j+\frac{1}{2}}^n = f(u_j^n, u_{j+1}^n) = \frac{f(u_j^n) + f(u_{j+1}^n)}{2} \quad (100)$$

It gives the standard central difference scheme for the conservation law (94) end up being unconditionally unstable even for linear equations. The conservation equations are presented in the discretization below: Using finite volume framework, we assume that  $\bar{u}^n \approx \bar{u}(x_i, t^n), \bar{f}(\bar{u}^n) \approx \bar{f}(\bar{u}(x_i, t^n))$  and  $\bar{G}_i^n \approx \bar{G}(\bar{u}(x_i, t^n))$ . The fully discrete finite volume method [18] to get the numerical solution to the balance laws (94) is

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x} \left( \bar{F}_{j+\frac{1}{2}}^n - \bar{F}_{j-\frac{1}{2}}^n \right) \quad (101)$$

We now proceed to describe our staggered finite volume scheme for (82). The mass conservative approximation for (82) at the cell  $x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}$  is

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} \left( F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \right) \quad \forall j \in \mathbb{Z} \quad (102)$$

Where

$$F_{j+\frac{1}{2}}^n = \bar{h}_{j+\frac{1}{2}}^n u_{j+\frac{1}{2}}^n \bar{h}_{j+\frac{1}{2}}^n := \begin{cases} h_j^n & \text{if } u_{j+\frac{1}{2}}^n > 0, \\ h_{j+1}^n & \text{if } u_{j+\frac{1}{2}}^n < 0 \end{cases} \quad (103)$$

The upwind approximation (103) is a direct consequence of considering the flow direction-when the flow is to the right ( $u_{j-\frac{1}{2}} \geq 0$ ) we take the left flux  $h_j u_{j+\frac{1}{2}}$ ; and when the flow is to the left ( $u_{j+\frac{1}{2}} < 0$ ) we take the right flux  $h_{j+1} u_{j+\frac{1}{2}}$ .

The approximated momentum (83) and surfactant conservation (84) are

$$\begin{aligned} h_{j+\frac{1}{2}}^{n+1} u_{j+\frac{1}{2}}^{n+1} &= h_{j+\frac{1}{2}}^n u_{j+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x} [F_{j+1}^n \bar{u}_{j+1}^n - F_j^n \bar{u}_j^n + \\ &\frac{2Ma}{Re\epsilon} (h_{j+1}^{n+1} - c_j^{n+1}) - \frac{4}{Re} \left( h_{j+1}^{n+1} \frac{u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n}{\Delta x} - \right. \\ &\left. h_j^{n+1} \frac{u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n}{\Delta x} \right)] - \frac{g}{Fr} \Delta t h_{j+\frac{1}{2}}^{n+1} e_{g_x} \quad (104) \end{aligned}$$

$$\begin{aligned} h_{j+\frac{1}{2}}^{n+1} c_{j+\frac{1}{2}}^{n+1} &= h_{j+\frac{1}{2}}^n c_{j+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x} \left[ F_{j+1}^n \bar{c}_{j+1}^n - F_j^n \bar{c}_j^n + \right. \\ &\left. \frac{\alpha}{\alpha+\beta} c_{j+\frac{1}{2}}^{n+1} (u_{j+1}^n - u_j^n) - \frac{1}{\alpha+\beta} h_{j+1}^{n+1} \frac{c_{j+\frac{1}{2}}^n - c_{j-\frac{1}{2}}^n}{\Delta x} \right] \quad (105) \end{aligned}$$

Where

$$h_{j+\frac{1}{2}}^n = \frac{1}{2}(h_j^n + h_{j+1}^n), \quad \forall j \in \mathbb{Z} \tag{106}$$

$$F_j^n = \frac{1}{2}\left(F_{j-\frac{1}{2}}^n + F_{j+\frac{1}{2}}^n\right), \tag{107}$$

$$\bar{u}_j^n = \begin{cases} u_{j-\frac{1}{2}}^n & \text{if } F_j^n > 0, \\ u_{j+\frac{1}{2}}^n & \text{if } F_j^n < 0, \end{cases} \quad \forall j \in \mathbb{Z} \tag{108}$$

$$\bar{c}_j^n = \begin{cases} c_{j-\frac{1}{2}}^n & u_j^n > 0 \\ c_{j+\frac{1}{2}}^n & u_j^n < 0 \end{cases} \quad \forall j \in \mathbb{Z} \tag{109}$$

$h_{j+\frac{1}{2}}^n, F_j^n, \bar{u}_j^n$  are as in (106), (107) and (108).

We are now ready to present our results of numerical simulations of the thin film drainage model.

#### IV. SIMULATIONS

In the previous sections, we have derived and analyzed models for the description and simulation of the dynamical behavior of a thin film foam. We now present and discuss results of simulation of the velocity profiles of the interface of the surfactant-fuel. This will depict the drainage of the aqueous films confined surfactant-fuel thin film in order to arrest fuel spill explosions.

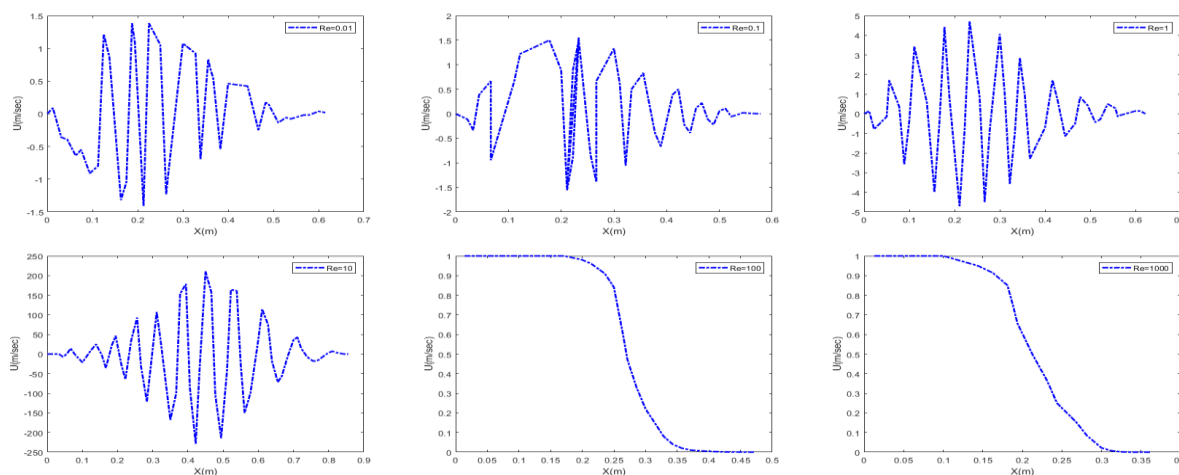


Fig. 2. Velocity profile of the thin-film of surfactant-fuel foam. The first row from left to right is  $Re = 0.01, 0.1, 1$ . The second row from left to right  $Re = 10, 100, 1000$

Fig. 2 indicated that for  $Re = 0.01$ , the velocity profile stabilizes after  $\approx 0.5$  and the flow continues. The plot also indicate that for this  $Re$  value, the speed becomes constant after  $\approx 0.5$  simulated distance.

Fig. 2 indicated that for  $Re = 0.1$ , the velocity profile stabilizes after  $\approx 0.5$  and the flow continues. This observation is similar to when  $Re = 0.01$ . However, unlike when  $Re = 0.01, 0.1$ , the maximum and minimum velocities are less than the previous figure. The plot also indicates that for this  $Re$  value, the speed becomes constant after  $\approx 0.5$  simulated distance.

Fig. 2 indicated that for  $Re = 1$ , the velocity profile stabilizes after  $\approx 0.5$  and the flow continues. This observation is similar to when  $Re = 0.01$  and  $Re = 0.1$ . However, unlike when  $Re = 0.01, 0.1$ , the maximum and minimum velocities

#### A. Numerical Simulations

In this section, we present some numerical solutions of the one dimensional problem. The one-dimensional problems are computed on the interval  $I = [0,1]$ . The boundary conditions applied are  $x_R = 1$  as discussed earlier in the chapter. We prescribe  $h_0 = 1$  and  $u_0 = 0$  as that initial condition for the thickness and the velocity. Numerical plots of the film velocity profile for the established discretized equation presented in (104)-(105) using finite volume method with varied  $Re = [0.01, 0.1, 1, 10, 100, 1000]$  and the results presented in Figs. 2-4. The simulations are based on the evolution of a film with the characteristic parameters  $\epsilon = 0.01, Ma = 100, S = 2$  and  $Pe = 1$ . The initial thickness is in this case given by  $h_0 = 1$ . The surfactant concentration at time  $t = 0$  is set to  $C_0 = 1$ , i.e. we start with a uniform distribution as in a freshly formed foam. Moreover, the film is motionless in the beginning, that is  $u_0 = 0$ .

#### B. Surfactant-Fuel Thin-film Velocity Profile

We now turn to the question of the influence of a surface active agent on the stability of a film spread over a fuel spill surface. We present velocity profiles of thin-film against distance under varied Reynolds number in Figs. 2.

are greater than the previous figures. The plot also indicates that for this  $Re$  value, the speed becomes constant after  $\approx 0.5$  simulated distance.

Fig. 2 indicated that for  $Re = 10$ , the velocity profile stabilizes after  $\approx 0.8$  and the flow continues. This observation is unlike when  $Re = 0.01, 0.1$ . However, unlike when  $Re = 0.01, 0.1$ , the maximum and minimum velocities are greater than the previous figures. At the commencement of the simulations, the velocity tends to be constant until after  $\approx 0.2m$ . The plot also indicates that for this  $Re$  value, the speed becomes constant after  $\approx 0.8m$  simulated distance.

For Fig. 2 the behavior of the solution is quite complicated. The initial disturbances compress in some parts and expand in some other parts. This leads to a combination of shocks and

rarefactions. The oscillations decay after  $x \approx 0.5, 0.52, 0.55, 0.57 & 0.79$ , respectively in Fig. 2. For high Reynolds numbers,  $Re = 100, 1000$  (see Fig. 2) the interfaces fails to oscillate, rather reaches equilibrium state. However, at  $Re = 100$  besides the motion being not oscillatory, it captures the discontinuity quite sharply.

Fig. 2 indicated that for  $Re = 100$ , the velocity profile is constant at the beginning of simulation then falls rapidly at  $\approx 0.25m$  to  $U = 0m/s$ . The observation is unlike when  $Re = 0.01, 0.1, 1, 10$ . However, unlike when  $Re = 0.01, 0.1, 1, 10$ , the maximum and minimum velocities are  $1m/s$  and  $0m/s$  respectively. The zero speed is reached at  $\approx 0.38m$ . This imply that at  $Re \gg 1$ , the surfactant-fuel thin-film is depleted faster than  $Re \ll 1$ . The observation suggests that in order to suppress the gas vapors and control the fire incidences, then  $Re \neq \gg 1$ .

Fig. 2 indicated that for  $Re = 1000$ , the velocity profile is constant at the beginning of simulation then falls rapidly at  $\approx 0.15m$  to  $U = 0m/s$ . The observation is unlike when  $Re = 0.01, 0.1, 1, 10$ . However, unlike when  $Re = 0.01, 0.1, 1, 10$ , the maximum and minimum velocities are  $1m/s$  and  $0m/s$  respectively.

respectively. The zero speed is reached at  $\approx 0.3m$ . This imply that at  $Re \gg 1$ , the surfactant-fuel thin-film is depleted faster than  $Re \ll 1$ . The observation suggests that in order to suppress the gas vapors and control the fire incidences, then  $Re \neq \gg 1$ . The observation when  $Re = 100, 1000$  indicate that the higher the  $Re$  the faster the depletion of the surfactant-fuel thin-film foam.

### C. Drainage of the Film

We can study how the drainage of the thin film proceeds. For each value of the thin film thickness  $h$ , we set the surfactant concentration. We in this study simulate the drainage in thin liquid film spread over fuel spill surface. So, once the remaining parameters  $Pe, Re, S, Ma$  are specified, we can in principle calculate  $Q$  for each value of  $h$  and thus determine the rate at which the thin film drains. To demonstrate two classes of solutions to the model, we start by keeping  $Pe = 5, Ma = 5$  and  $S = 2$  fixed while varying  $c$ . Fig. 3 summarizes the drainage profile of surfactant-fuel thin film profile

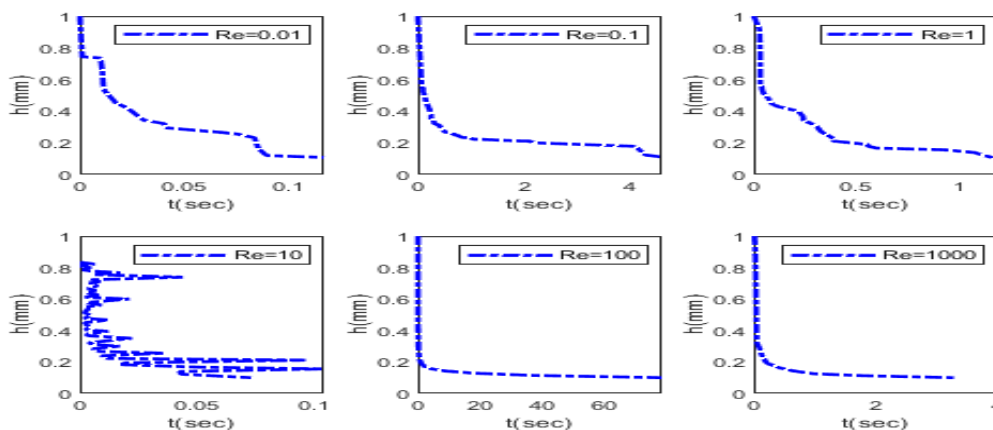


Fig. 3. Drainage profile beginning with  $Re = 0.01$  to  $Re = 1000$  with  $h$ (surfactant thickness ranging from  $H = 1mm$  to  $H_0 = 0.1mm$ ).

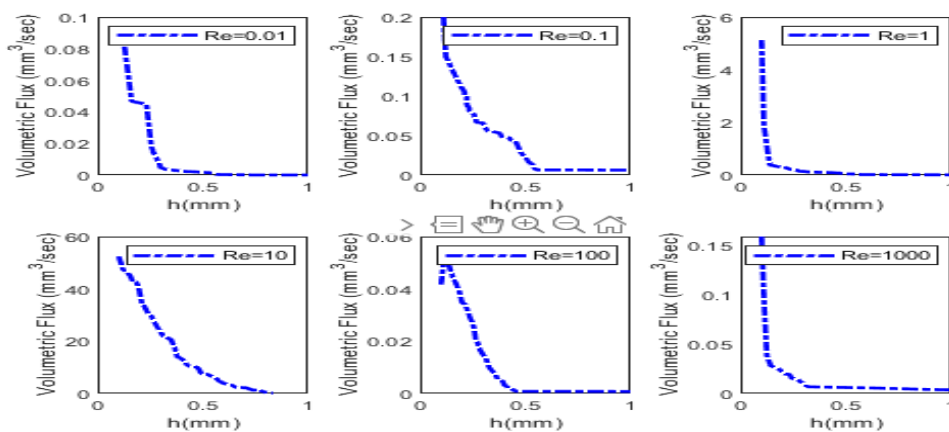


Fig. 4. Volumetric flux profile analogous to drainage profile beginning with  $Re = 0.01$  to  $Re = 1000$  with  $h$ (surfactant thickness ranging from  $H = 1mm$  to  $H_0 = 0.1mm$ ).

Fig. 3 indicate that at higher  $Re$ , the drainage is so quicker than the at lower  $Re$ .  $Re = \frac{\rho UL}{\mu}$  where  $\rho$  is the thin-film surfactant foam interface density,  $U$  is the velocity,  $L$  is the

characteristic length, and  $\mu$  is the viscosity.  $\rho, L, \mu$  is constant; thus, increasing  $Re$  increases the velocity of the thin film surfactant foam. The implication suggests that at higher  $Re$ , the

thin film surfactant foam drains faster. Thus, vice-versa is true if we look at  $Re = 0.01; 0.1; 1$ .

#### D. Volumetric Flux Profile

Fig. 4 indicate that at higher  $Re$ , the volumetric flux falls so quicker than the at lower  $Re$ . The observation is similar to those made in Figure 3. The volumetric flow rate is higher when  $Re$  is high and lower when  $Re$  is low. Figure 3 and Figure 4 suggest that at higher  $Re$  the surfactant drainage is faster and the fuel spill explodes.

In the beginning, a constant surfactant concentration where viscous forces is dominant show the film thinning very fast. Due to the flow of liquid and surfactant into the shock regions, a gradient in the surfactant concentration arises, ultimately stopping the thinning. Due to inertial effects, an equilibrium is not immediately reached. Still, the thin film shows an oscillatory behaviour in which liquid alternatively flows back into the thin film and out again until a quasi-equilibrium state is reached.

#### V. CONCLUSION

The estimation of the depletion of surfactant-fuel thin-film foam is an enormous task that has elicited many pieces of research. The chapter tries to estimate the distance before depletion of surfactant-fuel thin-film foam. We varied  $Re$  to show that given different interface Reynolds numbers (inertial forces to viscous forces), the surfactant-fuel thin-film foam will show different velocity profiles.

The simulation of the study has shown that for  $Re = 0.01, 0.1, 1$ , the velocity profile stabilizes after  $\approx 0.5$  and the flow continues. Surfactant-fuel thin-film foam with  $Re = 1$  has the maximum and minimum velocities are less than that of  $Re = 0.01$  &  $0.1$ .  $Re = 10$ , the velocity profile stabilizes after  $\approx 0.8$ , and the maximum and minimum velocities are greater than the previous values of  $Re$ . The higher  $Re$  that is, at  $Re \gg 1$ , the surfactant-fuel thin-film is depleted faster than  $Re \ll 1$ . Thus, to suppress the gas vapors and control the fire incidences,  $Re \neq \gg 1$ . Future studies in the area need to consider the effect of variation in temperature, velocity, besides Reynolds number, in determining the lifetime of a surfactant-fuel thin foam.

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