Volume 5, Issue 5, pp. 8-18, 2021.
International Journal of Scientific Engineering and Science
ISSN (Online): 2456-7361

A New Mixture of Gamma Shape Distributions: Properties and Applications

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Abstract— *Over the last few decades, mixture distributions are used in creating population from two or more distributions. Mixture distributions are a good application in the applications of medical science, biology, engineering, finance and economics. Gaussian mixture models have broad utility, including their usage for model-based clustering framework. Recently, there are indications to use of non-Gaussian mixture distributions to skewed and asymmetric data. We propose a mixture model of inverse power Gamma shape distributions (MIPGSD) to analyze positive data. Basic structural properties such raw and central moments and hazard rate function are obtained. Different estimation methods are studied to estimate the proposed model parameters. Simulation studies is done to present the performance and behavior of the different estimates of the proposed model parameters. A real data set is provided to compare the reliability of the new model with other models.*

Keywords— *Gamma shape distribution, Inverse power distribution, Mixture Gamma, mixture of Gamma shape distributions, Goodness of fit.*

I. INTRODUCTION

If we have positive data, right-skewed, and assumed to come from a mixture distribution, then the use of a Gamma density is a logical choice. There are many papers discussed finite mixtures of Gamma. John [17] discussed finite mixtures of Gamma of a two-component model with both the method of moments and maximum likelihood. Gharib [13] studied two characterizations for a mixture of two Gamma distributions. Huang and Chang [16] showed that the Lukacstype characterization for the sum of independent Gamma random variables can be represented as a particular mixture of Gamma. Mixtures of Gamma have also been presented as applied models for various applications for example, characterizing rates across sites of molecular sequence evolution Mayrose et al [20], modeling Internet traffic Almhana et al. [1], and modeling extremes in various hydrological phenomena Evin et al. [12].

The objective of this paper is to consider seven different estimators for the parameters of our proposed distribution and evaluates their performance in simulation and applications studies. Many authors have compared several classical estimation methods for estimating the parameters of well-known distributions. For example, Rodrigues et al. [22] for Poisson–exponential distribution, Karamikabir et al. [18] for a new extended generalized Gompertz distribution, Dey et al. [10] for exponentiated Chen distribution and Sharma et al. [23] for the generalized inverse Lindley distribution.

In this paper, we are motivated to introduce the MIPGSD model because (i) it contains a mixture of another lifetime sub model; (ii) this model reveals upside down bathtub-shaped hazard rate which occurs in most real life systems and very useful in survival analysis; (iii) the proposed model can be considered as a suitable model for fitting the positive data with a longer right tail which can be used in various fields such as survival analysis and biomedical studies; and (iv) the MIPGSD model outperforms most well-known lifetime models with respect to two real data sets.

For our paper, let X is continuous random variable follow the Gamma distribution with parameters λ and θ, then the probability density function (pdf) is given by

$$f(y;\lambda,\theta)=\frac{\theta^{\lambda}}{\Gamma(\lambda)}y^{\lambda-1}e^{-\theta y};\quad y>0,\lambda,\theta>0 \qquad (1)$$

where $\Gamma(a)=\int_0^\infty y^{a-1}e^{-y}dy$ is (complete) Gamma function.

Here, λ is a shape parameter and θ is a an inverse scale parameter called a rate parameter for the Gamma density. We denote this distribution by $G(\lambda,\theta)$ and the cumulative distribution function (cdf) can be written as

$$F(y;\lambda,\theta)=\frac{\gamma(\lambda,\theta y)}{\Gamma(\lambda)} \qquad (2)$$

where $\gamma(a,x)=\int_0^x y^{a-1}e^{-y}dy$ is the lower incomplete gamma function.

Let Y be a random variable having pdf (1), then the random variable $X=Y^{-1/\alpha}$ is said to follow an inverse power Gamma (IPG) distribution, shown as $X\sim IPG(\alpha,\lambda,\theta)$, with pdf and corresponding cdf defined, respectively, by

$$f(x;\alpha,\lambda,\theta)=\frac{\alpha\theta^{\lambda}}{\Gamma(\lambda)}x^{-(1+\alpha\lambda)}e^{-\theta x^{-\alpha}};\quad x>0,\alpha,\lambda,\theta>0 \qquad (3)$$

$$F(x;\lambda,\theta,\alpha)=\frac{\Gamma(\lambda,\theta x^{-\alpha})}{\Gamma(\lambda)} \qquad (4)$$

where $\Gamma(a,x)=\int_x^\infty y^{a-1}e^{-y}dy$ is the upper incomplete gamma function. It can be noticed that the inverse Gamma distribution is a special case of IPG when $\alpha=1$.

Suppose a mixture distribution consisting of k components ($i=1,2,...,k$) and the distribution of the ith individual component follows an IPG distribution. The generated mixture distribution represents the inverse power Gamma shape mixture IPGSM distribution with pdf and cdf defined, respectively, by

$$f\left(y;\boldsymbol{\pi},\alpha,\theta\right)=\sum_{i=1}^{k}\pi_i f_i\left(x;\alpha,\theta\right) \quad (5)$$

$$F\left(y;\boldsymbol{\pi},\alpha,\theta\right)=\sum_{i=1}^{k}\pi_i F_i\left(x;\alpha,\theta\right) \quad (6)$$

where $f_i\left(x;\alpha,\theta\right)=f\left(x;\alpha,i,\theta\right)$ and $F_i\left(x;\alpha,\theta\right)=F\left(x;\alpha,i,\theta\right)$ denote respectively, the pdf and cdf of an inverse power Gamma $IPG\left(\alpha,i,\theta\right)$ random variable. Let k is known and fixed, whereas $\boldsymbol{\pi}=\left(\pi_1,...,\pi_k\right)$ is a vector of mixture weights (proportions) that satisfy the conditions

(i) $0<\pi_i<1\ \forall i=1,2,...,k$ and

(ii) $\sum_{i=1}^{k}\pi_i=1$.

The aim of this paper is to define and study a new finite mixture distribution called the inverse power Gamma shape mixture (MIPGSD) distribution with its mathematical properties. These include the reliability measurers such as survival and hazard rate function. The moments and moment generating function are provided. Maximum likelihood estimation of the model parameters and confidence interval are derived. Application of the model to a real data set is finally presented and compared to the fit attained by some other well-known distributions.

II. THE MIPGSD MODEL AND STATISTICAL PROPERTIES

The formulas in (5) and (6) can simply be rewritten using next theorem.

Theorem 1. Let X be a random variable that follows the MIPGSD model, then the pdf and cdf can be written, respectively, as

$$f\left(x\right)=\frac{\alpha\theta^k}{\Gamma\left(k,\theta\right)}\frac{\left(1+x^{\alpha}\right)^{k-1}}{x^{\alpha k+1}}e^{-\left(1+x^{-\alpha}\right)\theta};\quad y>0,\theta,\alpha>0 \quad (7)$$

$$F\left(x\right)=\frac{\Gamma\left(k,\left(1+x^{-\alpha}\right)\theta\right)}{\Gamma\left(k,\theta\right)} \quad (8)$$

where $k=1,2,3,...$

proof

$$f\left(x\right)=\frac{\alpha\theta^k}{\Gamma\left(k,\theta\right)}\frac{\left(1+x^{\alpha}\right)^{k-1}}{x^{\alpha k+1}}e^{-\theta\left(x^{-\alpha}+1\right)}=\frac{\alpha\theta^k e^{-\theta}}{\Gamma\left(k,\theta\right)x^{\alpha k+1}}\sum_{i=1}^{k}\binom{k-1}{i-1}x^{\alpha(k-i)}e^{-\theta x^{-\alpha}}$$

where $\left(1+x^{\alpha}\right)^{k-1}=\sum_{i=1}^{k}\binom{k-1}{i-1}x^{\alpha(k-i)}$

$$f\left(x\right)=\frac{\Gamma\left(k\right)e^{-\theta}}{\Gamma\left(k,\theta\right)}\sum_{i=1}^{k}\frac{\theta^{k-i}}{\left(k-i\right)!}\frac{\alpha\theta^i}{\Gamma\left(i\right)}x^{-(\alpha i+1)}e^{-\theta x^{-\alpha}}=\frac{\Gamma\left(k\right)e^{-\theta}}{\Gamma\left(k,\theta\right)}\sum_{i=1}^{k}\frac{\theta^{k-i}}{\left(k-i\right)!}f_i\left(x;\alpha,\theta\right) \quad (9)$$

From the definition of the upper incomplete gamma function, we have

$$\Gamma\left(k,\theta\right)=\int_0^{\infty}\left(\theta+u\right)^{k-1}e^{-\left(\theta+u\right)}du$$

where $k=1,2,3,...$

$$\Gamma\left(k,\theta\right)=e^{-\theta}\int_0^{\infty}\sum_{i=1}^{k}\binom{k-1}{i-1}\theta^{k-i}u^{i-1}e^{-u}du$$

$$=e^{-\theta}\sum_{i=1}^{k}\binom{k-1}{i-1}\theta^{k-i}\int_0^{\infty}u^{i-1}e^{-u}\,du$$

$$\Gamma\left(k,\theta\right)=e^{-\theta}\int_0^{\infty}\sum_{i=1}^{k}\binom{k-1}{i-1}\theta^{k-i}u^{i-1}e^{-u}du=e^{-\theta}\sum_{i=1}^{k}\frac{\Gamma\left(k\right)\theta^{k-i}}{\left(k-i\right)!\Gamma\left(i\right)}\Gamma\left(i\right)$$

$$\Gamma\left(k,\theta\right)=\Gamma\left(k\right)e^{-\theta}\sum_{i=1}^{k}\frac{\theta^{k-i}}{\left(k-i\right)!} \quad (10)$$

From (9) and (10), we have

$$\pi_i=\frac{\Gamma\left(k\right)\theta^{k-i}e^{-\theta}}{\Gamma\left(k,\theta\right)\left(k-i\right)!} \quad (11)$$

where π_i is restricted to be positive and sum to unity ($\pi_i>0$ and $\sum_{i=1}^{k}\pi_i=1$).

The cumulative distribution function (cdf) of the MIPGSD model is given by

$$F\left(x\right)=\int_0^{x}f\left(z\right)dz$$

$$=\frac{\alpha\theta^k}{\Gamma\left(k,\theta\right)}\int_0^{x}\frac{\left(1+z^{\alpha}\right)^{k-1}}{z^{\alpha k+1}}e^{-\left(1+z^{-\alpha}\right)\theta}dz$$

letting $y=\left(1+z^{-\alpha}\right)\theta$ and after simplification the expression, we get the following

$$F\left(x\right)=\frac{1}{\Gamma\left(k,\theta\right)}\int_{\left(1+x^{-\alpha}\right)\theta}^{\infty}y^{k-1}e^{-y}dy=\frac{\Gamma\left(k,\left(1+x^{-\alpha}\right)\theta\right)}{\Gamma\left(k,\theta\right)}$$

A. Behaviour of the density function

The behaviors of the density function of the MIPGSD model at $x=0$ and $x=\infty$, respectively, are given by

$$\lim_{x\to0}f\left(x\right)=\frac{\alpha\theta^k}{\Gamma\left(k,\theta\right)}\frac{\left(\lim_{x\to0}\left(1+x^{\alpha}\right)^{k-1}\right)\left(\lim_{x\to0}e^{-\left(1+x^{-\alpha}\right)\theta}\right)}{\left(\lim_{x\to0}x^{\alpha k+1}\right)}$$

putting $y=x^{-1}$

$$\lim_{x\to0}f\left(x\right)=\lim_{y\to\infty}f\left(y\right)=\frac{\alpha\theta^k}{\Gamma\left(k,\theta\right)}\lim_{x\to\infty}\left(\frac{y^{\alpha k+1}}{e^{\left(1+y^{\alpha}\right)\theta}}\right)$$

$$=0$$

and

$$\lim_{x\to\infty} f(x) = \lim_{y\to 0} f(y) = \frac{\alpha\theta^k}{\Gamma(k,\theta)} \lim_{x\to 0}\left(\frac{y^{\alpha k+1}\left(1+y^\alpha\right)^{k-1}}{e^{\left(1+y^\alpha\right)\theta}\, y^{\alpha(k-1)}}\right)$$

$$= \frac{\alpha\theta^k}{\Gamma(k,\theta)}\lim_{x\to 0}\left(\frac{y^\alpha\left(1+y^\alpha\right)^{k-1}}{e^{\left(1+y^\alpha\right)\theta}}\right)$$

$$= 0$$

Theorem 2. The probability density function of the MIPGSD model is unimodal shaped in x

Proof. The first derivative of $f(x)$ is given by

$$f'(x) = -\frac{\psi(x)}{g(x)} f(x)$$

where $\psi(x) = ax^{2\alpha} + bx^\alpha + c, g(x) = \left(1+x^\alpha\right)x^{\alpha+1}$

with $a = 1+\alpha, b = 1+(k-\theta)\alpha, c = -\alpha\theta$

It is clear that $\psi(x)$ is a unimodal quadratic function and that the mode of $f(x)$ implies $\psi(x)=0$. Let $D = \left(b^2 - 4ac\right)$ be the discriminant of $\psi(x)$, the second derivative of $f(x)$ given by

$$f''(x) = -\frac{1}{g(x)}\left[\left(g'(x)+\psi(x)\right)f'(x) + \psi'(x) f(x)\right]$$

where $g'(x) = \left(1+\alpha+(1+2\alpha)x^\alpha\right)x^\alpha$ and $\psi'(x) = 2a\alpha x^{2\alpha-1} + b\alpha x^{\alpha-1}$.

Clearly, $D>0$ and $\psi(x)$ has maximum value at the point x_0 where

$$x_0 = \left(\frac{-b+\sqrt{D}}{2a}\right)^{\frac{1}{\alpha}}$$

since, $f''(x_0) = -\left(\sqrt{D}/g(x_0)\right) f(x_0) < 0$, $f(x)$ has a global maximum at x_0 ; hence, the mode of $f(x)$ is given by

$$x_0 = \left(\frac{-1-(k-\theta)\alpha+\sqrt{\left(1+(k-\theta)\alpha\right)^2 + 4\alpha\theta(1+\alpha)}}{2(1+\alpha)}\right)^{1/\alpha}$$

In Fig. 1, we plot the behavior of pdf for the MIPGSD model for some values of θ, α .

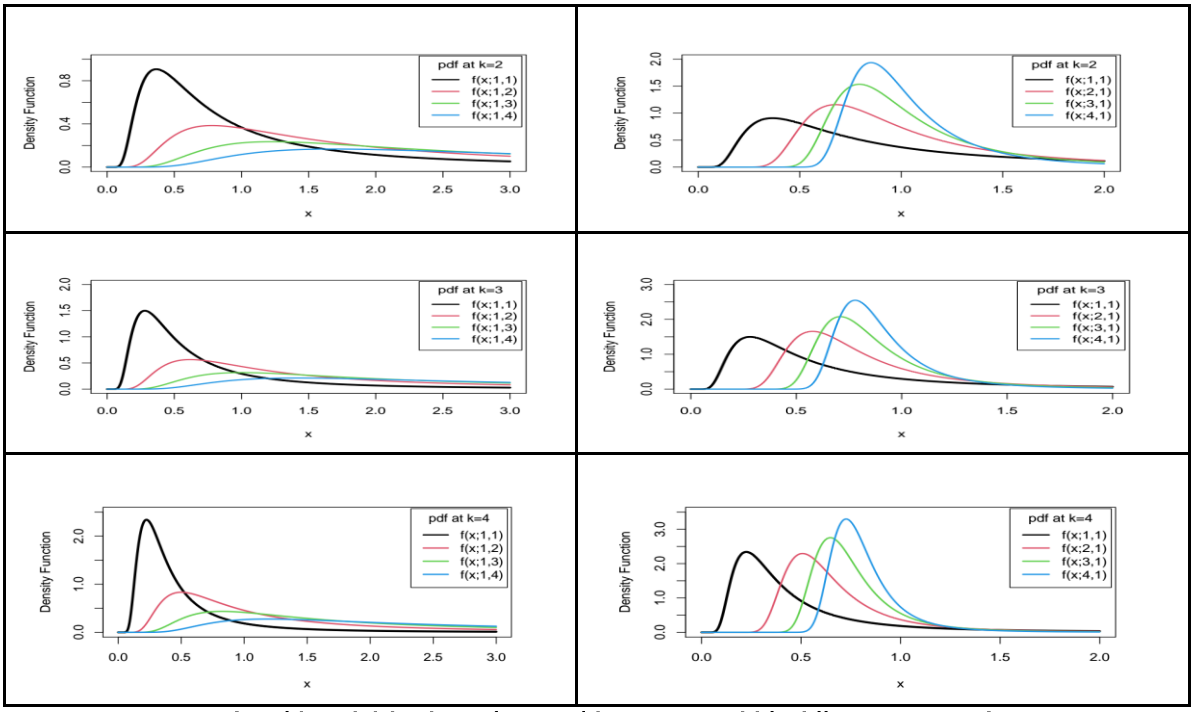

Fig.1. Plots of the probability density function of the MIPGSD model for different parameter values.

B. *Behaviour of the hazard rate function*

The hazard rate function (hf) of the proposed model is obtained as

$$h(x) = \frac{f(x)}{s(x)} = \frac{\alpha\theta^k}{\Gamma(k,\theta)-\Gamma\left(k,\left(1+x^{-\alpha}\right)\theta\right)} \frac{\left(1+x^\alpha\right)^{k-1}}{x^{\alpha k+1}} e^{-\left(1+x^{-\alpha}\right)\theta} \qquad (12)$$

where $x>0$ and $\alpha, \theta > 0$.

Fig. 2 shows the hrf plots of the MIPGSD model for different values of the parameters θ and α . Fig. 2 reveals that the hrf of proposed model is upside down bathtub shaped

International Journal of Scientific Engineering and Science
Volume 5, Issue 5, pp. 8-18, 2021.
ISSN (Online): 2456-7361

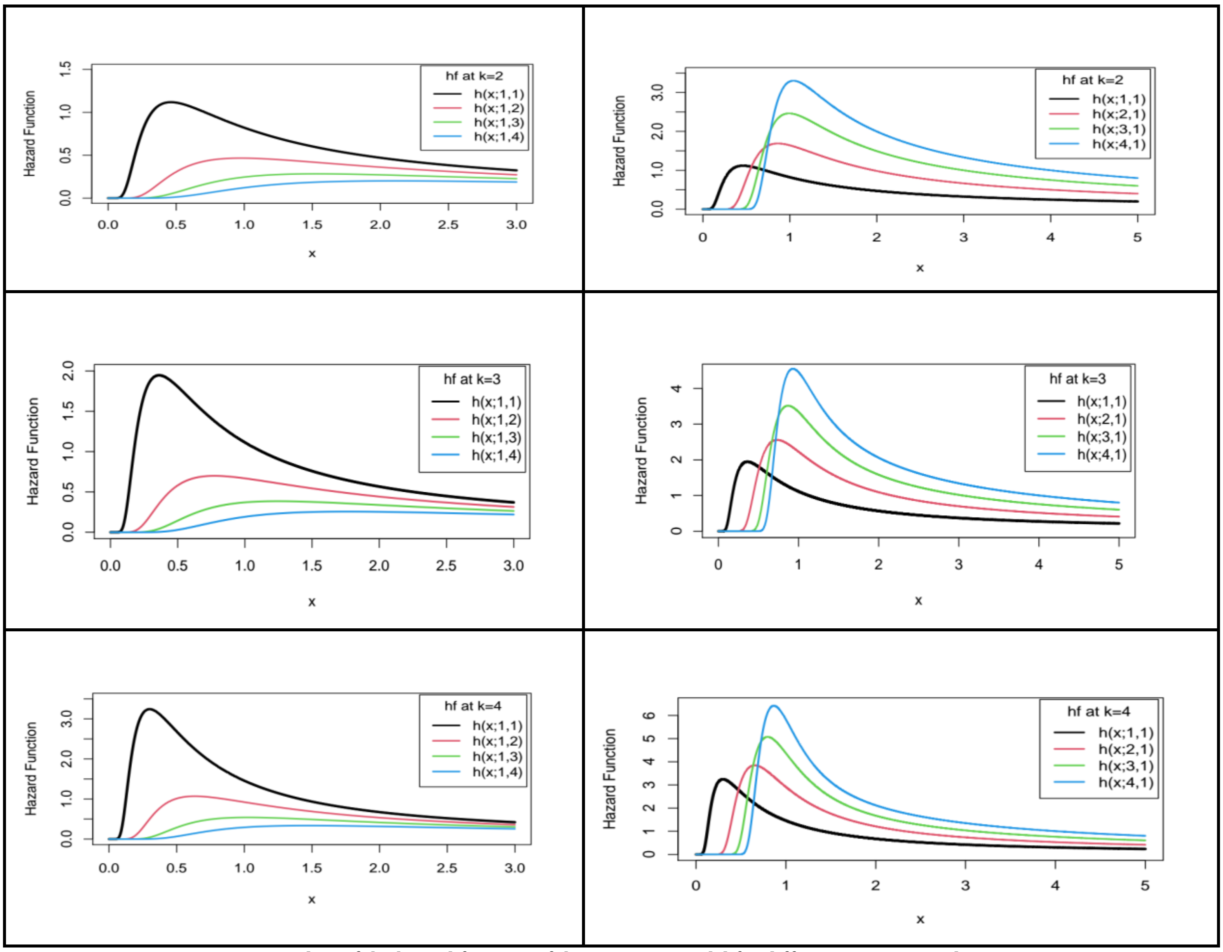

Fig. 2. Plots of the hazard function of the MIPGSD model for different parameter values

C. *Moments and related measures*

Let X be a random variable that follows the MIPGSD model with pdf as in (7), then the rth raw moment (about the origin) is given by

$$\mu'_r = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)}\sum_{i=1}^{k}\frac{\theta^{k+\frac{r}{\alpha}-i}}{(k-i)!\Gamma(i)}\Gamma\left(\frac{i\alpha-r}{\alpha}\right), i>\frac{r}{\alpha} \qquad (13)$$

The mean of the MIPGSD distribution is given by

$$\mu = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)}\sum_{i=1}^{k}\frac{\theta^{k+\frac{1}{\alpha}-i}}{(k-i)!\Gamma(i)}\Gamma\left(\frac{i\alpha-1}{\alpha}\right), i>\frac{1}{\alpha}$$

The nth central moments of the proposed model are given by

$$\mu_n = E(X-\mu)^n = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)}\sum_{j=0}^{n}\sum_{i=1}^{k}\binom{n}{j}\frac{(-\mu)^{n-j}\theta^{k+\frac{j}{\alpha}-i}}{(k-i)!\Gamma(i)}\Gamma\left(\frac{i\alpha-j}{\alpha}\right), i>\frac{j}{\alpha} \qquad (14)$$

The variance, coefficient of skewness, kurtosis and variation measures can be obtained from the expressions

$$\sigma^2 = \mu_2 = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)}\sum_{j=0}^{2}\sum_{i=1}^{k}\binom{n}{j}\frac{(-\mu)^{n-j}\theta^{k+\frac{j}{\alpha}-i}}{(k-i)!\Gamma(i)}\Gamma\left(\frac{i\alpha-j}{\alpha}\right), i>\frac{j}{\alpha}$$

$$\beta_1 = \frac{\mu_3^{\,2}}{\mu_2^{\,3}}, \beta_2 = \frac{\mu_4}{\mu_2^{\,2}} \text{ and } CV = \frac{\sigma}{\mu}\times 100$$

upon substituting for the central moments in (19).

Raw moment of the MIPGSD model will exist only when $i>\frac{r}{\alpha}$. Therefore, the evaluation of inverse moments may be of interest. The rth raw inverse moment (about the origin) is given by

$$\mu'_{r^{-1}} = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)}\sum_{i=1}^{k}\frac{\theta^{k-\left(\frac{r}{\alpha}+i\right)}}{(k-i)!\Gamma(i)}\Gamma\left(\frac{i\alpha+r}{\alpha}\right) \qquad (15)$$

The harmonic mean of the proposed distribution is obtained by

$$H = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)}\sum_{i=1}^{k}\frac{\theta^{k-\frac{1}{\alpha}-i}}{(k-i)!\Gamma(i)}\Gamma\left(\frac{i\alpha+1}{\alpha}\right) \qquad (16)$$

From the empirical relation among mean, median and mode, the median (M) of the proposed distribution can be written as

$$M = \frac{1}{3}x_0 + \frac{2}{3}\mu \qquad (17)$$

Table 1 shows some important measures of the MIPGSD model at different parameter combination and it is observed

International Journal of Scientific Engineering and Science
Volume 5, Issue 5, pp. 8-18, 2021. ISSN (Online): 2456-7361

that the shape of the proposed distribution is right skewed for values of k, α and θ.

TABLE 1.Values of some important measures of the MIPGSD model.

Moments	μ	σ^2	β_1	β_2	x_0	M	CV	H
k	$\alpha=6,\theta=4$							
2	1.37	0.11	8.100	25.19	1.18	1.31	24.79	0.76
4	1.26	0.09	9.02	27.41	1.10	1.21	23.73	0.82
6	1.15	0.06	11.06	32.87	1.02	1.11	21.33	0.89
8	1.04	0.03	14.19	43.81	0.96	1.01	17.24	0.97
10	0.96	0.01	14.67	55.06	0.914	0.94	12.60	1.04
α	$k=4,\theta=2$							
6	1.05	0.04	11.16	33.63	0.93	1.01	20.84	0.98
8	1.03	0.02	7.05	19.33	0.95	1.00	14.85	0.98
10	1.02	0.01	5.51	14.98	0.96	1.00	11.56	0.98
12	1.02	0.00	4.72	12.91	0.97	1.00	9.47	0.98
14	1.01	0.00	4.23	11.71	0.97	1.00	8.02	0.98
θ	$k=2,\alpha=6$							
1	1.03	0.05	9.20	28.16	0.90	0.99	23.23	1.00
2	1.19	0.08	8.49	26.15	1.03	1.14	24.30	0.87
3	1.29	0.10	8.22	25.49	1.12	1.24	24.64	0.80
4	1.37	0.11	8.10	25.19	1.18	1.31	24.79	0.76
5	1.43	0.12	8.03	25.03	1.23	1.36	24.87	0.72

III. ESTIMATION AND INFERENCE OF THE PARAMETERS

The main aim of this section is to study different estimation methods of the unknown parameters of the MIPGSD model.

A. *Maximum likelihood method*

The most widely method used for parameter estimation is maximum likelihood method. Let $x_1, x_2, ..., x_n$ be a random sample from the MIPGSD prmodel with pdf (11). The log-likelihood function is given by

$$L=-n\theta+n\ln(\alpha)+kn\ln(\theta)-n\ln\Gamma(k,\theta)-(1+\alpha)\sum_{i=1}^{n}x_i+$$
$$(k-1)\sum_{i=1}^{n}\ln\left(1+x_i^{-\alpha}\right)-\theta\sum_{i=1}^{n}\ln\left(x_i^{-\alpha}\right)$$

The maximum likelihood estimators (MLEs) of α,θ denoted by α_{MLE} and θ_{MLE} can be obtained by solving the following system of non-linear equations

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\ln(x_i)-\theta\sum_{i=1}^{n}-x_i^{-\alpha}\ln(x_i)+(k-1)\sum_{i=1}^{n}-\frac{x_i^{-\alpha}\ln(x_i)}{1+x_i^{-\alpha}}=0$$

$$\frac{\partial L}{\partial\theta}=-n+\frac{kn}{\theta}+\frac{n\theta^{k-1}e^{-\theta}}{\Gamma(k,\theta)}-\sum_{i=1}^{n}x_i^{-\alpha}=0$$

We used non-linear maximization techniques to get the solution of the MLE's of the parameters. For interval estimation of the parameter vector $\Theta=(\alpha,\theta)^T$, we derive Fisher information matrix for constructing $100(1-\psi)\%$ asymptotic confidence interval for the parameters using large sample theory. The Fisher information matrix can be obtained by using log-likelihood function as

$$I(\alpha,\theta)=-E\begin{pmatrix}\frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\theta}\\ \frac{\partial^2 L}{\partial\alpha\theta} & \frac{\partial^2 L}{\partial\theta^2}\end{pmatrix}$$

where

$$\frac{\partial^2 L}{\partial\alpha^2}=-\frac{n}{\alpha^2}-\theta\sum_{i=1}^{n}\left(\ln(x_i)\right)^2x_i^{-\alpha}+(k-1)\sum_{i=1}^{n}\left(\frac{x_i^{-\alpha}\left(\ln(x_i)\right)^2}{1+x_i^{-\alpha}}-\frac{x_i^{-2\alpha}\left(\ln(x_i)\right)^2}{\left(1+x_i^{-\alpha}\right)^2}\right)$$

$$\frac{\partial^2 L}{\partial\theta^2}=-\frac{kn}{\theta^2}-n\left(-\frac{\theta^{2k-2}e^{-2\theta}}{\Gamma(k,\theta)}-\frac{(k-1)\theta^{k-2}e^{-\theta}}{\Gamma(k,\theta)}+\frac{\theta^{k-1}e^{-\theta}}{\Gamma(k,\theta)}\right)$$

$$\frac{\partial^2 L}{\partial\alpha\theta}=\frac{\partial^2 L}{\partial\theta\alpha}=-\sum_{i=1}^{n}-x_i^{-\alpha}\ln(x_i)$$

The diagonal elements of the inverse of the Fisher information matrix $I^{-1}(\alpha,\theta)$ provide asymptotic variance of α and θ respectively. The corresponding asymptotic $100(1-\psi)\%$ confidence interval of θ and α, are given by

$$\alpha\mp Z_{1-\frac{\psi}{2}}\sqrt{Var(\alpha)},\ \theta\mp Z_{1-\frac{\psi}{2}}\sqrt{Var(\theta)}$$

respectively.

B. *Least squares and weighted least squares methods*

The least squares (LSE) and the weighted least squares (WLSE) methods are used to find the minimum distance between theoretical cumulative distribution and the empirical cumulative distribution. These methods were introduced by Swain et al. [24] to estimate the parameters of Beta distributions. Let $F\left(X_{(i)}\right)$ be the distribution function of the ordered random variables $X_{(1)}<X_{(2)}<...<X_{(n)}$ where $\{X_1, X_2, ..., X_2\}$ is a random sample of size n from a distribution function $F(.)$. Then, the expectation of the empirical cumulative distribution function is defined as

$$E\left[F\left(X_{(i)}\right)\right]=\frac{i}{n+i};i=1,2,...,n$$

The LSEs of α and θ denoted by α_{LSE} and θ_{LSE} can be obtained by minimizing the following function

$$LS(\alpha,\theta)=\sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta\right)-\frac{i}{n+i}\right)^2 \quad (18)$$

with respect to α and θ, where $F(.)$ is given by (8).

therefore α_{LSE} and θ_{LSE} can be obtained as the solution of the following system of non-linear equations:

$$\frac{\partial LS(\alpha,\theta)}{\partial\alpha}=\sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta\right)-\frac{i}{n+i}\right)F'_{\alpha}(x;\alpha,\theta)=0 \quad (19)$$

$$\frac{\partial LS(\alpha,\theta)}{\partial\theta}=\sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta\right)-\frac{i}{n+i}\right)F'_{\theta}(x;\alpha,\theta)=0 \quad (20)$$

Gupta & Kundu [15] introduced the following weighted function

$$\omega_i = \frac{1}{Var\left(x_{(i)}\right)} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$$

The WLSEs of α and θ denoted by α_{WLSE} and θ_{WLSE} can be obtained by minimizing

$$WLS(\alpha,\theta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F\left(x_{(i)};\alpha,\theta\right) - \frac{i}{n+i}\right)^2 \quad (21)$$

with respect to α and θ, therefore these estimators can also be obtained by solving:

$$\frac{\partial WLS(\alpha,\theta)}{\partial\alpha} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F\left(x_{(i)};\alpha,\theta\right) - \frac{i}{n+i}\right)\times F'_{\alpha}(x;\alpha,\theta) = 0 \quad (22)$$

$$\frac{\partial WLS(\alpha,\theta)}{\partial\theta} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F\left(x_{(i)};\alpha,\theta\right) - \frac{i}{n+i}\right)\times F'_{\theta}(x;\alpha,\theta) = 0 \quad (23)$$

where

$$F'_{\alpha}(x;\alpha,\theta) = \frac{\theta^k}{\Gamma(k,\theta)} \frac{\left(1+x^{\alpha}\right)^{k-1}}{x^{\alpha k}} e^{-\left(1+x^{-\alpha}\right)\theta} \ln(x)$$

and

$$F'_{\theta}(x;\alpha,\theta) = \frac{\theta^{k-1}\left(\Gamma(k,\theta) + \Gamma\left(k,\left(1+x^{-\alpha}\right)\theta\right) - \left(1+x^{-\alpha}\right)^k e^{-\theta x^{-\alpha}}\right)e^{-\theta}}{\left[\Gamma(k,\theta)\right]^2}$$

D. Cramer-von-Mises estimator

The Cramer-von Mises (CME) method is a type of minimum distance estimation method introduced by Choi and Bulgren [9]. This method based on the Cramer-von Mises statistics given by

$$W^2 = n\int_0^{\infty}\left[F(x_i) - E\left[F\left(x_{(i)}\right)\right]\right]^2 dF(x_i)$$

Boos [6] proved that the Cramer-von Mises statistics can be given by

$$C(\alpha,\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta\right) - \frac{2i-1}{2n}\right)^2 \quad (24)$$

Then the CME estimators α_{CME} and θ_{CME} of α and θ are obtained by minimizing (24) with respect to α and θ. These estimators can also be obtained by solving the following non-linear equations:

$$\frac{\partial C(\alpha,\theta)}{\partial\alpha} = \sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta\right) - \frac{2i-1}{2n}\right)F'_{\alpha}(x;\alpha,\theta) = 0 \quad (25)$$

$$\frac{\partial C(\alpha,\theta)}{\partial\theta} = \sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta\right) - \frac{2i-1}{2n}\right)F'_{\theta}(x;\alpha,\theta) = 0 \quad (26)$$

E. Maximum product spacing method

Cheng and Amin [7] introduced the maximum product spacing (MPS) and showed that the MPS method can be used as an alternative to MLE to estimate the parameters of continuous univariate distributions. This method assumes that differences (spacings) between the cdf values should be identically distributed at consecutive data points. Let the difference is defined as

$$D_i(\alpha,\theta) = F\left(x_{(i)};\alpha,\theta\right) - F\left(x_{(i-1)};\alpha,\theta\right), \quad i = 1,2,\ldots,n \quad (27)$$

where $F\left(x_{(0)};\alpha,\theta\right) = 0$ and $F\left(x_{(n+1)};\alpha,\theta\right) = 1$. The geometric mean of the differences can be written as

$$G(\alpha,\theta) = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i(\alpha,\theta)} \quad (28)$$

Substituting (8) in (28) and maximizing the above expression, we have

$$g(\alpha,\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left(F\left(x_{(i)};\alpha,\theta\right) - F\left(x_{(i-1)};\alpha,\theta\right)\right) \quad (29)$$

Cheng & Stephens [8] showed that finding the maximum of the geometric mean of the spacings is the same as finding the minimum of the Moran's statistics, the Moran's statistics given by

$$M(\alpha,\theta) = -\sqrt[n+1]{\prod_{i=1}^{n+1} D_i(\alpha,\theta)} \quad (30)$$

The MPSEs α_{MPS} and θ_{MPS} of α and θ are obtained as the simultaneous solution of the following nonlinear equations:

$$\frac{\partial \log G(\alpha,\theta)}{\partial\alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{F'_{\alpha}\left(x_{(i)};\alpha,\theta\right) - F'_{\alpha}\left(x_{(i-1)};\alpha,\theta\right)}{F\left(x_{(i)};\alpha,\theta\right) - F\left(x_{(i-1)};\alpha,\theta\right)}\right) = 0 \quad (31)$$

$$\frac{\partial \log G(\alpha,\theta)}{\partial\theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{F'_{\theta}\left(x_{(i)};\alpha,\theta\right) - F'_{\theta}\left(x_{(i-1)};\alpha,\theta\right)}{F\left(x_{(i)};\alpha,\theta\right) - F\left(x_{(i-1)};\alpha,\theta\right)}\right) = 0 \quad (32)$$

where $F'_{\alpha}(x;\alpha,\theta)$ and $F'_{\theta}(x;\alpha,\theta)$ are defined above.

F. Anderson–Darling and right-tail Anderson–Darling methods

Another type of minimum distance estimation method is the method of Anderson-Darling (AD). This method was introduced by Anderson and Darling [3, 4] and is based on an Anderson-Darling statistic. The Anderson-Darling statistic is given by

$$A^2 = n\int_0^{\infty}\frac{\left(F(x_i) - E\left[F\left(x_{(i)}\right)\right]\right)^2}{F(x_i)\left(1-F(x_i)\right)}dF(x_i)$$

Boos [6] proved that the Anderson-Darling statistic has computational form which is given by

$$A(\alpha,\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{(i)};\alpha,\theta\right) + \log\left(1-F\left(x_{(i)};\alpha,\theta\right)\right)\right] \quad (33)$$

Therefore, the ADs α_{AD} and θ_{AD} of α and θ can be determined by minimizing (36) with respect to α and θ. These estimators can also be obtained by solving the non-linear equations

IJSES

International Journal of Scientific Engineering and Science
Volume 5, Issue 5, pp. 8-18, 2021. ISSN (Online): 2456-7361

$$\frac{\partial A(\alpha,\theta)}{\partial \alpha}=-\frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\left(\frac{F_{\alpha}^{'}\left(x_{(i)};\alpha,\theta\right)}{F\left(x_{(i)};\alpha,\theta\right)}-\frac{F_{\alpha}^{'}\left(x_{(n-i+1)};\alpha,\theta\right)}{1-F\left(x_{(n-i+1)};\alpha,\theta\right)}\right)=0 \quad (34)$$

$$\frac{\partial A(\alpha,\theta)}{\partial \theta}=-\frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\left(\frac{F_{\theta}^{'}\left(x_{(i)};\alpha,\theta\right)}{F\left(x_{(i)};\alpha,\theta\right)}-\frac{F_{\theta}^{'}\left(x_{(n-i+1)};\alpha,\theta\right)}{1-F\left(x_{(n-i+1)};\alpha,\theta\right)}\right)=0 \quad (35)$$

Luceno [19] provides some motivation about AD statistics and also introduces a modification, namely Right-tail Anderson–Darling statistics. The Right-tail AD statistics given by

$$RA^{2}=n\int_{0}^{\infty}\frac{\left(F\left(x_{i}\right)-E\left[F\left(x_{(i)}\right)\right]\right)^{2}}{1-F\left(x_{i}\right)}dF\left(x_{i}\right)$$

Also, the Right-tail AD has computational form which is given by

$$RA(\alpha,\theta)=\frac{n}{2}-2\sum_{i=1}^{n}F\left(x_{(i)};\alpha,\theta\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(1-F\left(x_{(i)};\alpha,\theta\right)\right) \quad (36)$$

Hence, the RADs α_{RAD} and θ_{RAD} of α and θ are obtained by minimizing (36) with respect to α and θ. These estimators can also be determined by solving the non-linear equations

$$\frac{\partial RA(\alpha,\theta)}{\partial \alpha}=-n\sum_{i=0}^{n}\frac{F_{\alpha}^{'}\left(x_{(i)};\alpha,\theta\right)}{F\left(x_{(i)};\alpha,\theta\right)}+\frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\frac{F_{\alpha}^{'}\left(x_{(n-i+1)};\alpha,\theta\right)}{1-F\left(x_{(n-i+1)};\alpha,\theta\right)}=0 \quad (37)$$

$$\frac{\partial RA(\alpha,\theta)}{\partial \theta}=-n\sum_{i=0}^{n}\frac{F_{\theta}^{'}\left(x_{(i)};\alpha,\theta\right)}{F\left(x_{(i)};\alpha,\theta\right)}+\frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\frac{F_{\theta}^{'}\left(x_{(n-i+1)};\alpha,\theta\right)}{1-F\left(x_{(n-i+1)};\alpha,\theta\right)}=0 \quad (38)$$

IV. SIMULATION

Here, a simulation study is performed to examine the performance of the different estimates presented above. The following procedure for evaluating the efficiency of the estimators is adopted as follow:

1. Generate random sample with size n from the MIPGSD model.
2. The values obtained in step 1are used to compute the $\Theta=(\alpha,\theta)$ considering the MLE, LSE, WLSE, CME, MPS, AD and RAD estimators.
3. Repeat the steps 1 and 2 N times.
4. Using $\Theta=(\alpha,\theta)$ and $\Theta=(\alpha,\theta)$, compute the Bias and the mean square errors (MSE).

The results are computed using the nlminb function (in the stat package) and Nelder-Mead method in R software. The chosen values to perform this procedure are $\Theta=(1.5,0.8)$, $N=5,000$ and $n=(50,80,120,200,300)$. The simulation studies are put under the same conditions (initial values and random samples) for different estimation methods.

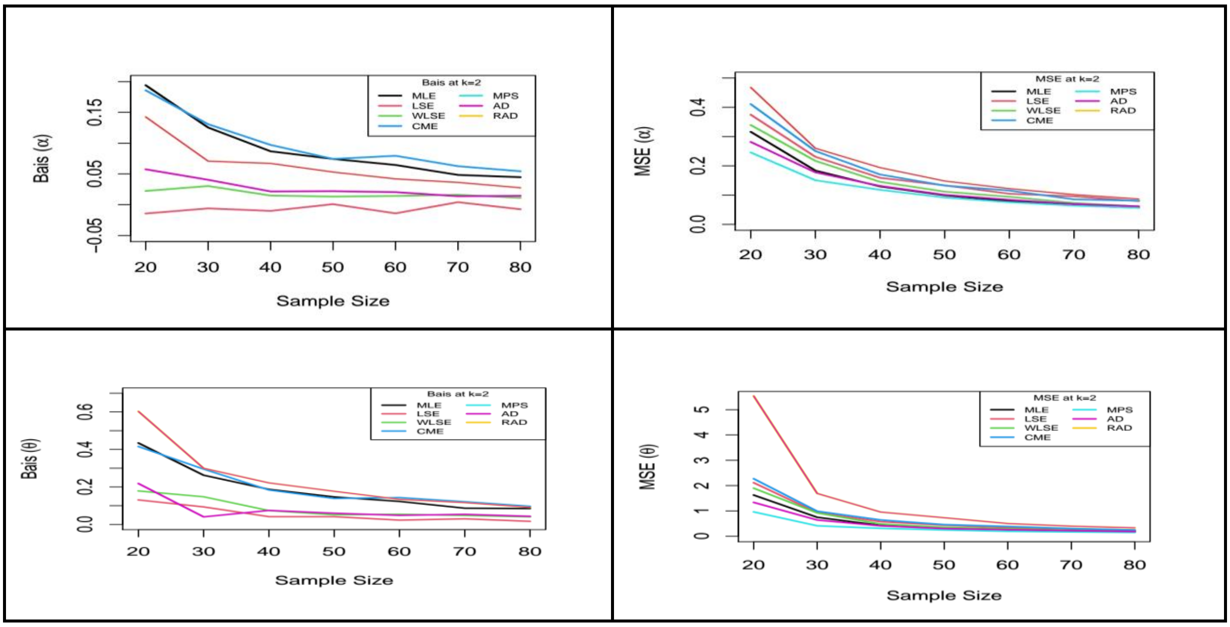

Fig. 3. Bias and MSEs, for the estimates of $\alpha=2.5$ and $\theta=3.5$ versus n when $k=2$ for the estimation methods

International Journal of Scientific Engineering and Science
Volume 5, Issue 5, pp. 8-18, 2021.
ISSN (Online): 2456-7361

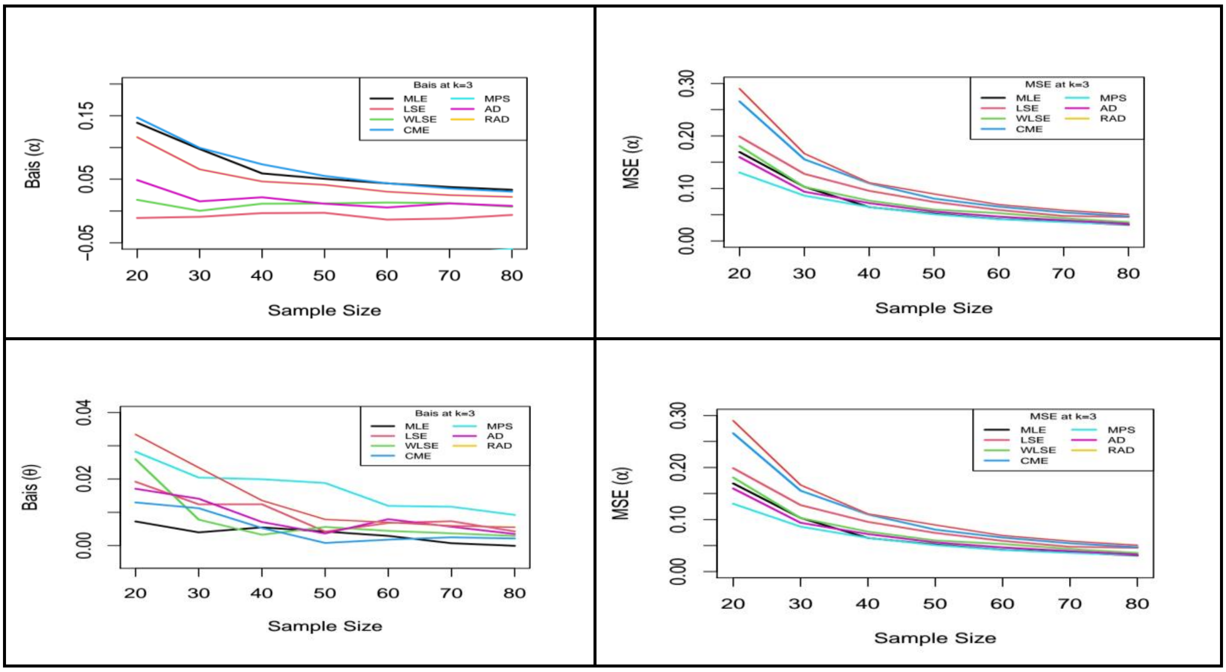

Fig. 4. Bias and MSEs, for the estimates of $\alpha = 2$ and $\theta = 1.5$ versus n when $k = 3$ for the estimation methods

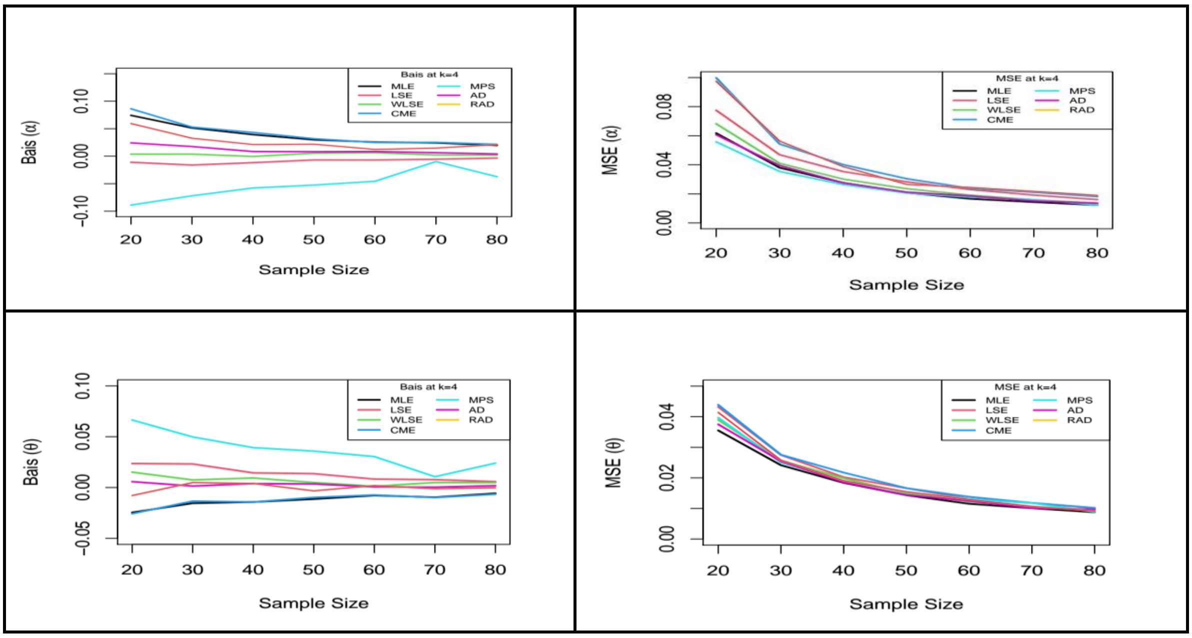

Fig. 5. Bias and MSEs, for the estimates of $\alpha = 1.5$ and $\theta = 1$ versus n when $k = 4$ for the estimation methods.

Figures 3-5 show how the seven biases, mean squared errors vary with respect to sample size for $k = 2$, 3 and 4. As expected, the Biases and MSEs of estimated parameters converge to zero as n increases

V. Application

In this section, we use maximum likelihood estimate of the parameters to perform the goodness of fit of the MIPGSD model for a data set to know the potentiality of the new model as compared to some other existing models.

1. The data set represent the relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark[14]. The observed values are

1.1	1.4	1.3	1.7	1.9
1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4
3.0	1.7	2.3	1.6	2.0

The data set is used to compare the MIPGSD model for values of $k = 2$ and 3 with four competitive models such as:

- Inverted exponentiated gamma (IEG) model (Yadav [27])

International Journal of Scientific Engineering and Science
Volume 5, Issue 5, pp. 8-18, 2021. ISSN (Online): 2456-7361

$$f(x)=\frac{\theta}{x^3}\left(1-\left(1+\frac{1}{x}\right)\exp\left(\frac{-1}{x}\right)\right)^{\theta-1}\exp\left(\frac{-1}{x}\right)$$

where $x,\theta>0$

- Inverse Gompertz (IG) model (Eliwa et al [11])

$$f(x)=\frac{\alpha}{x^2}\exp\left(\frac{-\alpha}{\beta}\left(\exp\left(\frac{\beta}{x}\right)-1\right)+\frac{\beta}{x}\right)$$

where $x,\alpha,\beta>0$

- Inverted xgamma (IXG) model (Yadav et al [28])

$$f(x)=\frac{\theta^2}{(1+\theta)}.\frac{1}{x^2}\left(1+\frac{\theta}{2}.\frac{1}{x^2}\right)\exp\left(\frac{-\theta}{x}\right)$$

where $x,\theta>0$

- Exponentiated inverse Rayleigh (EIR) model (Ul Haq [30])

$$f(x)=\frac{2\alpha\theta}{x^3}\exp\left(\frac{-\alpha\theta}{x^2}\right)$$

where $x,\alpha,\theta>0$.

For more simplification, let $MIPGSD_k(\alpha,\theta)=PM_k(\alpha,\theta)$ and to compare the models, we take the following goodness of fit measures into consideration: the log likelihood function (-L), Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) defined by:

$$AIC=-2L+2q$$

$$BIC=-2L+q\ln(n)$$

where q is the number of parameters, n is the sample size, and $F(x_i;\Theta)$ estimated cumulative distribution function of theoretical models. The model with the lowest values of goodness of fit measures provides the best fit for data set.

Tests statistics such as Cramér–von Mises W_n^2, Anderson–Darling A_n^2, Watson U_n^2, Liao-Shimokawa L_n and Kolmogrov- Smirnov $K-S$ with its respective p-value are considered in order to verify which distribution fits better to each data set. These tests display the differences between the proposed cumulative distribution function and the empirical cumulative distribution function from the data to verify the fit of the distributions (p-value > 0.05). For more details about above tests statistics see Al-Zahrani [2].

TABLE 2. The goodness of fit measures for the data set.

Models	Measures				
	MLEs		− L	AIC	BIC
$PM_{k=2}(\alpha,\theta)$	3.98	6.71	15.41	34.82	36.81
$PM_{k=3}(\alpha,\theta)$	3.94	7.41	15.41	34.83	36.82
$IEG(\theta)$	---	0.44	38.19	80.38	82.37
$IG(\alpha,\beta)$	0.110	6.14	16.39	36.78	38.77
$IXG(\theta)$	---	2.72	33.63	71.27	73.26
$EIR(\alpha,\theta)$	1.31	2.09	21.18	46.36	48.35

TABLE 3. The goodness-of-fit test statistics for the data set.

Models	Statistics					
	W_n^2	A_n^2	U_n^2	L_n	$K-S$	p-value
$PM_{k=2}(\alpha,\theta)$	0.02	0.15	4.54	0.61	0.10	0.98
$PM_{k=3}(\alpha,\theta)$	0.02	0.15	4.54	0.61	0.10	0.98
$IEG(\theta)$	1.21	5.72	5.61	2.13	0.47	0.00
$IG(\alpha,\beta)$	0.05	0.32	4.55	0.77	0.14	0.81
$IXG(\theta)$	1.05	5.08	5.68	2.06	0.40	0.00
$EIR(\alpha,\theta)$	0.39	2.06	4.99	1.41	0.25	0.14

Table 2 provides the values of the goodness of fit measures for the fitted models to the data set. The MIPGSD model provides the lowest values for all measures among all fitted models. The tests shown in Table 3 presents that the proposed model, IG model and EIR model fit the data set (p-value > 0.05) and the proposed model shows the lowest test statistics with the largest p-values. Thus, The MIPGSD model fits well the data set and can be considered as a good competitor against the other models.

Furthermore, seven estimation methods are used to estimate the unknown parameters of MIPGSD distribution. Table 4 display the estimates of the MIPGSD parameters using these estimation methods with its rank and the values of $K-S$ and its p-value for the data set, respectively. We can conclude that the CME estimation method is recommended to estimate the MIPGSD parameters for the data set.

TABLE 4. The parameter estimates of the MIPGSD model , $K-S$ and p-value for the data set at $k=2$.

Est. Meth.	Est. Par.		$K-S$	p-value	Rank
	α	θ			
MLE	3.98	6.71	0.10	0.98	5
LSE	3.87	6.46	0.10	0.98	4
WLSE	3.64	5.81	0.1	0.97	6
CME	4.23	7.6	0.09	0.99	1
MPS	3.41	5.23	0.10	0.97	7
AD	3.95	6.67	0.10	0.98	3
RAD	4.06	7.05	0.09	0.99	2

Figure 6 shows the probability-probability (P-P) plot of the fitted models for the data set, whereas Figure 7 shows fitted pdfs of the considered models for the data set. We can conclude that the proposed distribution was the one which best adjusted to the two data sets.

VI. Conclusion

In this paper, we proposed a mixture model of inverse power Gamma shape distributions and studied in detail. Some statistical expression for its properties are obtained. The estimation of distribution parameters by using seven estimation methods are performed. We present a simulation study to illustrate the performance of the estimates. Two data sets also presented for the demonstration of enhanced flexibility and better fit of the observed model as compared to some other well-known existing models.

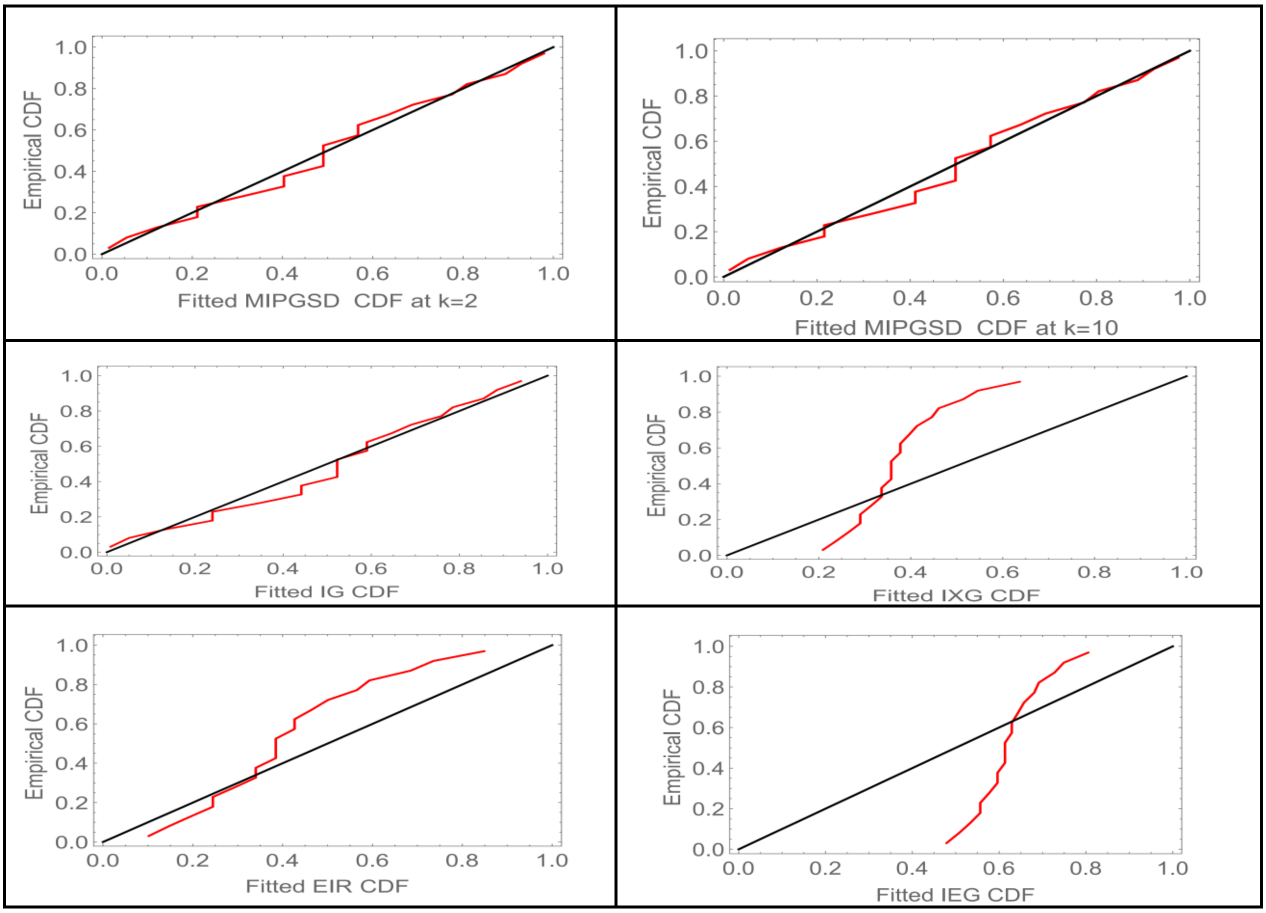

Fig.6. P-P plots for the first data set.

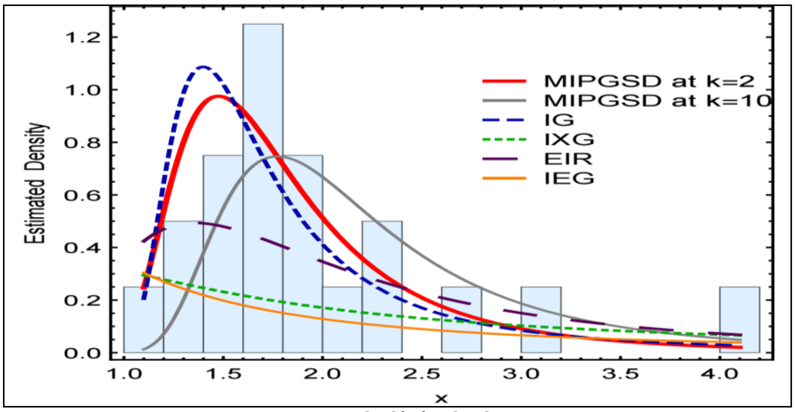

Fig.7. Estimated pdfs for the data set 1.

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