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Lower and Upper Solutions of Second Order Non-Linear Ordinary Differential Equations

[1]S.E. Aniaku, [2]E.C. Mbah, [3]C.C. Asogwa, [4]P.U. Nwokoro
[1,2,3,4]Department of Mathematics, University of Nigeria Nsukka Nigeria

Abstracts— *A lower solution α(t) and upper solution β(t) technique is used for second order boundary value problem if the function f (t, u, u^1) involved satisfies* $|f\ (t, u\ , u^1)\ \_\ f\ (t, u, v^1)\ | \le\ L(u^1 -\ v^1)|$*. Such function must also satisfy Nagumo condition. It was observed that unlike in first order ordinary differential equations (ODE) there is ordering in the lower and upper solutions of second order ODE.*

Keywords— *Nagumo condition, Lipschitz constant, lower solution; upper solution compact space.*

I. Introduction

When a differential equation is given, the question of whether the solution of such equation exists may be asked. In some cases, the question of the multiplicity of the solution may be asked, and even the non-existence of solution of that equation in a given domain may also be asked. In the literature, there are various existence results for differential equation of a given order. See for instances [1], [2], [3], [4], [5], [6], [7], [8], [9], [10] [11], and the references there in. We want to, in this paper, review some of these existence results using the lower and upper solutions techniques for second order non-linear differential equations. We also study some methods of constructing lower and upper solutions to second order differential equations.

Consider the boundary value problem.

u'' = f (t, u, u') (1.1)

a_1 u (a) – a_2 u' (a) = A (1.2)

b_1 u (b) + b_2 u' (b) = B (1.3)

where f: $[a, b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous, A, B ∈ ℝ; a_1, b_1, ∈ ℝ; a_2, b_2 ∈ $\mathbb{R}^+$.

Our interest is on the existence of the solution of (1.1) – (1.3) by the method of lower and upper solution technique. We start with the following notations and definitions;

1.1 Notations and Definitions

1.1. a Notations

I = $[a, b]$

ℝ = Set of real numbers

ODE= Ordinary Differential Equation.

1.1.b **Definitions**

Definition 1.1 Lower solution [8]

A function $\alpha \in C^2\ [a, b]$ will be called a lower solution of (1.1) - (1.3) if

$\alpha'' \ge$ f (t, α, α') on $[a, b]$ and

$a_1\ \alpha$ (a) – $a_2\ \alpha'$ (a) ≤ A

$b_1\alpha$(b) + $b_2\ \alpha'$ (b) ≤ B

Definition 1.2 Upper solution [8]

A function $\beta \in C^2\ [a, b]$ will be called an upper solution of (1.1) – (1.3) if

$\beta'' \le$ f (t, β, β') on $[a, b]$ and

$a_1\ \beta$ (a) + $a_2\ \beta$ (a) ≥ A

$a_1\ \beta$ (b) + $b_2\ \beta$ (b) ≥ B.

Definition 1.3 Definition of E

Let

E: = { (t, u, u') ∈ $[a, b] \times \mathbb{R}^2$: α (t) ≤ u (t) ≤ β (t)} (1.4)

Some Conditions

Nagumo condition [6] Let h : $\mathbb{R}^+ \longrightarrow \mathbb{R}$ be positive continuous function satisfying

$\int_0^\infty \frac{s}{h(s)}$ ds = ∞ (1.5)

The function f: E ⟶ ℝ is said to satisfy a Nagumo condition if

$|f(t, u, u')|\ \le$ h ($|u'|$) ∀ (t, u, u') ∈ E (1.6)

Other conditions

(A_0): There exist a lower solution α (t) of (1.1) – (1.3) and an upper solution β(t) of (1.1) – (1.3) with α (t) ≤ β (t) on $[a, b]$

(A_1) f (t, u, u') satisfies a Lipschitz condition with respect to u and u' on the set of E

(A_2) f(t, u, u') satisfies a Nagumo condition on the set E.

(A_3) For any $(t_0, u_0, u_0') \in E$, the solution of (1.1) – (1.3) satisfying the initial condition $u(t_0) = u_0, u'(t_0) = u_0'$ is unique.

Special Consideration

Now, let u(t) and v(t) be two linearly independent solutions of

$u'' = 0$ (i)

such that u satisfies (1.2) with B = 0. We define the function G(t,s) on the square I x I by

$$G(t, s) := \begin{cases} \frac{1}{c} v(t)u(s), & a \le s \le t \\ \frac{1}{c} u(t)v(s), & t \le s \le b \end{cases}$$

where c = u (t) v' (t) – u' (t) v (t) ≠ 0. It is known that any integrable function h(t), the solution u (t) of

u'' = h (t) (ii)

which satisfies (1.2) – (1.3) may be written as

u (t) = $\int_a^b G\ (t, s)$ h (s) ds + Q (t) (1.7)

where Q'' = 0, and Q satisfies boundary conditions (1.2) - (1.3) and is a solution of the integral equation

u (t) = $\int_a^b G\ (t, s)$ f (s, u(s), u'(s)) ds + Q (t) (1.8)

and conversely

Let us consider the following useful lemma

Lemma 1.1
Let there exist a constant M > 0 such that $|f(t,u,u')| \leq$ M for all (t, u, u') $\in$ I $\times \mathbb{R}^2$. Then the boundary value problem (1.1) – (1.3) has a solution.
Proof
Let $\mathfrak{B}$ = C^1 (I) for u $\in \mathfrak{B}$, define
$\|u\| = \sup_{t \in I} |u(t)| + \sup_{t \in I} |u'(t)|$. Then ($\mathfrak{B}$, ‖.‖) is a Benach space. Define a mapping T: $\mathfrak{B} \longrightarrow \mathfrak{B}$ by setting for each u $\in \mathfrak{B}$
$Tu(t) = \int_a^b G(t,s)\, f(s, u(s), u'(s))\, ds + Q(t)$
Set $N = \sup_{I \times I} |G(t,s)|$ (b-a)
$N^1 = \sup_{I \times I} |G_t(t,s)|$ (b-a)
$L = \sup_{I} |Q(t)|$
and $L^1 = \sup_{I} |Q'(t)|$
Then we can show that $|Tu(t)| \leq$ NM + L and $|(Tu)'(t)| \leq N^1 M + L^1$. Therefore, T maps the closed, bounded convex set $\mathfrak{B}_1$ = {u $\in \mathfrak{B}$: $|u(t)| \leq$ NM + L, $|(u'(t)| \leq N^1 M + L^1$ into itself. Therefore $\overline{T\beta}$ is compact. We now apply Schauder-Tychonoff theorem to conclude that T has a fixed point in $\mathfrak{B}$. Fixed points of T, however, are solutions of (1.8).

II. Existence of Lower and Upper Solutions

Since the main aim of this review is to study the existence of solution to (1.1) under suitable boundary conditions by the method of lower and upper solutions, we start with this theorem
Theorem 2.1
Suppose there exist a lower solution α (t) and upper solution β(t) of (1.1) – (1.3) such that $\alpha(t) \leq \beta(t)$, $\forall$ t $\in [a, b]$ and the conditions (A_1) and (A_2) hold, then there exist at least one solution u(t) of (1.1) – (1.3) satisfying $\alpha(t) \leq u(t) \leq \beta(t)$.
Proof
Define the function F (t, u, u') on $[a, b] \times \mathbb{R}^2$ by setting

$$F(t,u,u') := \begin{cases} f(t,\beta, u') + \frac{u-\beta(t)}{1+u^2}, & if\ u > \beta(t) \\ f(t,u,u'), & if\ \alpha(t) \leq u(t) \leq \beta(t) \\ f(t,\alpha,u') + \frac{u-\alpha(t)}{1+u^2}, & if\ u < \alpha(t) \end{cases}$$

Since f is bounded, F is also bounded. Hence by Lemma 2.1, there exists a solution u(t) of (1.1) – (1.3). We now show that (t, u, u^1) $\in$ E of definition 1.3, which of course means that u (t) is a solution of (1.1) – (1.3). Assume u (t) $> \beta(t)$ on t $\in [a, b]$, then there exist points $a \leq t_1 < t_2 \leq b$ such that $u(t_1) = \beta(t_i)$, i = 1, 2, and $u(t) > \beta(t)$, $t_1, < t < t_2$. The difference $u(t) - \beta(t)$ therefore will assume a positive maximum at a point t_0, $t_1 < t_0 < t_2$. We see $u'(t) = \beta'(t)$ and $u''(t_0) - \beta''(t_0) = 0$. But a computation, however shows that

$$u''(t_0) - \beta''(t_0) \geq F(t_0, u(t_0), u'(t_0)) - F(t_0, \beta(t_0), \beta'(t_0))$$
$$\geq f\big(t_0,\beta(t_0),\beta'(t_0)\big) + \frac{u(t_0)-\beta(t_0)}{1+u^2(t_0)} - f\big(t_0,\beta(t_0),\beta'(t_0)\big) > 0.$$

which is a contradiction.
Next, we show that $\alpha(t) \leq u(t)$. We assume, by contradiction that $u(t) < \alpha(t), t \in [a,b]$. Then there exist points $a \leq t_1 < t_2 \leq b$ such that $u(t_i) = \alpha(t_i), i = 1, 2;$ $and\ u(t) < \alpha(t), t_1 < t < t_2$. The difference $u(t) - \alpha(t)$ therefore will assume a negative maximum value at a point t_0, $t_1 < t_0 < t_2$ and $u'(t_0) = \alpha'(t_0)$ and $u''(t_0) - \alpha''(t_0) \geq 0$.
A computation, however shows that

$$u''(t_0) - \alpha''(t_0) = F\big(t_0,u(t_0),u'(t_0) - F(t_0,\alpha(t_0),\alpha'(t_0)\big)$$
$$\leq f\big(t_0,\alpha(t_0),\alpha'(t_0)\big) + \frac{u(t_0)-\alpha(t_0)}{1+U^2(t_0)} - f\big(t_0,\alpha(t_0),\alpha'(t_0)\big) < 0.$$

This is also a contradiction. These showed that $\alpha(t) \leq u(t) \leq \beta(t)$.
So, with regards to the above theorem, we shall be interested in the existence of such lower solution $\alpha(t)$ and upper solution $\beta(t)$.
Let us seek for conditions under which the functions $\alpha(t) and\ \beta(t)$ exist such that

$$\left.\begin{aligned} &\alpha''(t) \geq f\big(t,\alpha(t),\alpha'(t)\big), t \in [a,b]. \\ &\beta''(t) \leq f\big(t,\beta(t),\beta'(t)\big), t \in [a,b]. \\ &\alpha(t) \leq \beta(t). \\ &\alpha(a) \leq 0 \leq \beta(a);\ \alpha(b) \leq 0 \leq \beta(b). \end{aligned}\right\} \quad (2.1)$$

Theorem 2.2
Suppose $uf(t,u,u') > |u|Q(|u'|)$ for all u where Q$\in$ $C'([0,\infty),\mathbb{R}^+)$ and $\int_0^\infty \frac{ds}{Q(s)} > b - a$. Then there exist solutions $\alpha(t) and\ \beta(t)$ satisfying (2.1)
Proof.
Let $\beta(t)$ be the unique solution to
$\beta''(t) - Q(|\beta'(t)|), \beta(a) = M, \beta'(a) = 0.$
Then $\beta'|_b^a = -\int_a^t Q(|\beta'[s)|)ds$
Let $-\int_a^t Q(|\beta'[s)|)ds = \mathbb{Z}.$ $Then\ \beta'(t) = \mathbb{Z}$. So $\beta(t) - \beta(a) = \int_a^t \mathbb{Z}(s)ds \Rightarrow \beta(t) - M = \int_a^t \mathbb{Z}(s)ds \Rightarrow \beta(t) \geq M\ on\ [a,b]$.
Then $\beta''(t) = -Q(|\beta'(t)|), < f(t,\beta(t),\beta'(t)$. If we take $a(t) = -\beta(t)$, we see that (2.1) is also satisfied.
Corollary 2.1
Suppose
(a) $f(t,u_1,v) \leq f(t, u_2, v)$ for $u_2 > u_1$.
and
(b) $|f(t,u,v_1) - f(t,u,v_2)| \leq L|v_1 - v_2|$, then there exist $\alpha(t)$ and $\beta(t)$ satisfying (2.1), where L > 0 is a Lipschitz constant.
Proof
For u $\geq$ 0; $f(t,u,u') \geq f(t,0,u') \geq f(t,0,0) - L|u'| \geq A - L|u'|$.
where $|f(t,u,,u^1)| \leq A\ on\ [a,b]$.
For $u \leq 0, f(t,u,u') \leq A + L|u'|$
So, we may take Q(s) = $A + L|u'|$ and proceed as in Theorem 2.2.
Corollary 2.2
Let f satisfy the following conditions.
(i) $f(u,-c,0) \leq 0 \leq f(t,c,0)\ for\ any\ c \geq 0$ (2.2)
and
(ii) $|f(t,u,u') - f(t,u,v')| \leq L|u' - v'|$ (2.3)
for any u, a $\leq$ t $\leq$ b, L $\geq$ 0 a positive constant, then each of the problem (1.1) subject to (2.4), (2.5) and (2.6) has a solution, where
$u(a) = A, u(b) = B$ (2.4)

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$u(a) = A, u(b) = 0$ (2.5)
$u'(a) = 0, u'(b) = B$ (2.6)

Proof

Using (2.2), one may find lower and upper solutions. One may take
$\beta = Max\{0, A, B\}, \alpha = Min\{0, A, B\}$ for (1.1) subject to (2.4)
$\beta = Max\{0, A\};\ \alpha = Min\{0, A\}$ for (1.1) subject to (2.5)
$\beta = Max\{0, B\};\ \alpha = Min\{0, B\}$ for (1.1) subject to (2.6).
In each of these problems, a solution u(t) may be chosen so that
$\alpha(t) \leq u(t) \leq \beta(t)$.

Some Examples

Example 2.1

Consider $u'' = -|u|^{\frac{1}{2}} + t, t \in [1,2], u \geq 0$.
$u(1) = 0, u'(2) = 1$.
We note that since $-|U|^{\frac{1}{2}}$ is a decreasing function, $\left|-|u|^{\frac{1}{2}} + t\right| \leq 2\ \forall\ t \in [1,2]$.
So, we seek for solution $\beta(t)$ to $u'' = -2, u(1) = 0, u'(2) = 1$; to get $\beta(t) = -t^2 + 5t + 4$
For lower solution $\alpha(t)$, we consider the solution to $u'' = 2, u(1) = 0, u'(2) = 1$
Solving this, we get $\alpha(t) = t^2 - 3t + 2$.

Example 2.2

Consider $u'' = -u^3 + t; t \in [0,1]\ u(0) = 0;\ u'(1) = 1$.
We note that in this case, $f(t,u) = -u^3 + t$. Since $-u^3$ is a decreasing function, we see that $(-u^3 + t) \leq 1\ \forall\ t \in [0,1]$.
So for upper solution $\beta(t)$, we look for solution to the boundary value problem
$u'' = -1; u(0) = 0;\ u'(1) = 1$. to get $\beta(t) = \frac{t^2}{2} + 2t$.
For lower solution $\alpha(t)$, we look for solution of the boundary value problem
$u'' = 1; u(0) = 0;\ u'(1) = 1$. Solving this, we get $\alpha(t) = \frac{t^2}{2}$.
We also note that $\alpha(t) \leq \beta(t) \forall\ t \in [0,1]$.

Example 2.3

Consider
$u'' -\ u' = t^2; t \in [0,1]$;
u(0) = 1, $u'(1) = 0$.
For upper solution $\beta(t)$ of this problem, we consider the solution to
$u'' + L|u'| - M = 0;\ u(0) = 1, u'(1) = 0$ where L > 0 is a Lipschitz constant.
and m = $\inf_{t \in [0,1]} f(t, 0,0) = 0$.
By Lipschitz rule application, we know that there exists a constant K > 0 such that
$|u'| \leq$ k
Then $u'' = -L|u'| + m$ becomes
$u'' = -LK$.
Without loss of generality, let L = - 1. Then we shall be considering
$u'' = K; u(0) = 1; u(1) = 0$
Solving this, we get $\beta(t) = \frac{-Kt^2}{2} + Kt + 1$.
For lower solution $\alpha(t)$ of the problem, we consider the solution to
$u'' = -L|u'| - M = 0; u(0) = 1; u'(1) = 0$ where M = $\sup_{t \in [0,1]} f(t, 0,0) = 1$.
Following the same way as above, we see that we shall be looking for solution to
$u'' = K + 1; u(0) = 1; u'(1) = 0$.
Solving this, we get
$\alpha(t) = (K + 1)\frac{t^2}{2} - (K + 1)t + 1$. We note that $\alpha(t) \leq \beta(t) \forall\ t \in [0,1]$

III. Conclusion

The use of lower solution $\alpha(t)$ and upper solution $\beta(t)$ in the search of solution u(t) of second order ordinary differential equation (ODE) is possible provided such function satisfies Lipschitn condition in the second and third arguments. The satisfaction of Nagumo condition by such function also played enormous role. Lower solution $\alpha(t)$ and upper solution $\beta(t)$ of second order ODE help a lot in the search of the solution u(t) of such equation, provided $\alpha(t) \leq u(t) \leq \beta(t)$ for all $\forall\ t \in [a, b]$.

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