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Lower and Upper Solutions of First Order Non-Linear Ordinary Differential Equations

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Abstract— *In this paper, the conditions for the existence of lower and upper solutions of first order non- linear ordinary differential equations were considered. Also considered was the technique for constructing lower and upper solutions of the first order non-linear ordinary differential equation. Examples illustrating the use of lower and upper solutions were given.*

Keywords— *Lower solution, upper solution, Cauchy problem, fixed point.*

I. INTRODUCTION

Given a differential equation, a natural question to ask is either that of existence, multiplicity or even the non-existence of solutions of it in a given domain. In the literature, there are various existence results for differential equation of a given order. See for instance [1], [2] [3] [4] [5], [6], and the references there in. We shall therefore, in this power, review some of these existence results using lower and upper solutions techniques. We hope to study the existence of a C^1-solution using the method of lower and upper solutions. We shall also study some methods of constructing lower and upper solutions to first order ordinary differential equations. See also [7], [8], [9], [10]. The ordering of the lower and upper solutions of first order ordinary differential equation was also considered in the existence results.

Consider the periodic problem

u' (t) = f(t, u(t) (1.1)

u(a) = u (b) (1.2)

where a < b and $f:[a,b] \times R$ is a continuous function. We wish to establish the existence of solution to (1.1) - (1.2) provided we can find lower and upper solutions. We start with the following notations and definitions.

1.1 Notations and Definitions

Notations

$\mathbb{R}$ = the set of real numbers

Ω = an open region

$\partial\Omega$ = the boundary of the region Ω

I = [a, b], where a and b are in $\mathbb{R}$

ODE = Ordinary differential equation.

C^1 [a, b] = set of functions whose first derivatives is continuous.

Definitions:

Definition 1.1 Lower solution [4]

A function $\alpha \in C^1$ ($[a,b]$) is a lower solution of the periodic problem (1.1) - (1.2) if

(a) for all t $\in [a,b]$, α'(t) $\leq$ (f (t, α (t)), and

(b) α(a) $\leq$ α(b).

Definition 1.2 Upper solutions [4]

A function $\beta \in [a,b]$ is an upper solution of the periodic problem (1.1) - (1.2) if

(a) For all t $\in$ [a, b], β'(t) $\geq$ f(t, β(t)) and

(b) β (a) $\geq$ β (b).

Definition 1.3 . Definition P.

Let P be defines as

P: = {(t, u (t) $\in [a,b] \times \mathbb{R}$: α (t) $\leq$ u (t) $\leq$ β (t)} (1.3)

Definition 1.4 Definition of Q

Let Q be defined as

Q ; = {(t, u (t) $\in [a,b] \times \mathbb{R}$: β (t) $\leq$ u (t) $\leq$ α (t)} (1.4)

II. PROPOSITIONS AND THEOREMS

We have the following propositions and theorems concerning the lower and upper solutions of first order non-linear ODE. We also include some remarks

Theorem 2.1 [7]

Let $\propto$ and β be lower and upper solutions of (1.1) and (1.2) such that $\alpha \leq \beta$. Assume f is continuous on P (1.3) and solution of the Cauchy problem

u' (t) = f (t, u (t)), u (a) = u_0 (2.1)

with $u_0 \in [(\alpha(a), \beta$ (a)] is unique. The problem (1.1) _ (1.2) has at least one solution u $\in C^1$ (a, b) such that for all $t \in [a,b]$, α (t) $\leq$ u (t) $\leq$ β(t)

Proof.

Consider the modified problem

$$u' = f\big(t, \gamma(t,u)\big), u(a) = u(b) \tag{2.2}$$

where $\gamma: [a,b] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\gamma(t,u) := \begin{cases} \alpha(t), & if u(t) < \alpha(t) \\ u(t), & if\ \alpha(t) \leq u(t) \leq \beta(t) \\ \beta(t), & if\ u(t) > \beta(t) \end{cases} \tag{2.3}$$

Let u(t; u_0) denote a solution of the Cauchy problem.

$$u' = f\big(t, \gamma(t,u)\big), u(a) = u_0$$

We want to prove that if $u_0 \in [\alpha(a), \beta(a)]$. Then for all $t \in [a,b], \alpha(t) \leq u(t;u_0) \leq \beta(t)$.

Otherwise, if there exists $t_0 \in [a,b], with\ u(t_0) < \alpha(t_0)$, then there exists $t_1 \in [a,t_0]$ such that $u(t_1) < \alpha(t_1) and\ u'(t_1) < \alpha'(t_1)$. Hence, we have contradiction.

$0 > u'(t_1) - \alpha'(t_1) \geq f(t_1, \alpha(t)) - f\big(t_1, \alpha(t_1)\big) = 0.$

This prove that $\alpha(t) \leq u(t;u_0)\ on\ [a,b]$.

Next, we prove that $\beta(t) \geq u(t;u_0) on\ [a.b]$. Otherwise, if there exists $t_0 \in [a,b]\ with\ u(t_0) > \beta(t_0), then\ there\ exists\ t_2 \in [a,t_0]$ such that $u(t_2) > \beta(t_2) and\ u'(t_2) > \beta'(t_2)$. This will lead to contradiction.

$0 < u'(t_2) - \beta'(t_2) \le f(t_2, \beta(t_2)) - f(t_2, \beta(t_2)) = 0$
Thus, $u(t;u_0) \le f\beta(t)$ on [a,b]. Hence, $u(t;u_0)$ is a solution of (2.1). It is unique and defined on [a, b].
Let T: $[\alpha(a), \beta(a)] \to \mathbb{R}$ be defined by $Tu_0 := u(t;u_0)$. Using the boundary conditions on α and β, and the first part of the proof, we deduce that $\alpha(t) \le \alpha(b) \le Tu_o \le \beta(a) \le \beta(t)$.
Then by fixed point theorem of Brouwer, there exists $\bar{u}_0 \in [\alpha(a), \beta(a)] such\ that$
$T\bar{u}_0 = \bar{u}_0$. This $u(t; u_0)$ is a solution of (1.1) – (1.2).
Remark 2.1
A peculiarity of the first order ordinary differential equation (ODE) is that theorem 2.1 remains true it $\beta(a) \le \alpha(t)$. This remark gives rise to the following theorem.
Theorem 2.2. [5].
Let $a\ and\ \beta$ be lower and upper solutions of (1.1) – (1.2) such that $a(t) \ge \beta(t)$. Assume f is continuous on Q i.e. (1.4), and that the solution of the Cauchy problem.

$$u' = f(t, u), u(b) = u_0 \qquad (2.5)$$

with $u_0 \in [\beta(b), \alpha(b)]$ is unique. The problem (1.1) – (1.2) has at least solution $u \in C^1[a, b]$ such that for all $t \in [a, b], \beta(t) \le u(t) \le \alpha(t)$.
Proof:
Consider the problem

$$u' = -f(b + a - t, u) \qquad (2.6)$$

$\bar{\alpha}(t) = \beta(b + a - t) and\ \bar{\beta}(t) = \alpha(b + a - t, u)$ are lower and upper solutions of (2.6) – (1.2), where $\alpha(.)\ and\ \beta(.)$ are lower and upper solutions of (2.6) – (1.2).
This is so since
$\bar{\alpha}'(t) = -\beta'(b + a - t) \le -f(b + a - t, \beta(b + a - t))$.
Hence, we have $\beta'(b + a - t) \ge f(b + a - t, \beta(b + a - t))$.
Also $\bar{\beta}'(t) = -\alpha'(b + a - t) \ge -f(b + a - t, \alpha(b + a - t))$.
This gives $\alpha'(b + a - t) \le f(b + a - t, \alpha(b + a - t))$.
So far all $t \in [a, b], \bar{\alpha}(t) \ge \bar{\beta}(t)$. Hence by Theorem 2.1, the problem (2.6) – (1.2) has a solution $\bar{u}(t)$ such that $\bar{\alpha}(t) \ge \bar{u}(t) \ge \bar{\beta}(t)$.
It follows that u(t) = $\bar{u}(b + a - t)$ is the required solution of (1.1) – (1.2)
Examples
We have the following examples to illustrate Theorems 2.1 and 2.2.
Example 2.1
Consider the problem

$$u' + sinu = 1, u(0) = u(2\pi) \qquad (2.7)$$

There is a solution u(t) such that
$\frac{\pi}{2} \le u(t) \le \frac{3\pi}{2}$
where $\beta(t) = \frac{\pi}{2}\ and\ \alpha = \frac{3\pi}{2}$ are upper and lower solutions of (2.7).
This illustrates Theorem 2.2 about the non-ordering of lower and upper solutions of first order ODE.
The simplest example of lower and upper solutions are constants. This is illustrated below.
Corollary 2.1
Let f:[a,b] x $\mathbb{R} \to \mathbb{R}$ be continuous function such that for some constants $r_1 < r_2\ and\ for\ all\ t\epsilon[a, b]$. $f(t, r_1).(t, r_2) \le 0$. Then problem (1.1) – (1.2) has at least one solution $u\epsilon\ C^1[a, b]$ such that $r_1 \le u(t) \le r_2$.
Proof:
Let $f(t, r_1). f(t, r_2) \le 0$. This implies that either (1) $f(t, r_1) \ge 0\ and\ f(t, r_2) \le 0\ or$ (2) $f(t, r_1) \le 0$ and $f(t, r_2) \ge 0$
Let r_1 be a lower solution of (1.1). Then we have $r_1^1 \le f(t, r_1). Since\ r_1\ is$ a constant, it implies that $0 \le f(t, r_1)$ If r_2 is an upper solution, we have $r_2^1 \ge f(t, r_2)$. So, $0 \ge f(t, r_2)$. These gave rise to case (1) above.
If we have r_2 as a lower solution and $r_1$as an upper solution, we have case (2).
Therefore, since r_1 and r_2 being lower and upper solutions satisfy the hypotheses of the corollary, then by Theorem 2.1, there exists a solution u(t) such that $r_1 \le U(t) \le r_2$.
Example 2.2
Consider the problem

$$u' = u + \frac{t^2}{2}, u(0) = u(1)t \in [0,1], \qquad (2.8)$$

We note that $\alpha = 1$ is a lower solution and $\beta = -2$ is an upper solution of problem (2.8). Also, $f(t, 1). f(t, -2) \le 0$ for all $t \in [0,1]$. So there exists a solution u(t) of the above problem such that $-2 \le u(t) \le 1$

Example 2.3
Consider the problem

$$u' = -u + t^2 + 1; u(0) = u(1); t \in [0,1]. \qquad (2.9)$$

We note that $\alpha(t) = \frac{t}{2}$ and $\beta(t) = 2$ are lower and upper solutions of this example.

III. Construction of Lower and Upper Solutions of First Order Non Linear ODE

A possible way of constructing an upper and lower solutions to problem (1.1) (1.2) is as follows:
Let $M = \underset{t \in [a,b]}{Sup} f(t, 0)$, and consider the differential equation

$$u' + M = 0 \qquad (3.1)$$

We take $\beta(t)$ to be the solution of (3.1) which satisfies $\beta(a) \ge \beta(b)$. Such $\beta(t)$ is regarded as an upper solution of (1.1) – (1.2). We can then take the lower solution $\alpha(t)\ of\ (1.1) - (1.2)$ to be $-\beta(t)$.
Consider the following examples.
Example 3.1
Consider the problem

$$u' = -u + t^2; t \in [0,1]. u(0) = u(1) - 2 \qquad (3.2)$$

For upper solution $\beta(t)$ of (3.2), we consider
$M = \underset{t \in [0,1]}{Sup} f(t, 0) = 1.$ We then solve
$u' + 1 = 0, u(0) = -2$ to get
$\beta(t) = -t - 2$
For lower solution $\alpha(t)$ of (3.2), we consider $\alpha(t) = -\beta(t)$. i.e. $\alpha(t) = t + 2$.
Example 3.2
Consider the problem

$$u' = u + t^{\frac{1}{2}}, t \in [2,4], u(2) = u(4) = -8 \qquad (3.3)$$

Following the same way for the construction of upper solution, $\beta(t)$, and using the upper boundary condition, we see

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that $\beta(t)$ for (3.3) is $\beta(t) = -2t$. We can then regard as lower solution of (3.3) to be $\alpha(t) = 2t$

IV. Conclusion

The existence of lower and upper solutions of first order non-linear ODE assured us that a solution to its Cauchy problem exists and that it lies in between these solutions. For first order non-linear ODE, the ordering of lower solution and upper solutions does not matter.

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