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Generalized Inverse of Vague Matrix

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Abstract**—***In this paper the idea of generalized inverse of a vague matrix is presented. The notion of standard basis for the vague vectors is introduced and also derived some of the results related to generalized inverse of vague matrix. An illustrative example for generalized inverse is provided at the end of the paper*.

Keywords**—** *Vague matrix, vague permutation matrix, generalized inverse of vague matrix, minus ordering*.

I. INTRODUCTION

The generalization of the inverse of the matrix to the case of a singular matrix is the generalized inverse (g- inverse). g-inverse of a matrices plays an important role in various applications such as signal processing robotics and so on. Initially, fuzzy set theory was proposed by Zadeh [8]. The fuzzy matrix have been proposed to represent fuzzy relation in a system based on fuzzy set theory. Many authors have exhibited a number of results on fuzzy matrices [3,6,7], Cen [4] defined generalized inverse of fuzzy matrix and produced several results. Vague set theory was introduced by Gau and Buehrer [2]. In this paper we study the concept of generalized inverse of vague matrix and discussed some of the results and relation between minus- ordering and g- inverse of vague matrices. We also present an elementary algorithm to evaluate g- inverse with an illustrative example.

II. PRELIMINARIES

Definition 2.1:[2] A vague set A in the universe of discourse U is characterized by two membership functions given by:

(i) A true membership function $t_A : U \to [0,1]$ and

(ii) A false membership function $f_A : U \to [0,1]$

where $t_A(x)$ is a lower bound on the grade of membership of x derived from the "evidence for x", $f_A(x)$ is a lower bound on the negation of x derived from the "evidence for x", and $t_A(x) + f_A(x) \le 1$. Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(x), 1 - f_A(x)]$ of [0,1]. this indicates that if the actual grade of membership of x is μ(x), then, $t_A(x) \le \mu(x) \le 1 - f_A(x)$.The vague set A is written as $A = \{\langle x, [t_A(x), 1 - f_A(x)] \rangle / u \in U\}$ where the interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A, denoted by $V_A(x)$.

Definition 2.2:[1] Let A and B be VSs of the form $A = \{\langle x, [t_A(x), 1 - f_A(x)] \rangle / x \in X\}$ and $B = \{\langle x, [t_B(x), 1 - f_B(x)] \rangle / x \in X\}$ Then

(i) $A \subseteq B$ if and only if $t_A(x) \le t_B(x)$ and $1 - f_A(x) \le 1 - f_B(x)$ for all $x \in X$

(ii) A=B if and only if $A \subseteq B$ and $B \subseteq A$

(iii) $A^c = \{\langle x, f_A(x), 1 - t_A(x) \rangle / x \in X\}$

(iv) $A \cap B = \{\langle x, \min(t_A(x), t_B(x)), \min(1 - f_A(x), 1 - f_B(x)) \rangle / x \in X\}$

(v) $A \cup B = \{\langle x, (t_A(x) \vee t_B(x)), (1 - f_A(x) \vee 1 - f_B(x)) \rangle / x \in X\}$ or the sake of simplicity, we shall use the notation $A = \langle x, t_A, 1 - f_A \rangle$ instead of $A = \{\langle x, [t_A(x), 1 - f_A(x)] \rangle / x \in X\}$.

Definition 2.3:[5] A vague matrix A of order $m \times n$ is defined as $A = [x, [< t_{ij}, 1 - f_{ij} >]]_{m \times n}$ the function $t_{ij} : X \to [0,1]$ and $f_{ij} : X \to [0,1]$ define the degree of truth membership function and the degree of false membership function of x_{ij} in A respectively satisfying the condition $0 \le t_{ij} + f_{ij} \le 1$ for all i,j. The value of $\pi_{Aij}(x) = 1 - (t_{Aij}(x) + f_{Aij}(x))$ is called the vague hesitation degree of the element $x \in X$ respectively to the vague matrix A. For simplicity, we write $A = [a_{ij}]_{m \times n}$ where $a_{ij} = [a_{ij(t)}, a_{ij(i-f)}]$.

III. VAGUE MATRICES

In arithmetic operation, only the values of t_{ij} and $1 - f_{ij}$ are considered so from here we only consider the values of $a_{ij} = [a_{ij(t)}, a_{ij(i-f)}]$. To study several properties of vague matrices, we define two algebraic operations, such as component wise addition and multiplication of vague matrices as follows:

Definition 3.1: Let a and b be two elements of a vague matrix X such that $a = [a_{ij(t)}, a_{ij(i-f)}]$ and $b = [b_{ij(t)}, b_{ij(i-f)}]$ then component wise addition and multiplication are defined as,

$a + b = < \max\{a_{ij(t)}, b_{ij(t)}\}, \max\{a_{ij(1-f)}, b_{ij(1-f)}\} >$

and $a * b = < \min\{a_{ij(t)}, b_{ij(t)}\}, \min\{a_{ij(1-f)}, b_{ij(1-f)}\} >$

Definition 3.2: Let $A = [< a_{ij(t)}, a_{ij(1-f)} >] \in F_{m \times n}$ and $B = [< b_{ij(t)}, b_{ij(1-f)} >] \in F_{m \times n}$ then the matrix addition is given by $A + B = (< \max\{a_{ij(t)}, b_{ij(t)}\}, \max\{a_{ij(1-f)}, b_{ij(1-f)}\} >) \in F_{m \times n}$ and the product of $A = [< a_{ij(t)}, a_{ij(1-f)} >] \in F_{m \times p}$ and $B = [< b_{ij(t)}, b_{ij(1-f)} >] \in F_{p \times n}$ is given

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by $AB = (<\max_k\{\min\{a_{kj(t)}, b_{kj(t)}\}\}, \max_k\{\min\{a_{ik(1-f)}, b_{kj(1-f)}\}\}>)$, where k=1,2,…..p, i=1,2,….n.

Definition 3.3: A vague matrix is null if all elements of it are zero, that is all elements are [0,1] and it is denoted by $I_{[0,0]}$.

Definition 3.4: A vague identity matrix of order $n \times n$ is denoted by I_n and is defined by $<[\delta_{ij(t)}, \delta_{ij(1-f)}]>$ where,

$$I_n = \begin{cases} \delta_{ij(t)} = 1, \ \delta_{ij(1-f)} = 1 & \text{if} \quad i = j \\ \delta_{ij(t)} = 0, \ \delta_{ij(1-f)} = 0 & \text{if} \quad i \neq j \end{cases}$$

The universal matrix of order $m \times n$ is denoted by J and all the elements of it are <[1,1]>.

Definition 3.5: Let $A = [<a_{ij(t)}, a_{ij(1-f)}>] \in F_{m\times n}$ and $c \in V$ such that $0 \le c \le 1$, then the scalar multiplication is defined as $cA = [<\min\{c, a_{ij(t)}\}, \min\{c, a_{ij(1-f)}\}>] \in F_{m\times n}$.

Definition 3.6: Let $A, B \in F_{m\times n}$ such that $A = [<a_{ij(t)}, a_{ij(1-f)}>]$ and $B = [<b_{ij(t)}, b_{ij(1-f)}>]$, then we write $A \le B$ if, $a_{ij(t)} \le b_{ij(t)}$ and $a_{ij(1-f)} \le b_{ij(1-f)}$ for all i, j and we say that A is dominated by B or B dominates A. A and B are said to be comparable, if either $A \le B$ or $B \le A$.

Definition 3.7: The transpose A^T of a vague matrix $A = [a_{ij}]_{m\times n}$ is defined as $A^T = [a_{ji}]_{n\times m}$ where $a_{ji} = <[a_{ji(t)}, a_{ji(1-f)}]>$.

Definition 3.8: A square vague matrix is called vague permutation matrix (VPM), if every row and column contains exactly one [1,1] and all other entries are [0,0].

Definition 3.9: A vague matrix $A \in F_n$ is said to be invertible if and only if there exists another vague matrix $B \in F_n$ such that $AB = BA = I_n$.

Remark: Let P_n be the set of all vague matrices in F_n. If $A \in P_n$, then $AA^T = A^T A = I_n$. As the vague matrices are the only invertible matrices and then $P^{-1} = P^T$ where P is a vague permutation matrices.

IV. VAGUE VECTORS

Let V_n be the set of all n- tuples

$[<x_{1(t)}, x_{1(1-f)}>, <x_{2(t)}, x_{2(1-f)}>, ..., <x_{n(t)}, x_{n(1-f)}>]$

over <F>. An element of V_n is called a vague vector of dimension n, where $x_{i(t)}$ and $x_{i(1-f)}$ are the truth and false values of the component x_i.

The operations addition and multiplication are defined on V_n as follows: Let

$x = [<x_{1(t)}, x_{1(1-f)}>, <x_{2(t)}, x_{2(1-f)}>, ..., <x_{n(t)}, x_{n(1-f)}>]$ and $y = [<y_{1(t)}, y_{1(1-f)}>, <y_{2(t)}, y_{2(1-f)}>, ..., <y_{n(t)}, y_{n(1-f)}>]$ be two vague vectors in V_n, then

$$x + y = [<\max(x_{1(t)}, y_{1(t)}), \max(x_{1(1-f)}, y_{1(1-f)})>, <\max(x_{2(t)}, y_{2(t)}), \max(x_{2(1-f)}, y_{2(1-f)}) ....., <\max(x_{n(t)}, y_{n(t)}), \max(x_{n(1-f)}, y_{n(1-f)})>]$$

and

$$ax = [<\min(a, x_{1(t)}), \min(a, x_{1(1-f)})>, <\min(a, x_{2(t)}), \min(a, x_{2(1-f)})>] ....., <\min(a, x_{n(t)}), \max(a, x_{n(1-f)})>] \quad \textit{for} \ a \in [0,1]$$

This system V_n together with these operations of component wise addition and multiplication forms vague vectors space (VVS).

Definition 4.1: Let $S = \{a_1, a_2, ...a_p\}$ be a set of VVS of dimension n. The linear combination of elements of the set S is a finite sum $\sum_{i=1}^{p} c_i a_i$ where $a_i \in S$ and $c_i \in [0,1]$. The set of all linear combination of the elements of S is called the span S, denoted by $\langle S \rangle$.

An example of V_3 and its spanning set is given below.

Example 4.2: Let $S = \{a_1, a_2, a_3\}$ be a subset of V_3, where
$a_1 = [\langle 0.6,0.7\rangle, \langle 0.6,0.8\rangle, \langle 0.8,0.9\rangle]$
$a_2 = [\langle 0.5,0.7\rangle, \langle 0.5,0.9\rangle, \langle 0.4,0.8\rangle]$ and
$a_3 = [\langle 0.7,0.8\rangle, \langle 0.6,0.7\rangle, \langle 0.4,0.7\rangle]$. Then

$$\langle S \rangle = \{c_1[\langle 0.6,0.7\rangle, \langle 0.6,0.8\rangle, \langle 0.8,0.9\rangle] + c_2[\langle 0.5,0.7\rangle, \langle 0.5,0.9\rangle, \langle 0.4,0.8\rangle] + c_3[\langle 0.7,0.8\rangle, \langle 0.6,0.7\rangle, \langle 0.4,0.7\rangle]\}$$

Definition 4.3: Let W be a vague subspace of V_n and S be a subset of W such that the elements of S are independent. If every element of W can be expressed uniquely as a linear combination of the elements of S, then S is called a basis of W.

Definition 4.4: A basis B of a VVS W is a standard basis if and only if whenever $b_i = \sum_{j=1}^{n} a_{ij} b_j$ for $b_i, b_j \in B$ and $a_{ij} \in [0,1]$ then $a_{ii} b_i = b_i$.

Example 4.5: Let $S = \{a_1, a_2, a_3\}$ be a subset of V_3, where
$a_1 = [\langle 0.6,0.7\rangle, \langle 0.7,0.8\rangle, \langle 0.8,0.9\rangle]$
$a_2 = [\langle 0.5,0.7\rangle, \langle 0.6,0.8\rangle, \langle 0.7,0.8\rangle]$ and
$a_3 = [\langle 0.5,0.6\rangle, \langle 0.4,0.7\rangle, \langle 0.6,0.7\rangle]$. Here S is independent set, since $a_1 \neq c_1 a_1 + c_2 a_3, a_2 \neq c_3 a_1 + c_4 a_3$ *and* $a_3 \neq c_5 a_1 + c_6 a_2$. So

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$\{a_1, a_2, a_3\}$ is a basis for $\langle S \rangle$. Now this is a standard basis also, for $a_1 = c_{11}a_1 + c_{12}a_2 + c_{13}a_3$ holds if $c_{11} = 0.9, c_{12} = 0.8$ and $c_{13} = 0.7$. Also $a_1 = c_{11}a_1$. Similarly, for a_2 and a_3.

Definition 4.6: Let $A = [< a_{ij(t)}, a_{ij(1-f)} >] \in F_{m\times n}$ be a vague matrix. Then the element $< a_{ij(t)}, a_{ij(1-f)} >$ is the ij th entry of A. $A_{i*}(A_{*j})$ denotes the i th row (j th column) of A. The row space R(A) of A is the subspace of V_n generated by the rows $\{A_{i*}\}$ of A. The column space C(A) of A is the subspace of V_m generated by the columns $\{A_{*j}\}$ of A.

If each row of a vague matrix B can be expressed as a linear combiniation of the row of vague matrix A, then we write $R(B) \subseteq R(A)$.If $R(B) \subseteq R(A)$ and $R(A) \subseteq R(B)$, then we say that $R(A) = R(B)$.

Definition 4.7: A vague matrix $A \in F_{m\times n}$ is said to be regular if there exist another vague matrix, $X \in F_{m\times n}$ such that, $AXA = A$. In this case, X is called a generalized inverse (g-inverse) of A and it is denoted by A^-.

The g- inverse of a vague matrix is not unique that is, a vague matrix has many g-inverses. The set of all such g-inverses of A are denoted by A{1}.

Definition 4.8: For a vague matrix A of order $m \times n$ a vague matrix $G \in F_{n\times m}$ is said to be outer inverse of A , if $GAG = G$ and is denoted by A{2}.

G is said to be {1,2} inverse or semi inverse of A, if $AGA = A$ and $GAG = G$ and is denoted by A{1,2}.

The vague matrix G is said to be {1,3} inverse or least square g- inverse of A if, $AGA = A$ and $(AG)^T = AG$ and is denoted by A{1,3}.

Again G is said to be {1,4} inverse or minimum norm g-inverse of A if, $AGA = A$ and $(GA)^T = GA$ and is denoted by A{1,4}.

Theorem 4.9: Let $A, B \in F_{m\times n}$ be two vague matrices. If A is regular then,

(i) $R(B) \subseteq R(A)$ iff $B = BA^-A$ for each $A^- \in A\{1\}$.

(ii) $C(B) \subseteq C(A)$ iff $B = AA^-B$ for each $A^- \in A\{1\}$.

Proof: (i) Let $R(B) \subseteq R(A)$,then each row of B is a linear combination of the row of A. Hence $B_{i*} = \sum x_{ij}A_{j*}$, where $x_{ij} \in < F >$. That is, $B = XA$(for some $X \in F_m$) or $B = XAA^-A$ (since $A = AA^-A$) or $B = BA^-A$.

Conversely, if $B = BA^-A$, then $B = XAA^-A$(for some $X \in F_m$) or, $B = XA$ (since $A = AA^-A$). This implies that $R(B) \subseteq R(A)$.

(ii)Let $C(B) \subseteq C(A)$. Then $B = AY$(for some $Y \in F_n$) or $B = AA^-AY$.(as $A = AA^-A$) That is $B = AA^-B$.

Conversely, if $B = AA^-B$, then $B = AA^-AY$(for some $Y \in F_m$) or, $B = AY$ (as $A = AA^-A$). That is $C(B) \subseteq C(A)$.

Theorem 4.10: Let $A \in F_{m\times n}$ be a regular vague matrix and G be a g- inverse of A. Then,

(i) $G^T \in A^T\{1\}$.

(ii) If P and Q are vague permutation matrices, then $Q^TGP^T \in PAQ\{1\}$.

(iii) AG and GA are idempotent.

Proof: (i) Let G be a g- inverse of A. Then $AGA = A$ holds. Taking transpose on both sides, we get, $A^TG^TA^T = A^T$. This implies $G^T \in A^T\{1\}$.

(ii)Since P and Q are vague permutation matrices, P and Q are invertible and $P^{-1} = P^T$, $Q^{-1} = Q^T$. Now,

$PAQ(Q^TGP^T)PAQ = PA(QQ^T)G(P^TP)AQ = PAGAQ$(as $QQ^T = I, PP^T = I$)

$= PAQ$. (as $AGA = A$).

This implies $Q^TGP^T \in PAQ\{1\}$.

(iii) Again, $(AG)(AG) = (AGA)G = AG$(as $AGA = A$). Also $(GA)(GA) = (GAG)A = GA$(as $GAG = G$). Thus AG and GA are idempotent.

Theorem 4.11: Let A be a vague matrix, $Y, Z \in A\{1\}$ and $X = YAZ$. Then $X \in A\{1,2\}$, that is X is a semi- inverse of A.

Proof: Since $Y, Z \in A\{1\} \Rightarrow AYA = A$ and $AZA = A$. As $X = YAZ$ so, $AXA = A(YAZ)A = (AYA)ZA = AZA = A$. Also $XAX = (YAZ)A(YAZ) = Y(AZA)(YAZ)$ $= Y(AYA)Z = YAZ = X$. So X is a semi- inverse of the vague matrix A.

Theorem 4.12: If $A \in F_n$ be a symmetric and idempotent vague matrix then A itself a least square g- inverse.

Proof: Since A is symmetric, $A^T = A$ and A is idempotent, $A^2 = A$. Now $PA = A$ if $P = I_n$. Then $APA = AA = A^2 = A$. That is , $A \in A\{1\}$. Now $(AX)^T = X^TA^T = X^TA = A^TA$ (Taking X=A, as A itself a g- inverse) $= AA = AX$. This implies, $A \in A\{1,3\}$.

Theorem 4.13: If $A \in F_n$ be a symmetric and idempotent vague matrix then A itself a minimal norm g- inverse.

Proof: Here $A^T = A$ and $A^2 = A$. For $P = I_n, PA = A$. Then, $APA = AA = A^2 = A$. That is, $A \in A\{1\}$. Now $(XA)^T = A^TX^T = AX^T = AA^T$(Taking X=A, as A itself a g-inverse) $= AA = XA$. This implies, $A \in A\{1,4\}$.

Theorem 4.14: The set A {1,3} consists of all solutions for X of AX=AG, where G is a {1,3} inverse of A.

International Journal of Scientific Engineering and Science
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Proof: Since $G \in A\{1,3\}$, we have $AGA = A$ and $(AG)^T = AG$. For $X \in A\{1,3\}$, we have $AXA = A$ and $(AX)^T = AX$. Then,

$$AG = (AXA)G = (AX)(AG) = (AX)^T (AG)^T$$
$$= (X^T A^T)(G^T A^T) = X^T (A^T G^T A^T)$$
$$= X^T A^T = (AX)^T = AX$$. Hence X is a solution of AX=AG.

Conversely, let AG=AX with $G \in A\{1,3\}$. Then A=AGA implies A=AXA therefore $X \in A\{1\}$. Since $AG = AX \Rightarrow (AG)^T = (AX)^T \Rightarrow AG = (AX)^T \Rightarrow AX = (AX)^T$. That is. $X \in A\{3\}$. Therefore we have $X \in A\{1,3\}$.

Theorem 4.15: The set A{1,4} consists of all solutions for X of XA=GA, where G is a {1,4} inverse of A.
Proof: The proof is obvious.

V. MOORE- PENROSE INVERSE

Definition 5.1: For a vague matrix $A \in F_{m\times n}$, a vague matrix $G \in F_{m\times n}$ is said to be a Moore- Penrose inverse of A, if $AGA = A, GAG = G, (AG)^T = AG$ and $(GA)^T = GA$. The Moore-Penrose inverse of A is denoted by A^+.

Definition 5.2: Let A and B be two vague matrices of order $m \times n$. The minus ordering between A and B is denoted by $A \le^- B$. Then for some $A^- \in A\{1\}$ we say $A \le^- B$ if and only if $AA^- = BA^-$ and $A^-A = A^-B$.

Theorem 5.3: Let $A, B \in F_{m\times n}$ and A^+ exists, then the following are equivalent.
(i) $A \le^- B$,
(ii) $A^+A = A^+B$; $AA^+ = BA^+$,
(iii) $AA^+B = A = BA^+A$.

Proof: (i)$\Rightarrow$(ii) $A \le^- B$ implies $AA^- = BA^-$ and $A^-A = A^-B$ for some $A^- \in A\{1\}$. Now $A = AA^-A = AA^-B$ as $A^- \in A\{1\}$. So $A^+A = A^+AA^+B = A^+B$. Similarly, $AA^+ = BA^+$.
(ii)$\Rightarrow$(iii) $A^+A = A^+B$. This gives $A = AA^+A = AA^+B$. Also, from $AA^+ = BA^+, A = AA^+A = BA^+A$. Thus $A = AA^+B = BA^+A$.
(iii)$\Rightarrow$(i) Let $X = A^+AA^+$. Then $AXA = A(A^+AA^+)A = (AA^+A)A^+A = AA^+A = A$. Thus, X is a g- inverse of A. Now, $XA = (A^+AA^+)AA^+B = A^+(AA^+A)A^+B = (A^+AA^+)B = XB$. Similarly, $AX = BX$. Hence $A \le^- B$ for $X \in A\{1\}$.

Theorem 5.3: The following conditions are equivalent for the vague matrices A and B
(i) $A \le^- B$,
(ii) $A = AA^-B = BA^-A = BA^-B$.

Proof: (i)$\Rightarrow$(ii) $A \le^- B$ implies $AA^- = BA^-$ and $A^-A = A^-B$ for some $A^- \in A\{1\}$. Now
$A = A(A^-A) = AA^-B = (AA^-)A = BA^-A = B(A^-A) = BA^-B$.
(ii)$\Rightarrow$(iii) Let $X = A^-AA^-$ then $AXA = A(A^-AA^-)A = (AA^-A)A^-A = A$. This implies that X is a g- inverse. Now
$XA = (A^-AA^-)AA^-B = A^-(AA^-A)A^-B = (AA^-A)B = XB$.
Similarly, AX=BX. Hence $A \le^- B$ for $X \in A\{1\}$.

Theorem 5.4: Let A and B be two vague matrices. If $A \le^- B$ then $B\{1\} \subseteq A\{1\}$.

Proof: Here $A \le^- B \Rightarrow A = AA^-B = BA^-A = BA^-B$ (by the above theorem). For $B^- \in B\{1\}$.

$$AB^-A = (AA^-B)B^-(BA^-A) = AA^-(BB^-B)A^-A$$
$$= (AA^-B)A^-A = AA^-A = A$$.

Hence, $AB^-A = A$ for each $B^- \in B\{1\}$. Therefore, $B\{1\} \subseteq A\{1\}$.

Theorem 5.5: Let $A, B \in F_{m\times n}$ and A^+ exists, then the following are equivalent.
(i) $A \le^- B$,
(ii) $A^+A = A^+B$; $AA^+ = BA^+$,
(iii) $AA^+B = A = BA^+A$.

Proof: (i)$\Rightarrow$(ii) $A \le^- B$ implies $AA^- = BA^-$ and $A^-A = A^-B$ for some $A^- \in A\{1\}$. Now $A = AA^-A = AA^-B$ as $A^- \in A\{1\}$. So $A^+A = A^+AA^+B = A^+B$. Similarly, $AA^+ = BA^+$.
(ii)$\Rightarrow$(iii) $A^+A = A^+B$. This gives $A = AA^+A = AA^+B$. Also, from $AA^+ = BA^+, A = AA^+A = BA^+A$. Thus $A = AA^+B = BA^+A$.
(iii)$\Rightarrow$(i) Let $X = A^+AA^+$. Then
$AXA = A(A^+AA^+)A = (AA^+A)A^+A = AA^+A = A$. Thus, X is a g- inverse of A. Now,
$XA = (A^+AA^+)AA^+B = A^+(AA^+A)A^+B = (A^+AA^+)B = XB$.
Similarly, $AX = BX$. Hence $A \le^- B$ for $X \in A\{1\}$.

Theorem 5.6: If $A \le^- B$ and B is idempotent then B is a g- inverse of A. Also, if A^+ exist then B will be a g- inverse of A^+.

Proof: Since B is idempotent then B is regular and B itself is a g- inverse of B. Here $B \in B\{1\}$. Now $A \le^- B$ implies $A = AA^-B = BA^-A$. So,
$AB^-A = (AA^-B)B^-(BA^-A) = (AA^-B)A^-A = AA^-A = A$, for each $B^- \in B\{1\}$. This implies that $B\{1\} \subseteq A\{1\}$. Hence B is a g- inverse of A. Now if A^+ exist then, $A^+ = A^+AA^+ = A^+BA^+$ (since $A^+A = A^+B$). Hence $B \in A^+\{1\}$. Hence B is a g- inverse of A^+.

VI. AN APPLICATION OF G- INVERSE

Algorithm: (To find the g- inverse of vague matrix)

Step 1: Check whether the non- zero rows of a vague matrix A form a standard basis or not for the row space of A

Step 2: If the non- zero rows form a standard basis then find some vague permutation matrix P , such that APA=A.

Step 3: Choose a vague matrix R such that RA=A.

Volume 2, Issue 7, pp. 4-8, 2018. ISSN (Online): 2456-7361

Step 4: Then PR is a g- inverse of A.
The matrix PR is a g- inverse of A since A(PR)A=AP(RA)=APA=A.

Case Study:

Thyroid disorders are conditions that affects the thyroid gland, a butterfly- shaped gland in the front of the neck. The thyroid has important roles to regulate numerous metabolic processes throughout the body. Different types of thyroid disorders affect either its structure or function. Here we consider three types of thyroid disorders for our case study they are type 1 thyroid (Hypothyroidism d_1), type 2 thyroid (Hyperthyroidism d_2) and type 3 thyroid(Multinodular goiter d_3). The symptoms related to these disease are s_1 (fatigue, insomnia, weight and metabolism), s_2 (Pain, hair change, skin change) and s_3 (Mood change, eye change, voice change).

Step 1: Let us consider the vague matrix

$$A = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{bmatrix} [0.5,0.6] & [0.7,0.8] & [0.6,0.7] \\ [0.5,0.7] & [0.6,0.8] & [0.7,0.8] \\ [0.4,0.6] & [0.4,0.7] & [0.8,0.9] \end{bmatrix} \end{array}$$

Step 2: The rows of A are independent and they form a standard basis. Since $R_i = \sum a_{ij} R_j \in R$ for $R_j \in R$ (row space of A) $a_{ij} \in [0,1]$ and $a_{ij} R_i = R_i, i = 1,2,3$. Consider the vague permutation matrix,

$$P = \begin{bmatrix} [0,0] & [1,1] & [0,0] \\ [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [1,1] \end{bmatrix}$$

Now

$$APA = \begin{bmatrix} [0.5,0.6] & [0.7,0.8] & [0.6,0.7] \\ [0.5,0.7] & [0.6,0.8] & [0.7,0.8] \\ [0.4,0.6] & [0.4,0.7] & [0.8,0.9] \end{bmatrix} = A$$

Step 3: Now R is the relation between the patient and the type of thyroid.

$$R = \begin{bmatrix} [0.7,0.8] & [0.5,0.6] & [0.5,0.7] \\ [0.4,0.5] & [0.8,0.9] & [0.6,0.7] \\ [0.3,0.6] & [0.4,0.6] & [0.8,0.9] \end{bmatrix}$$

$$RA = \begin{bmatrix} [0.7,0.8] & [0.5,0.6] & [0.5,0.7] \\ [0.4,0.5] & [0.8,0.9] & [0.6,0.7] \\ [0.3,0.6] & [0.4,0.6] & [0.8,0.9] \end{bmatrix} \begin{bmatrix} [0.5,0.6] & [0.7,0.8] & [0.6,0.7] \\ [0.5,0.7] & [0.6,0.8] & [0.7,0.8] \\ [0.4,0.6] & [0.4,0.7] & [0.8,0.9] \end{bmatrix}$$

$$RA = \begin{bmatrix} [0.5,0.6] & [0.7,0.8] & [0.6,0.7] \\ [0.5,0.7] & [0.6,0.8] & [0.7,0.8] \\ [0.4,0.6] & [0.4,0.7] & [0.8,0.9] \end{bmatrix} = A$$

$\therefore RA = A$ *holds*

Step 4: So the g- inverse of A is ,

$$PR = \begin{bmatrix} [0,0] & [1,1] & [0,0] \\ [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [1,1] \end{bmatrix} \begin{bmatrix} [0.7,0.8] & [0.5,0.6] & [0.5,0.7] \\ [0.4,0.5] & [0.8,0.9] & [0.6,0.7] \\ [0.3,0.6] & [0.4,0.6] & [0.8,0.9] \end{bmatrix}$$

$$PR = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{bmatrix} [0.4,0.5] & [0.8,0.9] & [0.6,0.7] \\ [0.7,0.8] & [0.5,0.6] & [0.5,0.7] \\ [0.3,0.6] & [0.4,0.6] & [0.8,0.9] \end{bmatrix} \end{array} = X$$

which satisfies the relation AXA=A.

VII. Conclusion

In this case study we get a clear idea that patient P1 is suffering from type 2 thyroid, patient P2 is suffering from type 1 thyroid and patient P3 is suffering from type 3 thyroid.

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