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Generalized Stability of Cubic Functional Equation

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Abstract— *In this paper, we introduced the generalized Hyers-Ulam Stability of Cubic functional equation of the form*

$$g\left(nx+n^2y+2n^3z\right)+g\left(nx+n^2y-n^3z\right)=2g\left(nx+n^2y\right)+4\left[g(nx-n^3z)+g(nx-n^3z)+g(n^2y+n^3z)+g(n^2y-n^3z)\right]-8n^3g(x)-8n^6g(y)$$

Where $n\in\mathbb{N}$ *.Further, we have investigated the general solution and generalized Hyers-Ulam for the afforested defined functional equation using direct and fixed point method in Banach space consequence with the above given said in detail.*

MSC: 39B79, 32B72, 32B82

***Keywords*—** *Cubic functional equation, Banach space, fixed point*.

I. Introduction

In 1940, S.M.Ulam [22] proposed the following question concerning the stability of group homomorphism:

Let G_1 be a group and let G_2 be a metric group with the $d(.,.)$. Given $\varepsilon>0$, does there exists a $\delta>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d\left(h(x,y),h(x)*h(y)\right)<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d\left(h(x),H(x)\right)<\in$ for all $x\in G_1$ in other words, under what condition does there exists a homomorphism near an approximate homomorphism the concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D.H.Hyers [9] gave a first affirmative answer to the question of Ulam [22] for Banach spaces.

In this paper, the authors investigate generalized Hyers-Ulam Stability of Cubic functional equation of the form

$$g\left(nx+n^2y+2n^3z\right)+g\left(nx+n^2y-n^3z\right)=2g\left(nx+n^2y\right)+4\left[g(nx-n^3z)+g(nx-n^3z)+g(n^2y+n^3z)+g(n^2y-n^3z)\right] -8n^3g(x)-8n^6g(y) \tag{1.0}$$

K.W.Jun and H.M.Kim [23] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2\left[f(x+y)+f(x-y)\right]+12f(x) \tag{1.1}$$

$x,y\in X$, where f is a mapping from a real vector space X into a real vector space Y. They established the general solution and the generalized Hyers-Ulam-Rassias [14,15,16,17,18,19] stability for the functional equation (1.1), the function $f(x)=x^3$ satisfies the functional equation (1.1) which is the thus called a cubic functional equation. Every solution of a functional equation is said to be a cubic function. Jun and Kim proved that the mapping f from real vector space X and Y is solution of (1.11) if and only if there exists a unique function $C:X\times X\times X\to Y$ such that $f(x)=C(x,x,x)$ for all $x\in X$ and C is symmetric for one variable an additive for fixed two variables.

II. Stability of (1.0)

For completeness we will first investigate solutions of the functional equation (1.1) Let x and y be a real vector space.

Theorem 2.1. A function $g:x\to y$ satisfies the functional equation (1.0) if and only if $g:x\to y$ also satisfies the functional equation (1.1)

Proof: substituting (x,y) by (0,0) in (1.0) yields g(0)=0. Putting (x,y) in (1.0) that

$$g(-x)=-g(x) \tag{2.1}$$

for all $x\in X$, which implies that g is odd. Again setting (x,y) by (x,0) in (1,0), we obtain $g(2x)=8g(x)$ and replacing (x,y) by (x,x) in 1.0, that $g(3x)=27g(x)$ for all $x\in X$. We substitute (x,y) by $(nx,nx+n^2y)$ in (1.0), that

$$g\left(2nx+nx+n^2y\right)+g\left(2nx-(nx+n^2y)\right)=2g\left(nx+nx+n^2y\right)+2g\left(nx-(nx+n^2y)\right)+12f\left(nx\right)$$

$$g\left(3nx+n^2y\right)+g\left(nx-n^2y)\right)=2g\left(2nx+n^2y\right)+2g\left(-n^2y)\right)+12g\left(nx\right)$$

$$g\left(3nx+n^2y\right)+g\left(nx-n^2y\right)=2g\left(2nx+n^2y\right)-2g\left(n^2y\right)+12g\left(nx\right) \tag{2.2}$$

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for all $x, y \in X$. Again replacing (x,y) by $\left(nx,nx-n^2y\right)$ in (1.0), we get

$$g\left(2nx+nx-n^2y\right)+g\left(2nx-(nx-n^2y)\right)=2g\left(nx+nx-n^2y\right)+2g\left(nx-(nx-n^2y)\right)+12f\left(nx\right)$$

$$g\left(3nx-n^2y\right)+g\left(nx-n^2y\right)=2g\left(2nx-n^2y\right)+2g\left(-n^2y\right)+12f\left(nx\right) \quad (2.3)$$

for all $x, y \in X$. Adding (2.2) and (2.3) and than using (1.0), we see that

$$g\left(3nx+n^2y\right)+g\left(nx-n^2y\right)+g\left(3nx-n^2y\right)+g\left(nx+n^2y\right)=2g\left(2nx+n^2y\right)-2g\left(n^2y\right)+12g\left(nx\right)$$
$$+2g\left(2nx-n^2y\right)+2g\left(n^2y\right)+12f\left(nx\right)$$

$$g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)+g\left(nx+n^2y\right)+g\left(nx-n^2y\right)=2g\left(2nx+n^2y\right)-2g\left(2nx-n^2y\right)$$

$$g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)+g\left(nx+n^2y\right)+g\left(nx-n^2y\right)=\left[2g\left(2nx+n^2y\right)+2g\left(nx-n^2y\right)+12f\left(nx\right)\right]+24f(nx)$$

$$g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)+g\left(nx+n^2y\right)+g\left(nx-n^2y\right)=4g\left(nx+n^2y\right)+4g\left(nx-n^2y\right)+24f\left(nx\right)+24f(nx)$$

$$g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)=4g\left(nx+n^2y\right)+4g\left(nx-n^2y\right)-g\left(nx+n^2y\right)-g\left(nx-n^2y\right)+48f\left(nx\right)$$

$$g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)=3g\left(nx+n^2y\right)+3g\left(nx-n^2y\right)+48f\left(nx\right) \quad (2.4)$$

for all $x, y \in X$. Now putting (x,y) by (x+y, x-y) in (2.4) respectively, we have

$$g\left(3(nx+n^2y)+nx-n^2y\right)+g\left(3(nx-n^2y)-(nx-n^2y)\right)=3g\left(nx+n^2y+nx-n^2y\right)+3g\left(nx-n^2y-(nx-n^2y)\right)+48g\left(nx+n^2y\right)$$

$$g\left(3nx+3n^2y+nx-n^2y\right)+g\left(3nx-3n^2y-nx-n^2y\right)=3g\left(2nx\right)+3g\left(2n^2y\right)+48g\left(nx+n^2y\right)$$

$$g\left(4nx+2n^2y\right)+g\left(2nx+4n^2y\right)=3g\left(2nx\right)+3g\left(2n^2y\right)+48g\left(nx+n^2y\right)$$

for all $x, y \in X$. Which in view of the identity $g(2x)=8g(x)$, reduces to

$$g\left(4nx+2n^2y\right)+g\left(2nx+4n^2y\right)=3g.8g\left(nx\right)+3g.8g\left(n^2y\right)+48g\left(nx+n^2y\right)$$

$$8g\left(2nx+n^2y\right)+8g\left(nx+2n^2y\right)=24g\left(nx\right)+24g\left(n^2y\right)+48g\left(nx+n^2y\right)$$

for all $x, y \in X$ .Using the resultant of this result ,we lead to

$$g\left(2nx+n^2y\right)+g\left(nx+2n^2y\right)=3g\left(nx\right)+3g\left(n^2y\right)+6g\left(nx+n^2y\right) \quad (2.5)$$

for all $x, y \in X$. Replacing (x,y) by $\left(nx+3n^2y,nx-3n^2y\right)$ in (2.5), we arrive

$$g\left(2(nx+3n^2y),nx-3n^2y\right)+g\left(nx+3n^2y+2(nx-3n^2y)\right)=3g\left(nx+3n^2y\right)+nx-3n^2y+3g\left(nx-3n^2y\right)+6g\left(nx+3n^2y\right)$$

$$g\left(2nx+6n^2y+nx-3n^2y\right)+g\left(nx+3n^2y+2nx-6n^2y\right)=3g\left(nx+3n^2y\right)+3g\left(nx-3n^2y\right)+6g\left(nx\right)$$

$$27g\left(nx+n^2y\right)+27g\left(nx-n^2y\right)=3g\left(nx+3n^2y\right)+3g\left(nx-3n^2y\right)+48g\left(nx\right)$$

for all $x, y \in X$ .Using the identity of this result ,rearranging we arrive that

$$9g\left(nx+n^2y\right)+9g\left(nx-n^2y\right)=g\left(nx+3n^2y\right)+g\left(nx-3n^2y\right)+16g\left(nx\right) \quad (2.6)$$

for all $x, y \in X$. Let us interchange nx in n^2y and n^2y in nx in (2.6) to get the identity, then

$$9g\left(nx+n^2y\right)+9g\left(n^2y-nx\right)=g\left(3nx+n^2y\right)+g\left(n^2y-3nx\right)+16g\left(n^2y\right)$$

$$9g\left(nx+n^2y\right)-9g\left(nx-n^2y\right)=g\left(3nx+n^2y\right)-g\left(3nx-n^2y\right)+16g\left(n^2y\right) \quad (2.7)$$

for all $x, y \in X$. Then, by adding (2.6) and (2.7), we lead to

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$$9g\left(nx+n^2y\right)+9g\left(nx-n^2y\right)+9g\left(nx+n^2y\right)-9g\left(nx-n^2y\right)=g\left(nx+3n^2y\right)+g\left(nx-3n^2y\right)+16g\left(nx\right)$$

$$+g\left(3nx+n^2y\right)-g\left(3nx-n^2y\right)+16g\left(n^2y\right)$$

$$18g\left(nx+n^2y\right)=g\left(nx+3n^2y\right)+g\left(nx-3n^2y\right)+g\left(3nx+n^2y\right)-g\left(3nx-n^2y\right)+16g\left(nx\right)+16g\left(n^2y\right) \tag{2.8}$$

for all $x, y \in X$. Now we interchange nx in n^2y and n^2y in nx in (2.4), respectively we get

$$g\left(3n^2y+nx\right)+g\left(3n^2y-nx\right)=3g\left(3n^2y+nx\right)+3g\left(n^2y-nx\right)+49g\left(nx\right)$$

$$g\left(nx+3n^2y\right)-g\left(nx-3n^2y\right)=3g\left(nx+n^2y\right)-3g\left(nx-n^2y\right)+49g\left(n^2y\right) \tag{2.9}$$

for all $x, y \in X$. Hence according to (2.4) and (2.9), we arrive that

$$g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)=3g\left(nx+n^2y\right)+3g\left(nx-n^2y\right)+48g\left(nx\right)$$

$$g\left(nx+3n^2y\right)-g\left(nx-3n^2y\right)=3g\left(nx+n^2y\right)-3g\left(nx-n^2y\right)+48g\left(n^2y\right)$$

$$g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)+g\left(nx+3n^2y\right)-g\left(nx-3n^2y\right)=6g\left(nx+n^2y\right)+48g\left(nx\right)+48g\left(n^2y\right)$$

$$6g\left(nx+n^2y\right)=g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)+g\left(nx+3n^2y\right)-g\left(nx-3n^2y\right)-48g\left(nx\right)-48g\left(n^2y\right) \tag{2.10}$$

for all $x, y \in X$. Again by adding (2.8) and (2.10), we get

$$18g\left(nx+n^2y\right)=g\left(nx+3n^2y\right)+g\left(nx-3n^2y\right)+g\left(3nx+n^2y\right)-g\left(3nx-n^2y\right)+16g\left(nx\right)+16g\left(n^2y\right)$$

$$6g\left(nx+n^2y\right)=g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)+g\left(nx+3n^2y\right)-g\left(nx-3n^2y\right)-48g\left(nx\right)-48g\left(n^2y\right)$$

$$24g\left(nx+n^2y\right)=2g\left(nx+3n^2y\right)+2g\left(3nx+n^2y\right)-32g\left(nx\right)-32g\left(n^2y\right)$$

$$12g\left(nx+n^2y\right)=g\left(nx+3n^2y\right)+g\left(3nx+n^2y\right)-16g\left(nx\right)-16g\left(n^2y\right)$$

$$g\left(nx+3n^2y\right)+g\left(3nx+n^2y\right)=12g\left(nx+n^2y\right)+16g\left(nx\right)+16g\left(n^2y\right) \tag{2.11}$$

for all $x, y \in X$. Taking (2.4), we arrive that

$$g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)=3g\left(nx+n^2y\right)+3g\left(nx-n^2y\right)+48g\left(nx\right)$$

$$g\left(3nx+n^3z\right)+g\left(3nx-n^3z\right)=3g\left(nx+n^3z\right)+3g\left(nx-n^3z\right)+48g\left(nx\right)$$

$$g\left(3n^2y+n^3z\right)+g\left(3n^2y-n^3z\right)=3g\left(n^2y+n^3z\right)+3g\left(n^2y-n^3z\right)+48g\left(n^2y\right)$$

$$g\left(3nz+n^3z\right)+g\left(3nx-n^3z\right)+g\left(3n^2y+n^3z\right)+g\left(3n^2y-n^3z\right)=3g\left(nx+n^3z\right)+3g\left(n^2y+n^3z\right)+3g\left(n^2y-n^3z\right)+48\left(nx\right)+48g\left(n^2y\right)$$

$$16g\left(3nz+n^3z\right)+16g\left(3nx-n^3z\right)+16g\left(3n^2y+n^3z\right)+16g\left(3n^2y-n^3z\right)=48g\left(nx+n^3z\right)+48g\left(nx-n^3z\right)$$

$$+48g\left(n^2y-n^3z\right)+48\left(n^2y-n^3z\right)+768\left(nx\right)+768g\left(n^2y\right)$$

for all $x, y \in X$. Also setting $\left(nx,n^2y\right)$ by $\left(3nx+n^3z,3n^2y+n^3z\right)$ in (2.11), respectively we

$$g\left(nx+3n^2y\right)+g\left(3nx+n^2y\right)=12g\left(nx+n^2y\right)+16g\left(nx\right)+16g\left(n^2y\right)$$

$$g\left(3nx+n^3z+3(3n^3y+n^3z)\right)+g\left(3(3nx+n^3z)+3n^2y+n^3z\right)=12g\left(3nx+n^3z+3n^2y+n^3z\right)+16g\left(3nx+n^3z\right)+16g\left(3n^2y+n^3z\right)$$

$$g\left(3nx+n^3z+9n^2y+3n^3z\right)+g\left(9nx+3n^3z+3n^2y+n^3z\right)=12g\left(3nx+n^3z+3n^2y+n^3z\right)+16g\left(3nx+n^3z\right)+16g\left(3n^2y+n^3z\right)$$

$$g\left(3nx+9n^2y+4n^3z\right)+g\left(9nx+3n^2y+4n^3z\right)=12g\left(3nx+3n^2y+2n^3z\right)+16g\left(3nx+n^3z\right)+16g\left(3n^2y+n^3z\right) \tag{2.13}$$

for all $x, y \in X$. Replacing $\left(nx,n^2y\right)$ by $\left(3nx-n^3z,3n^2y-n^3z\right)$ in (2.11) we obtain

$$g\left(3nx-n^3z+3(3n^2y-n^3z)\right)+g\left(3(3nx-n^3z)+3n^2y-n^3z\right)=12g\left(3nx-n^3z\right)+16g\left(3nx-n^3z\right)+16g\left(3n^2y-n^3z\right)$$
$$g\left(3nx-n^3z+9n^2y-3n^3z)\right)+g\left(9nx-3n^3z+3n^2y-n^3z\right)=12g\left(3nx-n^3z+3n^2y-n^3z\right)+16g\left(3nx-n^3z\right)+16g\left(3n^2y-n^3z\right)$$
$$g\left(3nx+9n^2y-4n^3z)\right)+g\left(9nx+3n^2y-4n^3z\right)=12g\left(3nx+3n^2y-2n^3z\right)+16g\left(3nx-n^3z\right)+16g\left(3n^2y-n^3z\right)$$

for all $x, y \in X$. Using (2.13) and (2.14), we get the identity that

$$g\left(3nx+9n^2y+4n^3z)\right)+g\left(9nx+3n^2y+4n^3z\right)+g\left(3nx+9n^2y-4n^3z\right)+g\left(9nx+3n^2y-4n^3z\right)$$
$$=12g\left(3nx+3n^2y-2n^3z\right)+16g\left(3nz+n^3z\right)+16g\left(3n^2y+n^3z\right)+12g\left(3nz+3n^2y-2n^3z\right)+g\left(3nx+9n^2y+4n^3z)\right)$$
$$+g\left(9nx+3n^2y+4n^3z\right)+g\left(3nx+9n^2y-4n^3z\right)+16g\left(3nz-n^3z\right)+16g\left(3n^2y-n^3z\right)$$
$$g\left(3nx+9n^2y+4n^3z)\right)+g\left(9nx+3n^2y+4n^3z\right)+g\left(3nx+9n^2y-4n^3z\right)+g\left(9nx+3n^2y-4n^3z\right)$$
$$-12g\left(3nx+3n^2y-2n^3z\right)-12g\left(3nx+3n^2y-2n^3z\right)=16g\left(3nx+n^3z\right)+16g\left(3n^2y+n^3z\right)+16g\left(3n^2y-n^3z\right)$$

for all $x, y \in X$. Using (2.4) that

$$g\left(3(3nx+n^2y)\right)+g\left(3nx-n^2y\right)=3g\left(nx+n^2y\right)+3g\left(nx-n^2y\right)+48g\left(nx\right)$$
$$g\left(3(3nx+3n^2y)+4n^3z\right)+g\left(3(3nx+3n^2y)-4n^3z\right)=3g\left(nx+3n^2y+4n^3z\right)+3g\left(nx+3n^2y-4n^3z\right)+48g\left(nx+4n^2y\right)$$
$$g\left(3(3nx+3n^2y)+4n^3z\right)+g\left(3(3nx+3n^2y)-4n^3z\right)=3g\left(nx+3n^2y+4n^3z\right)+3g\left(nx+3n^2y-4n^3z\right)+48g\left(nx+3n^2y\right)$$
$$g\left(3nx+9n^2y+4n^3z\right)+g\left(3nx+9n^2y-4n^3z\right)=3g\left(nx+3n^2y+4n^3z\right)+3g\left(nx+3n^2y-4n^3z\right)+48g\left(nx+3n^2y\right) \tag{2.16}$$

for all $x, y \in X$. Again using (2.4) respectively, that

$$g\left(3(3nx+3n^2y)+4n^3z\right)+g\left(3(3nx+3n^2y)-4n^3z\right)=3g\left(3nx+n^2y+4n^3z\right)+3g\left(3nx+n^2y-4n^3z\right)+48g\left(3nx+n^2y\right)$$
$$g\left(9nx+3n^2y+4n^3z\right)+g\left(9nx+3n^2y-4n^3z\right)=3g\left(3nx+n^2y+4n^3z\right)+3g\left(3nx+n^2y-4n^3z\right)+48g\left(3nx+n^2y\right) \tag{2.17}$$

for all $x, y \in X$. Adding (2.16) and (2.17), we obtain

$$g\left(3nx+9n^2y+4n^3z\right)+g\left(3nx+9n^2y-4n^3z\right)+g\left(9nx+3n^2y+4n^3z\right)+g\left(9nx+3n^2y-4n^3z\right)=3g\left(nx+3n^2y+4n^3z\right)$$
$$+3g\left(nx+3n^2y-4n^3z\right)+48g\left(3nx+3n^2y\right)$$
$$=3g\left(nx+3n^2y+4n^3z\right)+3g\left(nx+3n^2y-4n^3z\right)+48g\left(3nx+3n^2y\right)+3g\left(3nx+n^2y+4n^3z\right)+3g\left(3nx+n^2y-4n^3z\right)+48g\left(3nx+n^2y\right)$$

for all $x, y \in X$. Then applying (2.18) in (2.15), we arrive

$$\begin{aligned}16g\left(3nx+n^3z\right)+16g\left(3n^2y+n^3z\right)+16g\left(3nx-cz\right)+16g\left(3n^2y-n^3z\right)=3g\left(nx+3n^2y+4n^3z\right)\\+3g\left(nx+3n^2y-4n^3z\right)+48g\left(3nx+3n^2y\right)+3g\left(3nx+n^2y+4n^3z\right)+3g\left(3nx+n^2y-4n^3z\right)\\+48g\left(3nx+n^2y\right)-12g\left(3nx+3n^2y-2n^3z\right)-12g\left(3nx+3n^2y-2n^3z\right)\end{aligned} \tag{2.18}$$

for all $x, y \in X$. From (2.4), we obtain

$$g\left(3nx+n^2y\right)+g\left(3n^2x-n^3y\right)=3g\left(nx+n^2y\right)+3g\left(nx-n^2y\right)+48g\left(nx\right)$$
$$g\left(3(nx+n^2y)+2n^3z\right)+g\left(3(nx+n^2y)-2n^3z\right)=3g\left(nx+n^2y+2n^3z\right)+3g\left(nx+n^2y-2n^3z\right)+48g\left(nx+n^2y\right)$$
$$g\left(3nx+3n^2y+2n^3z\right)+g\left(3nx+3n^2y-2n^3z\right)=3g\left(nx+n^2y+2n^3z\right)+3g\left(nx+n^2y-2n^3z\right)+48g\left(nx+n^2y\right) \tag{2.19}$$

for all $x, y \in X$. Using (2.19) in (2.18), that

$$16g\left(3nx+n^3z\right)+16g\left(3n^2y+n^3z\right)+16g\left(3nx-n^3z\right)+16g\left(3n^2y-n^3z\right)=3g\left(nx+3n^2y+4n^3z\right)$$

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$$+3g\left(nx+n^2y-4n^3z\right)+48g\left(nx+3n^2y\right)+3g\left(3nx+n^2y+4n^3z\right)$$
$$+3g\left(3nx+n^2y-4n^3z\right)+48g\left(3nx+n^2y\right)-12\left[3g\left(nx+n^2y+2n^3z\right)-3g\left(nx+n^2y-2n^3z\right)+48g\left(nx+n^2y\right)\right]$$

$$16g\left(3nx+n^3z\right)+16g\left(3n^2y+n^3z\right)+16g\left(3nx-n^3z\right)+16g\left(3n^2y-n^3z\right)=3g\left(nx+3n^2y+4n^3z\right)$$
$$+3g\left(nx+3n^2y-4n^3z\right)+48g\left(nx+3n^2y\right)+3g\left(3nx+n^2y+4n^3z\right)+3g\left(3nx+n^2y-4n^3z\right)+48g\left(3nx+n^2y\right) \quad (2.20)$$
$$+36g\left[nx+n^2y+2n^3z-36g\left(nx+n^2y-2n^3z\right)-576g\left(nx+n^2y\right)\right]$$

for all $x, y \in X$. Yields, by modifying of (2.12), the relation

$$3g\left(nx+3n^2y+4n^3z\right)+3g\left(nx+3n^2y-4n^3z\right)+48g\left(nx+3n^2y\right)+3g\left(3nx+n^2y+4n^3z\right)=3g\left(3nx+n^2y-4n^3z\right)$$
$$+48g\left(3nx+n^2y\right)-36g\left(nx+n^2y+2n^3z\right)-36g\left(nx+n^2y-2n^3z\right)$$
$$-576g\left(nx+n^2y\right)=48g\left(nx+n^3z\right)+48g\left(nx-n^3z\right)+768(nx)+48g\left(n^2y-n^3z\right)+48g\left(n^2y-n^3z\right)+768g(n^2y)$$

$$3g\left(nx+3n^2y+4n^3z\right)+3g\left(nx+3n^2y-4n^3z\right)+48g\left(nx+3n^2y\right)+3g\left(3nx+n^2y+4n^3z\right)+3g\left(3nx+n^2y-4n^3z\right)+48g\left(3nx+n^2y\right)$$
$$3g\left(nx+3n^2y+4n^3z\right)+3g\left(nx+3n^2y-4n^3z\right)=48g\left(3nx+n^2y\right)+48g\left(nx-n^3z\right)+768g\left(nx\right)+48g\left(n^2y+n^3z\right) \quad (2.21)$$
$$+48g\left(n^2y-n^3z\right)+768g\left(n^2y\right)+36g\left(nx+n^2y+2n^3z\right)+36\left(nx+n^2y-2n^3z\right)+576g\left(nx+n^2y\right)$$

for all $x, y \in X$. The concept of (2.11) and (2.14). the left side of (2.12) can be written in the form

From (2.4)

$$g\left(3nx+n^2y\right)+g\left(3nx-n^2y\right)=3g\left(nx+n^2y\right)+3g\left(nx-n^2y\right)+48g\left(nx\right)$$
$$g\left(3(3nx+n^2y)+2n^3z\right)+g\left(3(3nx+n^2y)-2n^3z)\right)=3g\left(3nx+n^2y+2n^3z\right)+3g\left(3nx+n^2y-2n^3z\right)+48g\left(3nx+n^2y\right)$$
$$g\left(9nx+3n^2y+2n^3z\right)+g\left(9nx+3n^2y-2n^3z)\right)=3g\left(3nx+n^2y+2n^3z\right)+3g\left(3nx+n^2y-2n^3z\right)+48g\left(3nx+n^2y\right)$$
$$g\left(3(nx+3n^2y)+2n^3z\right)+g\left(3(nx+3n^2y)-2n^3z)\right)=3g\left(nx+3n^2y+2n^3z\right)+3g\left(nx+3n^2y-2n^3z\right)+48g\left(3nx+3n^2y\right)$$
$$g\left(3nx+9n^2y+2n^3z\right)+g\left(3nx+9n^2y-2n^3z)\right)=3g\left(nx+3n^2y+2n^3z\right)+3g\left(nx+3n^2y-2n^3z\right)+48g\left(nx+3n^2y\right)$$
$$g\left(9nx+3n^2y+2n^3z\right)+g\left(9nx+3n^2y-2n^3z)\right)+g\left(3nx+9n^2y+2n^3z\right)+g\left(3nx+9n^2y-2n^3z)\right)=3g\left(3nx+3n^2y+2n^3z\right)$$
$$+3g\left(3nx+3n^2y-2n^3z\right)+3g\left(nx+3n^2y+2n^3z)\right)+3g\left(nx+3n^2y-2n^3z)\right)+48g\left(3nx+3n^2y\right)+48g\left(nx+3n^2y\right)$$
$$g\left((9nx+3n^2y)+2n^3z\right)+g\left(9nx+3n^2y-2n^3z)\right)+g\left(3nx+9n^2y+2n^3z\right)+g\left(3nx+3n^2y-2n^3z)\right)+48\left(nx+n^2y\right)+48g\left(nx+3n^2y\right)$$
$$=3g\left(3nx+3n^2y+2n^3z\right)+3g\left(nx+3n^2y-2n^3z\right)+3g\left(nx+3n^2y+2n^3z)\right)+3g\left(nx+3n^2y-2n^3z)\right)$$
$$g\left(9nx+3n^2y+2n^3z\right)+g\left(9nx+3n^2y-2n^3z)\right)+g\left(3nx+9n^2y+2n^3z\right)+g\left(3nx+3n^2y-2n^3z)\right)-12g\left(3nx+3n^2y\right)-12g\left(3nx+3n^2y\right)$$
$$=3g\left(nx+3n^2y+2n^3z\right)+3g\left(nx+3n^2y-2n^3z\right)+3g\left(nx+3n^2y+2n^3z\right)+3g\left(nx+3n^2y-2n^3z\right) \quad (2.22)$$

For all $x, y \in X$. Using (2.22), we get the identity that

$$16g\left(3nx+n^3z\right)+16g\left(3nx-n^3z)\right)+16g\left(3n^2y+n^3z\right)+16g\left(3n^2y-n^3z)\right)=3g\left(3nx+3n^2y+2n^3z\right)$$
$$+3g\left(3nx+n^2y-2n^3z\right)+3g\left(nx+3n^2y-2n^3z\right)+3g\left(nx+3n^2y+2n^3z\right)-648g\left(3nx+n^2y\right) \quad (2.23)$$
$$+48g\left(3nx+n^2y\right)+48g\left(nx+3n^2y\right)$$

For all $x, y \in X$. Replacing z by 2z in (2.23) and than using (2.21), we arrive

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$$16g\left(3nx+2n^3z\right)+16g\left(3nx-2n^3z)\right)+16g\left(3n^2y+2n^3z\right)+16g\left(3n^2y-2n^3z)\right)=3g\left(3nx+n^2y+4n^3z\right)$$
$$+3g\left(3nx+n^2y-4n^3z\right)+3g\left(nx+3n^2y-4n^3z\right)+3g\left(nx+3n^2y+4n^3z\right)-648g\left(nx+n^2y\right)+48g\left(3nx+n^2y\right)+48g\left(nx+3n^2y\right) \quad (2.24)$$

For all $x, y \in X$. Again making use of (2.11) and (2.21), we obtain

$$16g\left(3nx+2n^3z\right)+16g\left(3nx-2n^3z)\right)+16g\left(3n^2y+2n^3z\right)+16g\left(3n^2y-2n^3z)\right)=48g\left(nx+n^3z\right)+48g\left(nx-n^3z\right)$$
$$+768g\left(nz\right)+48g\left(n^2y+n^3z\right)+48g\left(n^2y-n^3z\right)+768g\left(n^2y\right)+36g\left(nx+n^2y+2n^3z\right)+36g\left(nx+n^2y-2n^3z\right) \quad (2.25)$$
$$+576g\left(nx+n^2y\right)-648\left(nx+n^2y\right)$$

For all $x, y \in X$. Again making use of (2.11) and (2.4), we get

$$16g\left(3nx+2n^3z\right)+16g\left(3nx-2n^3z)\right)+16g\left(3n^2y+2n^3z\right)+16g\left(3n^2y-2n^3z\right)=g\left(12nx+4n^3z\right)+g\left(12nx-4n^3z\right)+12g\left(6nx\right)$$
$$+g\left(12n^2y-4n^3z\right)+g\left(12n^2y-4n^3z\right)-12g\left(6n^2y\right)$$
$$=64g\left(3nx+n^3z\right)+64g\left(3nx-n^3z\right)+2592g\left(nx\right)+64g\left(3n^2y+n^3z\right)+64g\left(3n^2y-n^3z\right)-2592g\left(n^2y\right)$$
$$=64\left(g(3nx+n^3z)+g(3nx-n^3z)+g(3n^2y+n^3z)+g(3n^2y-n^3z)\right)-2592g\left(nx\right)-2592g(n^2y)$$
$$=64\left[g(3nx+n^3z)+g(3nx-n^3z)+g(3n^2y+n^3z)+g(3n^2y-n^3z)\right]-2592g\left(nx\right)-2592g(n^2y)$$
$$=64\left[3g(nx+n^3z)+3g(nx-n^3z)+48g\left(nx\right)+3g(n^2y+n^3z)+3g(n^2y-n^3z)+48(n^2y)\right]-2592g\left(nx\right)-2592g(n^2y) \quad (2.26)$$

For all $x, y \in X$. Using (2.26), to reduces that,

$$16g\left(3nx+2n^3z\right)+16g\left(3nx-2n^3z\right)+16g\left(3n^2y+2n^3z\right)+16g\left(3n^2y-n^3z\right)=192g\left(nx+n^3z\right)+192g\left(nx-n^3z\right)$$
$$+480g\left(nx\right)+192g\left(n^2y+n^3z\right)+192g\left(n^2y-n^3z\right)-480g(n^2y) \quad (2.27)$$

$$48g\left(nx+n^3z\right)+48g\left(nx-n^3z\right)+768g\left(nx\right)+48g\left(n^2y+n^3z\right)+48g\left(n^2y-n^3z\right)+768g\left(n^2y\right)+36g\left(nx+n^2y+2n^3z\right)$$
$$+36g\left(nx+n^2y-2n^3z\right)+576g\left(nx+n^2y\right)-678\left(nx+n^2y\right)$$
$$=12g\left(nx+n^3z\right)+192g\left(nx-n^3z\right)+480g\left(nx\right)+192g\left(n^2y+n^3z\right)+192g\left(n^2y-n^3z\right)+480g(n^2y)$$

$$36g\left(nx+n^2y+2n^3z\right)+36g\left(nx+n^2y-n^3z\right)=192g\left(nx+n^3z\right)+48g\left(nx+n^3z\right)+192g\left(nx-n^3z\right)-48g\left(nx-n^3z\right)$$
$$+480g\left(nx\right)-768g\left(nx\right)+192g\left(n^2y+n^3z\right)-48g\left(n^2y+n^3z\right)+192g\left(n^2y-n^3z\right)-48g\left(n^2y-n^3z\right)$$
$$+480g(n^2y)-768g\left(n^2y\right)+72\left(nx+n^2y\right)$$

$$36g\left(nx+n^2y+2n^3z\right)+36g\left(nx+n^2y-n^3z\right)=144g\left(nx+n^3z\right)+144g\left(nx-n^3z\right)+144g\left(n^2y+n^3z\right)+144g\left(n^2y-n^3z\right)$$
$$+72\left(nx+n^2y\right)-288g\left(nx\right)-288g\left(n^2y\right) \quad \div 36$$

$$g\left(nx+n^2y+2n^3z\right)+g\left(nx+n^2y-n^3z\right)=2g\left(nx+n^2z\right)+4g\left(nx-n^3z\right)+4g\left(nx-n^3z\right)+4g\left(n^2y+n^3z\right)$$
$$+4g\left(n^2y-n^3z\right)-8g\left(nx\right)-8g\left(n^2y\right) \quad (2.28)$$

for all $x, y, z \in X$. By considering $g\left(nx\right)=n^3g(x)$, gives that

$$g\left(nx+n^2y+2n^3z\right)+g\left(nx+n^2y-n^3z\right)=2g\left(nx+n^2y\right)+4\left[g(nx-n^3z)+g(nx-n^3z)+g(n^2y+n^3z)+g(n^2y-n^3z)\right]-8n^3g\left(x\right)$$
$$-8n^6g\left(y\right)$$

for all $x, y, z \in X$. Which implies that g is cubic.

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Conversely Suppose that $g: x \to y$ satisfies the functional equation (1.4). putting $x = y = z = 0$ in (1.5) yie $g(0) = 0$ setting (x, y, z) by $\left(\frac{-x}{n}, \frac{-x}{n}, \frac{x}{2n}\right)$ in the result we get $g(-x) = -g(x)$ which implies that g is odd.

Replacing y=0 in (1.5) and employing the fact that g is odd, we obtain

$$g\left(2nx+2n^3 z\right) + g\left(2nx-2n^3 z\right) = -6n^3 g\left(2x\right) + 4g\left(2nx+n^3 z\right) + 4g\left(2nx-n^3 z\right)$$

$$8g\left(nx+n^3 z\right) + 8g\left(nx-n^3 z\right) = -48n^3 g\left(x\right) + 4g\left(2nx+n^3 z\right) + 4g\left(2nx-n^3 z\right)$$

for all $x, y, z \in X$ .And again setting (x, y, z) by $\left(\frac{-x}{n}, \frac{-x}{n}, \frac{x}{2n}\right)$ from above identity, we get our desired result of (1.0)

III. Stability Results for (1.0): Direct Method

In this, we present the generalized Hyers-Ulam stability of the function (1.5).

Theorem :3.1. Let $j \in \{-1,1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that $\sum_{k=0}^{\infty} \frac{\alpha(n^{kj}x, n^{kj}y, n^{kj}z)}{n^{3kj}}$ converges in R and $\sum_{k=0}^{\infty} \frac{\alpha(n^{kj}x, n^{kj}y, n^{kj}z)}{n^{3kj}} = 0$ for all $x, y, z \in X$. Let $g: x \to y$ be an odd function satisfying the inequality $\|Dg(x,y,z)\| \le \alpha(x, y, z)$. for all $x, y, z \in X$. There exists cubic mapping $c: x \to y$ which satisfies the functional equation (1.5) and

$$\|g(x)-c(x)\| \le \frac{1}{8n^3} \sum_{k=\frac{1-i}{2}}^{\infty} \frac{\alpha(n^{kj}x, 0, 0)}{n^{3kj}} \qquad (3.1)$$

for all $x \in X$. The mapping $c(x)$ is defined by $c(x) = \lim_{n\to\infty} \frac{g\left(n^{kj}x\right)}{n^{3kj}}$ for all $x \in X$.

Corollary: 3.2. Let $x \in X$ and q be a non-negative real numbers. Let $g: x \to y$ satisfying the inequality

$$\|Dg(x,y,z)\| \le \begin{cases} \lambda; \\ \lambda\left\{\|x\|^q + \|y\|^q + \|z\|^q;\right. \\ \lambda\left\{\left(\|x\|^q \|y\|^q \|z\|^q + \|x\|^{3q} + \|y\|^{3q} + \|z\|^{3q}\right)\right\}; \end{cases} \qquad (3.2)$$

for all $x, y, z \in X$. Then there exists a unique cubic mapping $c: x \to y$ such that

$$\|g(x)-c(x)\| \le \begin{cases} \frac{1}{8}\left[\frac{1}{n^3-1}\right]; \\ \frac{\lambda}{8}\left[\frac{\|x\|^q}{\left|n^3-n^q\right|}\right]; \\ \frac{\lambda}{8}\left[\frac{\|x\|^{3q}}{\left|n^3-n^{3q}\right|}\right]; \end{cases} \qquad (3.3)$$

for all $x \in X$.

IV. Stability Results For [1.0]: Fixed Point Method

In this section, we investigate the generalized-Ulam –Hyers stability of the functional equation (1.1) for explicitly later use, the following theorem.

Theorem 4.1 (The alternative of fixed point) Suppose that we are given a complete generalized metric space (τ, d) and a strictly contractive mapping $T : \tau \to t$ with Lipchitz constant L. Then for each given $x \in \tau$, either

$d(T^n x, T^{n+1} x) = \infty$ for all $n \geq 0$ or there exists a natural number $\cap_0$, such that

i). $d\left(T^n x, T^{n+1} x\right) \leq 0$ for all $n \geq 0$.

ii). The sequence $(T^n x)$ is convergent to a fixed point y^* of T;

iii). y^* is the unique fixed point of T in the set $\Delta = \left\{y \in \tau; d(T^n x, y) < \infty\right\}$.

iv). $d\left(y, y^*\right) \leq \frac{1}{1-L} d\left(y, T_y\right)$ for all $y \in \Delta$.

Utilising the above mentioned fixed point alternative, we now obtained our main result, i.e)., the generalized Hyers-Ulam stability of the functional equation (1.5) from now on ,

Let x be a real vector space and y be a real given a mapping $g : x \to y$, we get

$$Dg\left(x, y, z\right) = g\left(nx + n^2 y + 2n^3 z\right) + g\left(nx + n^2 y - 2n^3 z\right) - 2g\left(nx + n^2 y\right) - 4g\left(nx + n^3 z\right) - 4g\left(nx - n^3 z\right)$$
$$- 4g\left(n^2 y + n^3 z\right) - 4g\left(n^2 y - n^3 z\right) + 8n^3 g\left(x\right) + 8n^6 g\left(y\right)$$

for all $x, y, z \in X$. Let $\psi : X \times X \times X \to \left[0, \infty\right)$ be a function such that

$$\lim_{h \to 0} \frac{\psi\left(\mu_i^k x, \mu_i^k y, \mu_i^k z\right)}{\mu_i^{3k}} = 0$$

for all $x, y, z \in X$, where $\mu_i = 2$ if i=0 and $\mu_i = \frac{1}{2}$ if i=1

Theorem: 4.2. suppose that a function $g : x \to y$ satisfies the function inequality

$\left\|Dg\left(x, y, z\right)\right\| \leq \psi(x, y, z)$ for all $x, y, z \in X$, if there exists $L = L(i)$ such that a function

$x \to \beta(x) = \frac{1}{2} \alpha\left(\frac{x}{n}, 0, 0\right)$ has the property $\frac{1}{\mu_i^3} \beta\left(\mu_i x\right) = L\beta(x)$

for all $x \in X$. Then there exists a unique cubic function $c : x \to y$ satisfies the functional equation (1.0) and $\left\|g(x) - c(x)\right\| \leq \frac{L^{1-i}}{1-L} \beta(x)$ for all $x \in X$.

Corollary 3.2. Let $g : x \to y$ be an mapping and there exists a real numbers γ and p such that

$$\left\|Dg(x, y, z)\right\| \leq \begin{cases} \gamma; \\ \gamma\left\{\|x\|^p + \|y\|^p + \|z\|^p\right\}; \\ \gamma\left\{\|x\|^p + \|y\|^p + \|z\|^p + \|x\|^{3p} + \|y\|^{3p} + \|z\|^{3p}\right\}; \end{cases} \quad (4.1)$$

for all $x, y, z \in X$. There exists a unique cubic mapping $c : x \to y$ such that $\left\|g(x) - c(x)\right\| \leq \begin{cases} \frac{\gamma}{8}\left[\frac{1}{n^3 - 1}\right]; \\ \frac{\gamma}{8}\left[\frac{\|x\|^p}{\left|n^3 - n^p\right|}\right]; \gamma \neq 3 \\ \frac{\gamma}{8}\left[\frac{\|x\|^{3q}}{\left|n^3 - n^{3q}\right|}\right]; \gamma \neq 1; \end{cases}$ (4.2) for all $x \in X$

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