International Journal of Scientific Engineering and Science
Volume 2, Issue 1, pp. 1-4, 2018. ISSN (Online): 2456-7361

# A Generalized Formulation of the Concept of the "Cube" – A Hidden Relationship of a Topological Nature between the Cube and Truncated Octahedron

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***Abstract***— *This first objective of this paper is to search for higher-dimensional cube structures and explore their relationship with certain Archimedean polyhedra. In particular, the research focuses on the relationship between the six-dimensional cube and the truncated octahedron. The problem is approached through the study of higher-dimensional hypercube projections in three-dimensional space and, subsequently, on a plane, and the finding that each n-hypercube can result from a composition of lower-dimensional cubes. This procedure basically leads to the generalization of the concept of the cube through its successive projections in different dimensions, and finally reveals a relationship of a topological nature between the cube and Archimedean solids, such as the truncated octahedron.*

***Keywords***—*Cube, polyhedra topology, truncated octahedron*.

## I. The Concept of the Cube in the Different Dimensions – Successive Projections

A three-dimensional cube can be projected parallel to any vector in a certain plane, thus providing an axonometric image of it. A lower-dimensional shape. According to Pohlke's theorem, if we map any axonometric image of a cube on a plane, then there is always a vector (d) parallel to which the cube was projected, thus providing this particular image. This means that a cube has infinite axonometric images on the plane, some of which provide a high level of monitoring of the cube, while others do not. Although Pohlke's theorem does not apply in general in the case of hypercubes, one can say that an n-hypercube has many faces when projected on an m-space of a lower dimension than n. As one reduces the dimension through successive projections, bodies occur, which belong each time to the respective dimension and consist of fragments of cubes of an even lower or equal dimension, and also constitute an image of the initial n-hypercube in this dimension. Thus there is constantly and in every dimension (k) a connection between the image of the n-hypercube in this dimension and its fragments, which constitute images of the hypercube in dimensions lower than (n) or images of it in dimension (k). Upon reaching dimension 3 through this procedure, a composite solid will arise, whose fragments will constitute different images of hypercubes in a smaller dimension than the initial cube, as well as axonometric images of a three-dimensional cube. This can be described as a transformation of a topological nature, which connects entities of different dimensions, not only through projections in the different dimensions, but also through intersections, through fragmentation of an entity in one dimension into simpler entities that would constitute a whole in a smaller dimension. Through this mechanism, the dimension from which the initial piece of information originates does not appear to be of great importance.

## II. The Example of N-Hypercubes

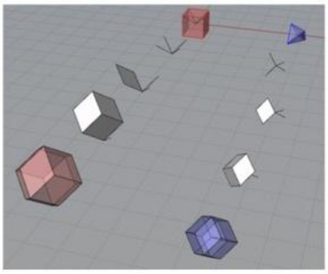

Fig. 1. From a cube to a hypercube model.

The Minkowski sum of n vectors in space results in a spatial lattice (a network) of vertices, edges, faces and cells, whose convex hull, which is determined by the faces of the outline, is a convex polyhedron with n*(n-1) faces, where n is the number of different vector directions (Fig. 1). If the vectors are equal in size, then the faces of the convex polyhedron are rhombuses and the polyhedron is an equilateral zonohedron. Equilateral zonohedra with n*(n-1) faces are considered to be three-dimensional projections of n-dimensional hypercubes. The more symmetrical the order of the initial vectors in space are, the more symmetrical the resulting zonohedra. Using the property of the three-dimensional cube, which can create a spatial tessellation and can – with the appropriate intersections – result in planar tessellations, as well as the corresponding property of the n-dimensional cube, which can fill the corresponding space of n dimensions, we will explore the possibility of creating spatial tessellation structures in the three dimensions and, by extension, planar tessellations resulting from hypercubes by working with their three-dimensional models, thus attempting a topological exploration of the transformation of this property when projected in different dimensions. This exploration is also based on the fact that each n-polar zonohedron and, consequently, each n-hypercube can result from the composition of zonohedra of a lower order. This means that the cube in any dimension can result from a composition of

Volume 2, Issue 1, pp. 1-4, 2018.
ISSN (Online): 2456-7361

cubes of smaller dimensions. Thus, for example, the rhombic triacontahedron (which is a 3D model of the 6-cube) can result as follows: Let us take the parallelohedra defined by two triads of the six vectors that determine the rhombic triacontahedron. The combination of two oblong and two oblate parallelohedra results in a rhombic dodecahedron (3D model of the 4-cube). This, along with the use of six additional parallelohedra, results in a rhombic icosahedron (3D model of the 5-cube). The rhombic icosahedron combined with 10 parallelohedra, results in a rhombic triacontahedron (which is a 3D model of the 6-cube) (Fig. 2).

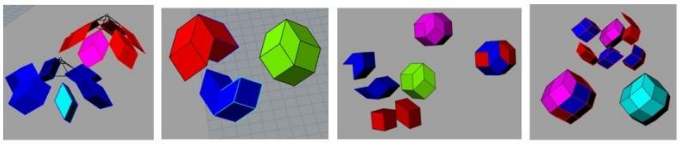
Fig. 2. A 3d model of the 6-cube.

## III. THE ALGORITHMIC APPROACH TO THE PROBLEM

The more the dimension of the hypercube increases, the greater the complexity of the lattice, thus making it difficult or even impossible to design it on a computer. An algorithm has been formulated in order to design the initial lattice of cells forming a hypercube. This algorithm results in the Minkowski sum of n vectors in space, however the representation is limited to the $n$=7 value. This algorithm follows a procedure with specific consecutive steps, and in each step the three-dimensional image of the fragment of the initial hypercube with all its internal cells is produced (Fig. 3). Each fragment constitutes an image of a hypercube of a smaller dimension.

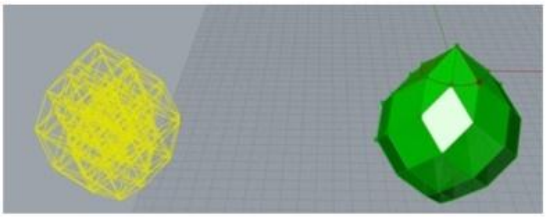
Fig. 3. The internal lattice of cells.

A second algorithm was designed to select, from the overall network of faces, those faces of the n-hypercube that belong to/define its convex hull (without the internal lattice of edges). This algorithm results in three-dimensional projections of hypercubes of large dimensions (Fig. 4). This algorithm, too, follows a similar gradual procedure to the previous one.

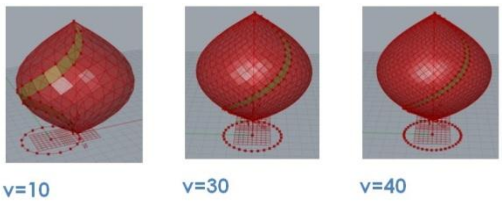

Fig. 4. Hypercubes of larger dimensions.

## IV. DESCRIPTION OF THE ALGORITHMS

The first algorithm was designed to create the Minkowski sum of (n) vectors and is graphically described in figure 5:

1. Three vectors are selected from the (n) vectors and the first rhombohedron (three-dimensional model of the 3-cube) is created.
2. The translation surfaces are created from this rhombohedron in the direction of the fourth vector.
3. This results in a convex polyhedron, which is a three-dimensional model of the 4-cube, together with the internal lattice of the individual rhombohedra.
4. From the resulting whole, the translation surfaces are created in the direction of the fifth vector.
5. This results in a convex polyhedron, which is a three-dimensional model of the 5-cube, together with the internal lattice of the individual rhombohedra.

The same procedure is repeated until the n-hypercube and for n=7 at the most.

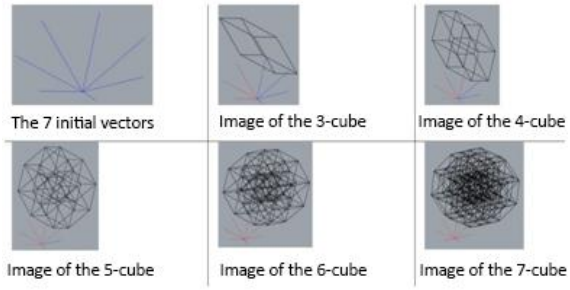

Fig. 5. The first algorithm.

The second algorithm was designed to create only the outer hull of the n-hypercube in the case of the polar zonohedron (orthogonal projection of the hypercube), without the internal lattice of cells:

1. From the n vectors creating the hypercube, select two and create the translation surface they define, which is in the shape of a rhombus.
2. Draw the translation surface from the edge of the resulting rhombus and lengthwise along the third vector.
3. From the resulting new rhombus, create the translation surface by transferring its edge along the fourth vector, and so on.

This results in the creation of a polyhedral helical surface. The axis of the helix runs parallel to the height of the initial pyramid from which the vectors have resulted. The polar zonohedron is then created (orthogonal projection of the n-hypercube) through circular replication of the polyhedral surface n times so as to achieve a full 360-degree rotation. This algorithm does not set any limits on the (n) dimension of the hypercube, while it only creates the outer hull of the zonohedron without the internal cells.

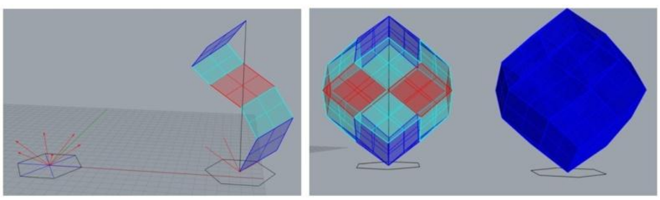
Fig. 6. Rhomboid faces around the axis.

Polar zonohedra form a unique category of zonohedra. Let

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us take a regular n-gon in the plane and line segments $\varepsilon_1$, $\varepsilon_2$,...$\varepsilon_v$, which connect the centre O of the polygon to its vertices. Then let us take the equal vectors $\delta_1$, $\delta_2$,…$\delta_v$, with O being the common starting point, which are projected in the plane of the n-gon by $\varepsilon_1$, $\varepsilon_2$,…$\varepsilon_v$. The zonohedron resulting from the Minkowski sum of vectors $\delta_1$, $\delta_2$,…$\delta_v$, which is called a polar zonohedron, is a convex polyhedron whose n-fold axis is the vertical line in the centre of the plane of the polygon with n(n-1) rhomboid faces laid out in zones around the axis (Fig.6). The second algorithm allows us to create polar zonohedra by controlling the number of vectors, their inclination to the plane of each polygon and their size. This results in a variety of forms. When several sides of a regular polygon are infinite, then the zonohedron leans towards a surface of revolution. It has been observed that when the angle of inclination of the vectors to the plane of the polygon is 36.264°, the n-polar zonohedron is considered a 3D orthogonal isometric projection of the n-hypercube.

## V. THE GENERALIAZATION

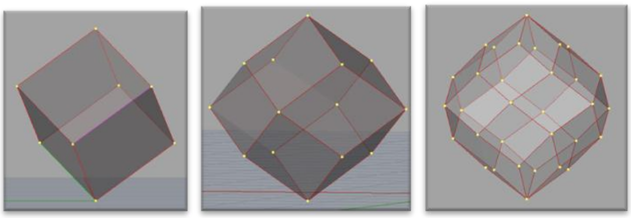

Fig. 7. 3d models of 3,4 and 6 dimensional cubes.

Up until this point we assume that the concept of the "cube" (Fig.7) determines the relationship between the points arranged in three-dimensional space. The repetition of this arrangement fills the space and provides the known result. We further assume that the cubic network of n-dimensional points fills the corresponding (n) space in a similar way. The discovery and formulation of a relationship between these three dimensions/components, which organise the arrangements in the form of vectors, as well as the generalization of this formulation in (n) dimensions, lead us to find new structures in three-dimensional space, which confirm the initial search criterion. The generalization of the formulation regarding this relationship leads to the generalization of the concept of the "cube" and allows for a notional transition from dimension to dimension, with new design results each time in the three-dimensional world, which however have consequences and a cause for existence in every dimension, to the point that dimension becomes meaningless.

## VI. FROM THE MACROCOSM TO THE MICROCOSM

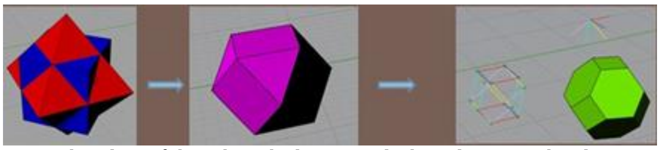

Fig. 8. The edges of the cuboctahedron provide the 6 directions that determine the 6-cube model, a truncated octahedron.

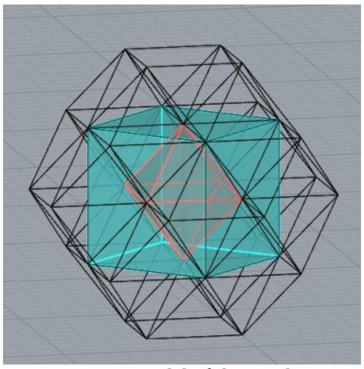

Fig. 9. A model of the 6-cube.

The truncated octahedron is an Archimedean polyhedron (4,6,6). The directions of its edges result from the intersection of a cube with an octahedron, i.e. from the edges of the octahedron (Fig.8). At the same time, this same solid constitutes a model of the 6-cube (Fig.9). If seen as a model of the 6-cube, then its lattice of vertices includes Archimedean polyhedra, either individually or combined so as to create a spatial tessellation, like those presented in figure 10. Otherwise it is an individual polyhedron.

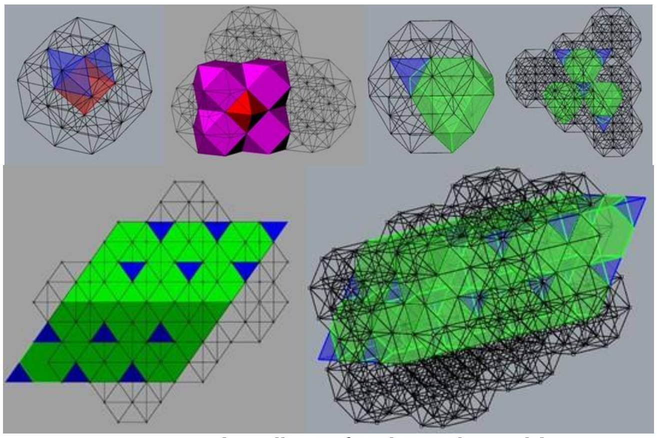

Fig. 10. Spatial tessellations from hypercubes models.

This formulation is a result of the generalization of the formulation of the concept of "truncated octahedron" or the generalization of the concept of the cube, considering that this is a model of the 6-cube, and leads us to wonder whether the truncated octahedron and cube have some hidden common properties. However, in order to formulate this question we first had to make the transition to another dimension.

## VII. CONCLUSION

Similar observations will be made if we examine the zonohedron which is a model of the 9-cube. This zonohedron consists of 9 edge-directions and is presented in figure 11. At the same time, its outer hull is the truncated cuboctahedron, another Archimedean polyhedron. The same edge-directions are derived from a cube and a single tetrahedron in the cube, or from a cube and its dual octahedron (Fig. 12). Therefore, if the truncated cuboctahedron is viewed as a model of the 9-cube, then its internal lattice of vertices (Fig. 13) and edges will feature other solids, such as those presented in figure 14.

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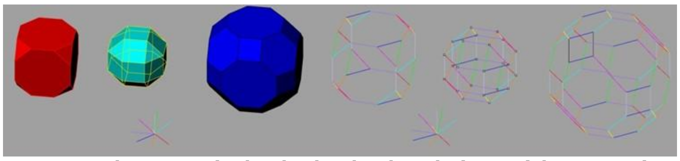

Fig. 11. The truncated cube, the rhombicuboctahedron and the truncated cuboctahedron give the same 9 vectors that form a model of the 9-cube.

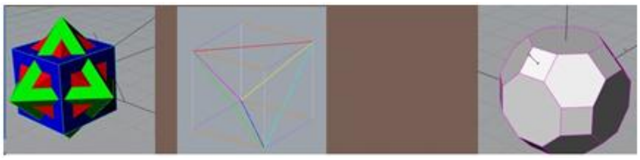

Fig. 12. The Minkowski sum of these vectors is the truncated cuboctahedron, which is a model of the 9-cube.

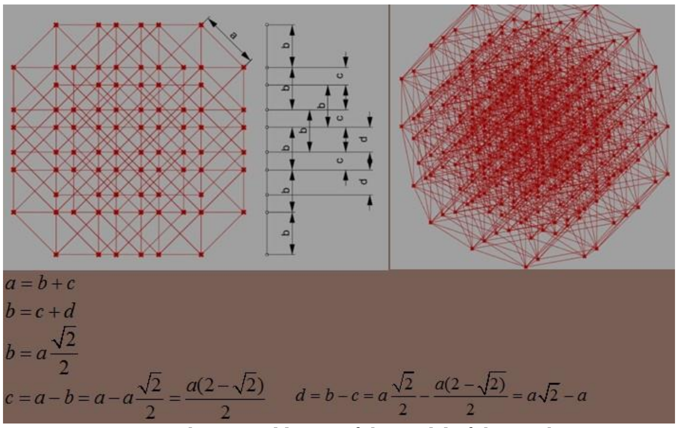


Fig. 13. The internal lattice of the model of the 9-cube.

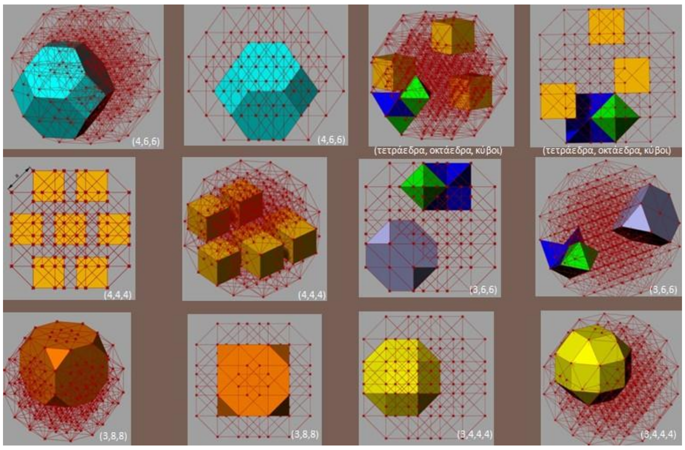


Fig. 14. Cubes in a cube.

## ACKNOWLEDGMENT

The study on hypercubes began and was completed – at the initial stage – as part of the research programme titled “THALES – Advanced Geometric Computing and Critical Applications”, under scientific coordinator I. Emiris. The author of this paper occupied the position of Postdoctoral Researcher in this programme. This paper is the further development of the initial study, which was presented in the Interdepartmental Postgraduate Study Programme of the School of Architecture at the National Technical University of Athens titled “Architecture – Spatial Design”, and in particular in the A2 Specialization: “Advanced Issues of Architectural Design” and in the course titled “Interaction Design: Contemporary Fields of Design – Construction / Methodological Analysis Tools for the Design of Strategic Integrated Urban Interventions” under scientific coordinator G. Parmenidis. (The images are the product of the study and personal designs of the author of this paper.).

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