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The Linearized Dispersion Relation of Navier-Stokes Equation in Shallow Water

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Abstract— *A few number of variants of Navier-Stokes equations and their higher order generalizations are derived to describe the two way propagation of small amplitude, wavelength, gravity waves on the surface of water in a canal. These systems are also seen to model the propagation of long-crested waves on large lakes or the ocean and in other contexts. Depending on linearized terms and positing a solution of the form* $e^{i(kx-wt)}$*, the wave frequency is formulated in terms of wave number. Finally, the phase speed has been established with viscous term for long wavelength where the first three terms correspond to the expansion of the full linearized dispersion relation.*

Keywords— *Dispersion relation :Navier-Stokes equations: Phase Speed: Shallow Water Waves.*

I. INTRODUCTION

In many field and laboratory studies and in engineering applications, the full Navier-Stokes equations appear complex situation for modeling at hand and consequently there have appeared many approximate models applying to restricted physical regimes. In the 1870s, Boussinesq derived some model evolution equations which are applicable in principle to describe motions that are sensibly two dimensional and which have the form of a perturbation of the one dimensional wave equation. J. L. Bona et al. [1] derived four parameter family of Boussinesq systems from the two dimensional Euler equations for free surface flow and formulate criteria to help decide which of these equations one might choose in a given modeling situation. J. L. Bona et al. [2] also established the first order correct models that are linearly well posed are in fact locally nonlinearly well posed. J. L. Bona and M. Chen [3] studied the systems which describes approximately the two dimensional propagation of surface waves in a uniform horizontal channel of length filled with an irrotational, incompressible, inviscid fluid. A. Ouahsine et al. [4] studied an innovative approach based on the finite elements method is presented to improve the dispersion relation. F. Marche [5] studied the derivation with asymptotic analysis of two dimensional viscous shallow water model in rotating framework with irregular topography, linear and quadratic bottom terms and capillary effects considering the three dimensional Navier-Stokes equations with a free moving surface boundary condition and hydrostatic approximation. D. Dutykh and F. Dias [6] showed how to express the vertical component of the velocity only in terms of the potential and free surface elevation. D. Dutykh [7] analysed dispersion relation properties of proposed models and also presented some computations with viscous Boussinesq equations solved by a Fourier type spectral method. R. Barros et al. [8] derived an approximate multidimensional model of dispersive waves propagating in two layer fluid with free surface and also introduced the notion of generalized vorticity and derived analogues of integrals of motion, such as Bernoulli integrals, which are well known in ideal Fluid Mechanics. J. L. Bona et al. [9] obtained new nonlinear systems describing the interaction of long water waves in both two and three dimensions. T. H. C. Herbers et al. [10] examined the nonlinear dispersion of random directionally spread surface gravity waves in shallow water with Boussinesq theory. Y. A. Li et al. [11] described a pseudo-spectral numerical method to solve the systems of one dimensional evolution equations for free surface waves in a homogeneous layer of an ideal fluid. P. L. –F. Liu and A. Orfila [12] derived sets of depth-integrated continuity and momentum equations for transient long wave propagation with viscous effects using a perturbation approach and the Boussinesq approximation. V. Duchene [13] derived asymptotic models for the propagation of two and three dimensional gravity waves at the free surface and the interface between two layers of immiscible fluids of different densities over an uneven bottom. D. Lannes [14] described the motion of the free surface and the evolution of the velocity field of a layer of perfect, incompressible, irrotational fluid under the influence of gravity for an ideal liquid of water wave problem. D. Bresch and B. Desjardins [15] constructed approximate solutions for the two dimensional viscous shallow water model and for compressible Navier-Stokes models. M. Chen et al. [16] investigated a water wave model with a nonlocal viscous term and the decay rate of solutions theoretically and numerically. J. L. Bona and H. Chen [17] derived various model equations considering a body of water of finite depth under the influence of gravity bounded below by a flat, impermeable surface. They [17] ignored viscous and surface tension effects and assumed that the flow is incompressible and irrotational where the fluid motion is governed by the Euler equations together with suitable boundary conditions on the rigid surface and on the air-water interface. In this paper, we formulate the wave frequency in terms of wave number depending on linearized terms and positing a solution of the form $e^{i(kx-wt)}$ and establish the phase speed with viscous term for long wavelength.

II. FORMULATION

Let Ω_t be the domain in R^3 which is occupied by viscous, incompressible fluid at time *t*. The system describing the motion of such a fluid is the Navier-Stokes equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+(\underline{v}.\nabla)\underline{v}\right)=\underline{F}-\nabla p+\mu\nabla^2\underline{v}\ ,\ \text{in}\quad \Omega_t \tag{1}$$

and

$$\nabla.\underline{v}=0\ ,\quad \text{in}\ \Omega_t \tag{2}$$

where $\underline{v}=u\hat{i}+v\hat{j}+w\hat{k}$ is the fluid velocity, $\hat{i},\ \hat{j},\ \hat{k}$ are the unit vectors along the $x,\ y,\ and\ z$ axis respectively, in R^3, $\underline{F}$ is the body forces (per unit volume) acting on the fluid, ρ is the fluid density, p is the forcing pressure on the free surface and μ is the fluid viscosity.

Under the assumption of incompressible and irrotational flow, the water wave motion is described by the velocity potential $\phi(x,y,z,t)$ and the free surface water elevation $\eta(x,y,t)$, $\underline{F}=-\rho\underline{g}$, where $\underline{g}$ is the gravitational acceleration and $\underline{g}=(0,0,g)$. For irrotational flow, $\nabla\times\underline{v}=0$, where $\underline{v}=\nabla\phi$, for some potential $\phi=\phi(x,y,z,t)$. Then it satisfies the Laplace equation

$$\nabla^2\phi=\phi_{xx}+\phi_{yy}+\phi_{zz}=0, at -h<z<\eta \tag{3}$$

Again, Eq.(1) can be written as

$$\rho\left(\frac{\partial}{\partial t}(\nabla\phi)+(\nabla\phi.\nabla)\nabla\phi\right)=-\rho g\hat{k}-\nabla p+\mu\nabla^2(\nabla\phi)$$

$$\therefore \rho\int\left(\frac{\partial}{\partial t}(\nabla\phi)+(\nabla\phi.\nabla)\nabla\phi\right).d\underline{r}=\int\left(-\rho g\hat{k}-\nabla p+\mu\nabla^2(\nabla\phi)\right)dr$$

$$\frac{\partial}{\partial t}\int d\phi+\int(\nabla\phi.\nabla)\nabla\phi.dr=-gz-\frac{1}{\rho}\int dp+\upsilon\int\nabla^2(\nabla\phi)dr$$, where $\frac{\mu}{\rho}=\upsilon$ is kinematic viscosity.

$$\frac{\partial\phi}{\partial t}+\int(\nabla\phi.\nabla)\nabla\phi.dr=-gz-\frac{p}{\rho}+\upsilon\nabla^2\phi,\ at\ z=\eta \tag{4}$$

Here, $(\nabla\phi.\nabla)\nabla\phi=\frac{1}{2}\nabla.(\nabla\phi)^2-\nabla\phi\times(\nabla\times\nabla\phi)\ =\frac{1}{2}\nabla.(\nabla\varphi)^2$

$$\therefore \int(\nabla\phi.\nabla)\nabla\phi.d\underline{r}=\frac{1}{2}\int\nabla.(\nabla\phi)^2.d\underline{r}\ =\frac{1}{2}(\nabla\varphi)^2 \tag{5}$$

Substituting this value in Eq. (4), we have

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2=-gz-\frac{p}{\rho}+\upsilon\nabla^2\phi,\ at\ z=\eta \tag{6}$$

Also the kinematic free surface boundary condition is

$$\eta_t+\phi_x\eta_x+\phi_y\eta_y-\phi_z=0,\ at\ z=\eta \tag{7}$$

On the fixed portion of the boundary, condition of impermeability is

$\underline{v}.\underline{n}=0$, $\underline{n}$ being the normal direction of the surface.

$$\phi_x h_x+\phi_y h_y+\phi_z=0,\ \ at\ z=-h \tag{8}$$

Suppose the bottom of the channel be flat and horizon and let h denote the depth of the liquid in its undisturbed state. Then Eqs. (3), (6), (7), and (8) can be written as

$$\left.\begin{aligned}&\phi_{xx}+\phi_{zz}=0,\ \ at -h<z<\eta\\&\phi_z=0,\ \ at\ z=-h\\&\eta_t+\phi_x\eta_x-\phi_z=0,\ at\ \ z=\eta\ and\\&\phi_t+\frac{1}{2}\left(\phi_x^2+\phi_z^2\right)+gz=\upsilon(\phi_{xx}+\phi_{zz}),\ \ at\ \ z=\eta,\quad \text{on the free surface}\ \ p=0\end{aligned}\right\} \tag{9}$$

In dimensionless form, the above variables are as follows:

$$x = lx^*,\ z = h(z^* - 1),\ \eta = a\eta^*, t = \frac{lt^*}{c_0},\ \phi = \frac{gal\phi^*}{c_0},\ \alpha = \frac{a}{h},\ \beta = \frac{h^2}{l^2},\ \upsilon = \alpha\beta c_0 l \upsilon^* \quad (10)$$

where $c_0 = \sqrt{gh}$. So Eq. (9) becomes

$$\beta\phi^*_{x^*x^*} + \phi^*_{z^*z^*} = 0,\ at\ 0 < z^* < 1 + \alpha\eta^* \quad (11)$$

$$\phi^*_{z^*} = 0 \quad (12)$$

$$\eta^*_{t^*} + \alpha\phi^*_{x^*}\eta^*_{x^*} - \frac{1}{\beta}\phi^*_{z^*} = 0,\ at\ z^* = 1 + \alpha\eta^* \quad (13)$$

and

$$\eta^* + \varphi^*_{t^*} + \frac{1}{2}\alpha\varphi^{*2}_{x^*} + \frac{1}{2}\frac{\alpha}{\beta}\varphi^{*2}_{z^*} = \alpha\beta\upsilon^*\left(\varphi^*_{x^*x^*} + \frac{1}{\beta}\varphi^*_{z^*z^*}\right),\ at\ z^* = 1 + \alpha\eta^* \quad (14)$$

The velocity potential $\phi(x,z,t)$ is assumed analytic and we can expand it in power series with respect to the vertical coordinates,

$$\phi(x,z,t) = \sum_{m=0}^{\infty} f_m(x,t) z^m \quad (15)$$

Substituting this value in Eq. (11) and dropping asterisk, we have,

$$(m+1)(m+2) f_{m+2}(x,t) = -\beta(f_m(x,t))_{xx}.\ \text{for}\ m = 0,1,2.............. \quad (16)$$

Let $F = \phi_0(x,t)$ denotes the velocity potential at the bottom $z = 0$ and using Eq. (16) repeatedly, we have

$$f_{2k}(x,t) = \frac{(-1)^k \beta^k}{(2k)!}\frac{\partial^{2k} F(x,t)}{\partial x^{2k}}, k = 0,1,2,..............$$

Also Eq. (12) implies that $f_1(x,t) = 0$, and so

$$f_{2k+1}(x,t) = 0, k = 0,1,2....................,$$

Hence,

$$\phi(x,z,t) = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!}\frac{\partial^{2k} F(x,t)}{\partial x^{2k}} z^{2k}.$$

$$= F - \frac{1}{2}\beta\frac{\partial^2 F}{\partial x^2}z^2 + \frac{1}{4!}\beta^2\frac{\partial^4 F}{\partial x^4}z^4 - .... \quad (17)$$

Substituting this value in Eqs. (13) and (14) and dropping asterisk, we obtain,

$$\eta_t + \alpha\eta_x \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!}\frac{\partial^{2k+1} F(x,t)}{\partial x^{2k+1}}(1+\alpha\eta)^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k+1)!}\frac{\partial^{2k+2} F(x,t)}{\partial x^{2k+2}}(1+\alpha\eta)^{2k+1} = 0. \quad (18)$$

and

$$\eta + \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!}\frac{\partial^{2k+1} F(x,t)}{\partial x^{2k}\partial t}(1+\alpha\eta)^{2k} + \frac{1}{2}\alpha\left\{\sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!}\frac{\partial^{2k+1} F(x,t)}{\partial x^{2k+1}}(1+\alpha\eta)^{2k}\right\}^2$$

$$+ \frac{1}{2}\alpha\beta\left\{\sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k+1)!}\frac{\partial^{2k+2} F(x,t)}{\partial x^{2k+2}}(1+\alpha\eta)^{2k+1}\right\}^2 \quad (19)$$

$$= \alpha\beta\upsilon\left(\sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!}\frac{\partial^{2k+2} F(x,t)}{\partial x^{2k+2}}(1+\alpha\eta)^{2k} + \beta\sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k+2)!}\frac{\partial^{2k+4} F(x,t)}{\partial x^{2k+4}}(1+\alpha\eta)^{2k+2}\right).$$

The parameters α *and* β have the same small order while F *and* η have been scaled so that they and their partial derivatives are of order one keeping only the terms of Eqs. (18) and (19) which are the lowest order, then the system becomes

$$\eta_t + \frac{\partial^2 F}{\partial x^2} = terms\,in\,\alpha,\beta$$

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$$\eta + \frac{\partial F}{\partial t} = terms\,in\,\alpha,\beta$$

Differentiating the second equation with respect to x and considering $\frac{\partial F}{\partial x} = u(x,t)$, the scaled horizontal velocity at the bottom of the channel, the above two equations become

$$\eta_t + u_x = terms\,in\,\alpha,\beta \quad (20)$$

$$\eta_x + u_t = terms\,in\,\alpha,\beta \quad (21)$$

When the terms of formal order α,β are ignored, then from Eqs. (20) and (21), we have

$$u_{xx} + u_{tt} = 0 \quad (22)$$

Which is simply the linear wave equation.

All the terms in Eqs. (18) and (19) are at most linear in α or β taking next order of approximation .Then the system becomes

$$\eta_t + \frac{\partial^2 F}{\partial x^2} + \alpha\eta_x\frac{\partial F}{\partial x} + \alpha\eta\frac{\partial^2 F}{\partial x^2} - \frac{1}{6}\beta\frac{\partial^4 F}{\partial x^4} = quadratic\ terms\ in\ \alpha,\beta$$

$$\eta + \frac{\partial F}{\partial t} - \frac{\beta}{2}\frac{\partial^3 F}{\partial x^2\partial t} + \frac{\alpha}{2}\left(\frac{\partial F}{\partial x}\right)^2 = quadratic\ terms\ in\ \alpha,\beta$$

Differentiating the second equation with respect to x and using $\frac{\partial F}{\partial x} = u(x,t)$, we have from the above two equations

$$\left.\begin{aligned} &\eta_t + u_x + \alpha\eta_x u + \alpha\eta u_x - \frac{1}{6}\beta u_{xxx} = quadratic\ terms\ in\ \alpha,\beta \\ &\eta_x + u_t - \frac{\beta}{2}u_{xxt} + \alpha u u_x = quadratic\ terms\ in\ \alpha,\beta \end{aligned}\right\} \quad (22)$$

For the second order case, Eqs. (18) and (19) can be rewritten as

$$\eta_t + \alpha\eta_x\left(\frac{\partial F}{\partial x} - \frac{\beta}{2}\frac{\partial^3 F}{\partial x^3}\right) + \frac{\partial^2 F}{\partial x^2}(1+\alpha\eta) - \frac{\beta}{6}\frac{\partial^4 F}{\partial x^4}(1+3\alpha\eta) + \frac{\beta^2}{120}\frac{\partial^6 F}{\partial x^6} = cubic\ terms\ in\ \alpha,\beta$$

$$\eta + \frac{\partial F}{\partial t} - \frac{\beta}{2}\frac{\partial^3 F}{\partial x^2\partial t}(1+2\alpha\eta) + \frac{\alpha}{2}\left\{\left(\frac{\partial F}{\partial x}\right)^2 - \beta\frac{\partial F}{\partial x}\frac{\partial^3 F}{\partial x^3}\right\} + \frac{\beta^2}{24}\frac{\partial^5 F}{\partial x^4\partial t} + \frac{1}{2}\alpha\beta\left(\frac{\partial^2 F}{\partial x^2}\right)^2 - \alpha\beta\upsilon\frac{\partial^2 F}{\partial x^2}$$
$$= cubic\ terms\ in\ \alpha,\beta$$

Again, differentiating the second equation with respect to x and using $\frac{\partial F}{\partial x} = u(x,t)$, we have

$$\eta_t + u_x + \alpha\eta_x u - \frac{1}{2}\alpha\beta\eta_x u_{xx} + \alpha\eta u_x - \frac{1}{6}\beta u_{xxx} - \frac{1}{2}\alpha\beta\eta u_{xxx} + \frac{\beta^2}{120}u_{xxxxx} = cubic\ terms\ in\ \alpha,\beta \quad (23)$$

$$\eta_x + u_t - \frac{\beta}{2}u_{xxt} - \alpha\beta\eta u_{xxt} - \alpha\beta\eta_x u_{xt} + \alpha u u_x - \frac{\alpha\beta}{2}u u_{xxx} + \frac{\alpha\beta}{2}u_x u_{xx} + \frac{\beta^2}{24}u_{xxxxt} - \alpha\beta\upsilon u_{xx} \quad (24)$$
$$= cubic\ \ terms\ in\ \alpha,\beta$$

For undisturbed surface, the depth is $(1-\theta)h$ below where $0 \le \theta \le 1$.When $\theta = 0$, it leads to $w = u$, the horizontal velocity at the bottom. A formal use of Taylor's formula with remainder shows and using Eq. (17), we get

$$\phi(x,z,t) = F - \frac{1}{2}\beta\frac{\partial^2 F}{\partial x^2}z^2 + \frac{1}{4!}\beta^2\frac{\partial^4 F}{\partial x^4}z^4 - \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore w = \left(\frac{\partial\phi}{\partial x}\right)_{z=\theta} = F_x - \frac{\beta}{2}\theta^2 F_{xxx} + \frac{\beta^2}{4!}\theta^4 F_{xxxxx} + o(\beta^3)$$

$$w = u - \frac{\beta}{2}\theta^2 u_{xx} + \frac{\beta^2}{4!}\theta^4 u_{xxxx} + o(\beta^3) \text{ as } \beta \to 0 \text{ and using } \frac{\partial F}{\partial x} = F_x = u(x,t).$$

By using Fourier transform of a function w of the special variable x is

$$\hat{w}(k) = \int_{-\infty}^{\infty} e^{-ikx} w(x)dx \quad \left[\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx\right] = \int_{-\infty}^{\infty} e^{-ikx}\left[u(x) - \frac{\beta}{2}\theta^2 u_{xx}(x) + \frac{\beta^2}{4!}\theta^4 u_{xxxx}(x) + o(\beta^3)\right]dx$$

$$= \left(1 + \frac{\beta}{2}\theta^2 k^2 + \frac{\beta^2}{4!}\theta^4 k^4\right)\hat{u} + o(\beta^3)$$

$$\therefore \hat{u}(k) = \left(1 + \frac{\beta}{2}\theta^2 k^2 + \frac{\beta^2}{4!}\theta^4 k^4\right)^{-1}\hat{w} + o(\beta^3) = \left[1 - \frac{1}{2}\beta\theta^2 k^2 + \frac{5}{24}\beta^2\theta^4 k^4\right]\hat{w} + o(\beta^3)$$

as $\beta \to 0$. Thus there appears the relationship

$$u = w + \frac{1}{2}\beta\theta^2 w_{xx} + \frac{5}{24}\beta^2\theta^4 w_{xxxx} + o(\beta^3) \tag{25}$$

Substituting this value in Eqs. (23) and (24), we have

$$\eta_t + w_x + \beta\left(\frac{1}{2}\theta^2 - \frac{1}{6}\right)w_{xxx} + \alpha(\eta w)_x + \frac{1}{2}\alpha\beta(\theta^2 - 1)(\eta w_{xx})_x + \frac{5}{24}\beta^2\left(\theta^2 - \frac{1}{5}\right)^2 w_{xxxxx} = \text{cubic terms in } \alpha, \beta \tag{26}$$

and

$$\eta_x + w_t + \frac{1}{2}\beta(\theta^2 - 1)w_{xxt} + \alpha w w_x - \alpha\beta\eta w_{xxt} - \alpha\beta\eta_x w_{xt} + \frac{1}{2}\alpha\beta(\theta^2 - 1)w w_{xxx} + \frac{1}{2}\alpha\beta(\theta^2 + 1)w_x w_{xx} + \frac{5}{24}\beta^2(\theta^2 - 1)\left(\theta^2 - \frac{1}{5}\right)w_{xxxxt} - \alpha\beta\upsilon w_{xx} = \text{cubic terms in } \alpha, \beta \tag{27}$$

Other system of equations correct to second order in α, β can be obtained using the lower order approximations. Keeping all the terms in Eqs. (26) and (27) quadratic in α, β to the right hand side, we have

$$\eta_t + w_x + \beta\left(\frac{1}{2}\theta^2 - \frac{1}{6}\right)w_{xxx} + \alpha(\eta w)_x = \text{quadratic terms in } \alpha, \beta \tag{28}$$

and

$$\eta_x + w_t + \frac{1}{2}\beta(\theta^2 - 1)w_{xxt} + \alpha w w_x = \text{quadratic terms in } \alpha, \beta \tag{29}$$

Differentiating Eq. (28) twice with respect to x, we have

$$w_{xxx} = -\eta_{xxt} - \beta\left(\frac{1}{2}\theta^2 - \frac{1}{6}\right)w_{xxxxx} - \alpha(\eta w)_{xxx} + \text{quadratic terms in } \alpha, \beta$$

Now, let $\lambda \in R$, we get

$$\beta w_{xxx} = \lambda\beta w_{xxx} + (1-\lambda)\beta w_{xxx} = \lambda\beta w_{xxx} + (1-\lambda)\beta\left[-\eta_{xxt} - \beta\left(\frac{1}{2}\theta^2 - \frac{1}{6}\right)w_{xxxxx} - \alpha(\eta w)_{xxx} + \text{cubic terms in } \alpha, \beta\right]$$

$$= \lambda\beta w_{xxx} - (1-\lambda)\beta\eta_{xxt} - (1-\lambda)\beta^2\left(\frac{1}{2}\theta^2 - \frac{1}{6}\right)w_{xxxxx} - \alpha\beta(1-\lambda)(\eta w)_{xxx} + \text{cubic terms in } \alpha, \beta$$

Neglecting higher order terms,

$$(1-\lambda)\beta w_{xxx} = -(1-\lambda)\beta\eta_{xxt} + \text{quadratic terms in } \alpha, \beta$$

$$\therefore \beta w_{xxx} = -\beta\eta_{xxt} + \text{quadratic terms in } \alpha, \beta$$

Therefore,

$$\beta^2 w_{xxxxx} = -\beta^2\eta_{xxxxt} + \text{cubic terms in } \alpha, \beta$$

Thus we may write

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$$\beta^2 w_{xxxxx} = \beta^2 \lambda_1 w_{xxxxx} - (1-\lambda_1)\beta^2 \eta_{xxxxt} + cubic\ terms\ in\ \alpha, \beta$$

Similarly, differentiate Eq. (29) twice with respect to x, we have

$$w_{xxt} = -\eta_{xxx} - \frac{1}{2}\beta(\theta^2 - 1)w_{xxxxt} + \alpha(ww_x)_{xx} + quadratic\ terms\ in\ \alpha, \beta$$

Now, let $\mu \in R$, we get

$$\beta w_{xxt} = (1-\mu)\beta w_{xxt} + \mu\beta w_{xxt}$$

$$= (1-\mu)\beta\left\{-\eta_{xxx} - \frac{1}{2}\beta(\theta^2-1)w_{xxxxt} + \alpha(ww_x)_{xx} + cubic\ terms\ in\ \alpha,\beta\right\} + \mu\beta w_{xxt}$$

$$= -(1-\mu)\beta\eta_{xxx} + \mu\beta w_{xxt} - \frac{1}{2}(1-\mu)\beta^2(\theta^2-1)w_{xxxxt} + (1-\mu)\alpha\beta(ww_x)_{xx} + cubic\ \ terms\ in\ \alpha,\beta$$

Again, neglecting higher order terms,

$$(1-\mu)\beta w_{xxt} = -(1-\mu)\beta\eta_{xxx} + quadratic\ terms\ in\ \alpha,\beta$$

$$\Rightarrow \beta w_{xxt} = -\beta\eta_{xxx} + quadratic\ \ terms\ \ in\ \ \alpha,\beta$$

Also, we may write

$$\beta^2 w_{xxxxt} = -\beta^2\eta_{xxxxx} + cubic\ \ terms\ \ in\ \ \alpha,\beta$$

and

$$\beta^2 w_{xxxxt} = \beta^2\mu_1 w_{xxxxt} - \beta^2(1-\mu_1)\eta_{xxxxx} + cubic\ terms\ in\ \alpha,\beta$$

Hence, Eqs. (26) and (27), we have

$$\eta_t + w_x + \alpha(\eta w)_x + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda\beta w_{xxx} + \frac{1}{2}\left(\theta^2-\frac{1}{3}\right)(1-\lambda)\beta\left[-\eta_{xxt} - \beta\left(\frac{1}{2}\theta^2 - \frac{1}{6}\right)w_{xxxxx} - \alpha(\eta w)_{xxx}\right]$$

$$+\frac{1}{2}\alpha\beta(\theta^2-1)(\eta w_{xx})_x + \frac{5}{24}\left(\theta^2-\frac{1}{5}\right)^2\beta^2\lambda_1 w_{xxxxx} - \frac{5}{24}\left(\theta^2-\frac{1}{5}\right)^2(1-\lambda_1)\beta^2\eta_{xxxxt} = cubic\ \ terms\ \ in\ \alpha,\beta$$

and

$$\eta_x + w_t + \frac{1}{2}(\theta^2-1)(1-\mu)\beta w_{xxt} + \frac{1}{2}(\theta^2-1)\mu\beta\left[-\eta_{xxx} - \frac{1}{2}\beta(\theta^2-1)w_{xxxxt} - \alpha(ww_x)_{xx}\right] + \alpha ww_x + \alpha\beta(\eta\eta_{xx})_x + \frac{1}{2}\alpha\beta(\theta^2-1)ww_{xxx}$$

$$+\frac{1}{2}\alpha\beta(\theta^2+1)w_x w_{xx} + \frac{5}{24}(\theta^2-1)\left(\theta^2-\frac{1}{5}\right)\beta^2\mu_1 w_{xxxxt} - \frac{5}{24}(\theta^2-1)\left(\theta^2-\frac{1}{5}\right)\beta^2(1-\mu_1)\eta_{xxxxx} - \alpha\beta\upsilon w_{xx} = cubic\ \ \ \ terms\ \ \ in\ \alpha,\beta$$

Neglecting cubic terms in α, β, we get

$$\eta_t - b\beta\eta_{xxt} + b_1\beta^2\eta_{xxxxt} = -w_x - \alpha(\eta w)_x - a\beta w_{xxx} + b\alpha\beta(\eta w)_{xxx} - \left(a+b-\frac{1}{3}\right)\alpha\beta(\eta w_{xx})_x - a_1\beta^2 w_{xxxxx} \tag{30}$$

and

$$w_t - d\beta w_{xxt} + d_1\beta^2 w_{xxxxt} = -\eta_x - c\beta\eta_{xxx} - \alpha ww_x - c\alpha\beta(ww_x)_{xx} - \alpha\beta(\eta\eta_{xx})_x + \alpha\beta(c+d)ww_{xxx} + \alpha\beta(c+d-1)w_x w_{xx} - c_1\beta^2\eta_{xxxxx} + \alpha\beta\upsilon w_{xx} \tag{31}$$

where,

$$a = \frac{1}{2}\left(\theta^2-\frac{1}{3}\right)\lambda,\ b = \frac{1}{2}\left(\theta^2-\frac{1}{3}\right)(1-\lambda), c = \frac{1}{2}(1-\theta^2)\mu\ and\ \ d = \frac{1}{2}(\theta^2-1)(\mu-1)$$

$$a_1 = -\frac{1}{4}\left(\theta^2-\frac{1}{3}\right)^2(1-\lambda) + \frac{5}{24}\left(\theta^2-\frac{1}{5}\right)^2\lambda_1$$

$$b_1 = -\frac{5}{24}\left(\theta^2-\frac{1}{5}\right)^2(1-\lambda_1),$$

$$c_1 = \frac{5}{24}\left(1-\theta^2\right)\left(\theta^2 - \frac{1}{5}\right)\left(1-\mu_1\right)$$
$$d_1 = -\frac{1}{4}\left(\theta^2 - 1\right)^2 \mu + \frac{5}{24}\left(\theta^2 - 1\right)\left(\theta^2 - \frac{1}{5}\right)\mu_1$$

Now the change of variables are

$$x = \beta^{\frac{1}{2}}\hat{x},\ t = \beta^{\frac{1}{2}}\hat{t},\ \eta = \alpha^{-1}\hat{\eta},\ w = \alpha^{-1}\hat{w}\ and\ \upsilon = \alpha^{-1}\beta^{-\frac{1}{2}}\hat{\upsilon}$$

In the new variables, Eqs.(30) and (31) can be rewritten as

$$\frac{\alpha^{-1}}{\beta^{\frac{1}{2}}}\hat{\eta}_{\hat{t}} - \frac{\alpha^{-1}}{\beta\beta^{\frac{1}{2}}} b\beta\hat{\eta}_{\hat{x}\hat{x}\hat{t}} + \frac{\alpha^{-1}}{\beta^2\beta^{\frac{1}{2}}} b_1\beta^2\hat{\eta}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{t}} = -\frac{\alpha^{-1}}{\beta^{\frac{1}{2}}}\hat{w}_{\hat{x}} - \frac{\alpha^{-2}}{\beta^{\frac{1}{2}}}\alpha\left(\hat{\eta}\hat{w}\right)_{\hat{x}} - a\beta\frac{\alpha^{-1}}{\beta\beta^{\frac{1}{2}}}\hat{w}_{\hat{x}\hat{x}\hat{x}} + b\frac{\alpha^{-2}}{\beta\beta^{\frac{1}{2}}}\alpha\beta\left(\hat{\eta}\hat{w}\right)_{\hat{x}\hat{x}\hat{x}}$$
$$-\frac{\alpha^{-2}}{\beta\beta^{\frac{1}{2}}}\alpha\beta\left(a+b-\frac{1}{3}\right)\left(\hat{\eta}\hat{w}_{\hat{x}\hat{x}}\right)_{\hat{x}} - a_1\beta^2\frac{\alpha^{-1}}{\beta^2\beta^{\frac{1}{2}}}\hat{w}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{x}}$$

$$\therefore \hat{\eta}_{\hat{t}} - b\hat{\eta}_{\hat{x}\hat{x}\hat{t}} + b_1\hat{\eta}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{t}} = -\hat{w}_{\hat{x}} - \left(\hat{\eta}\hat{w}\right)_{\hat{x}} - a\hat{w}_{\hat{x}\hat{x}\hat{x}} + b\left(\hat{\eta}\hat{w}\right)_{\hat{x}\hat{x}\hat{x}} - \left(a+b-\frac{1}{3}\right)\left(\hat{\eta}\hat{w}_{\hat{x}\hat{x}}\right)_{\hat{x}} - a_1\hat{w}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{x}} \quad (32)$$

and similarly,

$$\hat{w}_{\hat{t}} - d\hat{w}_{\hat{x}\hat{x}\hat{t}} + d_1\hat{w}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{t}} = -\hat{\eta}_{\hat{x}} - c\hat{\eta}_{\hat{x}\hat{x}\hat{x}} - \hat{w}\hat{w}_{\hat{x}} - c\left(\hat{w}\hat{w}_{\hat{x}}\right)_{\hat{x}\hat{x}} - \left(\hat{\eta}\hat{\eta}_{\hat{x}\hat{x}}\right)_{\hat{x}} + (c+d)\hat{w}\hat{w}_{\hat{x}\hat{x}\hat{x}}$$
$$+(c+d-1)\hat{w}_{\hat{x}}\hat{w}_{\hat{x}\hat{x}} - c_1\hat{\eta}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{x}} + \upsilon\hat{w}_{\hat{x}\hat{x}} \quad (33)$$

To determine the wave frequency in terms of wave number depending on linearized terms and positing a solution of the form $e^{i(kx-\omega t)}$, we have from Eq.(32),

$$-i\omega - bik^2\omega - b_1ik^4\omega = -ik + iak^3 - ia_1k^5$$
$$\Rightarrow \omega(k) = k\frac{1-ak^2+a_1k^4}{1+bk^2+b_1k^4}$$

and similarly from Eq.(33),

$$\omega(k) = k\frac{1-\upsilon ik - ck^2 + c_1k^4}{1+dk^2+d_1k^4}$$
$$\omega^2(k) = \left(k\frac{1-ak^2+a_1k^4}{1+bk^2+b_1k^4}\right)\left(k\frac{1-\upsilon ik-ck^2+c_1k^4}{1+dk^2+d_1k^4}\right) = k^2\frac{\left(1-ak^2+a_1k^4\right)\left(1-\upsilon ik-ck^2+c_1k^4\right)}{\left(1+bk^2+b_1k^4\right)\left(1+dk^2+d_1k^4\right)}$$

Hence the phase speed with viscous term for long wave length is

$$c^2(k) = \frac{\omega^2(k)}{k^2} = \frac{\left(1-ak^2+a_1k^4\right)\left(1-\upsilon ik-ck^2+c_1k^4\right)}{\left(1+bk^2+b_1k^4\right)\left(1+dk^2+d_1k^4\right)}$$
$$= 1-\upsilon i\left\{k-(a+b+d)k^3+\left(a_1+ab+ad-b_1-bd-d_1+b^2+2bd+d^2\right)k^5\right\}-(a+b+c+d)k^2$$
$$+\left(a_1+ac+c_1+ab+ad+bc+cd-b_1-bd-d_1+b^2+2bd+d^2\right)k^4$$
$$-\begin{pmatrix} a_1c+ac_1+a_1b+abc+bc_1+a_1d+acd+c_1d-ab_1-abd-ad_1-b_1c-bcd-cd_1+b_1d+bd_1+ab^2 \\ +2abd+ad^2+b^2c+2bcd+cd^2-2bb_1-2b^2d-2bd_1-2b_1d-2bd^2-2dd_1+b^3+3b^2d+3bd^2+d^3 \end{pmatrix}k^6+o\left(k^7\right)$$
$$\therefore c^2(k) = 1-\upsilon i\left\{k-(a+b+d)k^3+\left\{a_1+b(a+b+d)+ad-b_1-d_1+d^2\right\}k^5\right\}-(a+b+c+d)k^2$$
$$+\left\{a_1+ac+c_1+b(a+b+c+d)+ad+cd-b_1-d_1+d^2\right\}k^4$$
$$-\begin{pmatrix} -b_1(a+b+c+d)-d_1(a+b+c+d)+b^2(a+b+c+d)+d^2(a+b+c+d) \\ +bd(a+c)+ac(b+d)-bb_1-dd_1+a_1(b+c+d)+c_1(a+b+d) \end{pmatrix}k^6+o\left(k^7\right)$$

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$$=1-\frac{1}{3}k^2+\frac{2}{15}k^4-\upsilon i\left\{\begin{array}{l} k-\frac{1}{2}\left(\frac{2}{3}-\mu+\mu\theta^2\right)k^3 \\ +\frac{1}{120}\left[\begin{array}{l}\left(21-20\mu+30\mu^2-5\mu_1\right)+10\left(-3+2\mu-6\mu^2+3\mu_1\right)\theta^2 \\ +5\left(5+6\mu^2-5\mu_1\right)\theta^4\end{array}\right]k^5 \end{array}\right\}$$

$$-\frac{1}{720}\left[\begin{array}{l}\left(51+\lambda+\lambda_1+15\mu-15\mu_1\right)-\left(67+13\lambda+13\lambda_1+105\mu-105\mu_1\right)\theta^2 \\ +5\left(1+11\lambda+11\lambda_1+33\mu-33\mu_1\right)\theta^4+75\left(1-\lambda-\lambda_1-\mu+\mu_1\right)\theta^6\end{array}\right]k^6+0\left\{(k)^7\right\}$$

$$\therefore c^2(k)=\frac{\tanh kh}{k}-\upsilon i\left\{k+f(\mu,\theta)k^3+g(\mu,\mu_1,\theta)k^5\right\}+h(\lambda,\lambda_1,\mu,\mu_1,\theta)k^6+0\left\{(k)^7\right\} \tag{34}$$

Where

$$f(\mu,\theta)=-\frac{1}{2}\left(\frac{2}{3}-\mu+\mu\theta^2\right),$$

$$g(\mu,\mu_1,\theta)=\frac{1}{120}\left[\left(21-20\mu+30\mu^2-5\mu_1\right)+10\left(-3+2\mu-6\mu^2+3\mu_1\right)\theta^2+5\left(5+6\mu^2-5\mu_1\right)\theta^4\right]$$

$$h(\lambda,\lambda_1,\mu,\mu_1,\theta)=-\frac{1}{720}\left[\begin{array}{l}\left(51+\lambda+\lambda_1+15\mu-15\mu_1\right)-\left(67+13\lambda+13\lambda_1+105\mu-105\mu_1\right)\theta^2 \\ +5\left(1+11\lambda+11\lambda_1+33\mu-33\mu_1\right)\theta^4+75\left(1-\lambda-\lambda_1-\mu+\mu_1\right)\theta^6\end{array}\right]$$

From Eq. (34), it is seen that the first three terms which are independent of parameters $\theta, \lambda, \lambda_1, \mu, \mu_1$ correspond to the expansion of the full linearized dispersion relation of the Navier-Stokes equations.

III. Results and Discussions

Neglecting imaginary and higher order terms in equation (34), we obtain the following figure 1. It is obvious that the phase velocity is the wave of a single wavelength. This figure also represents a wave of a single wavelength of small surface elevation and small amplitude.

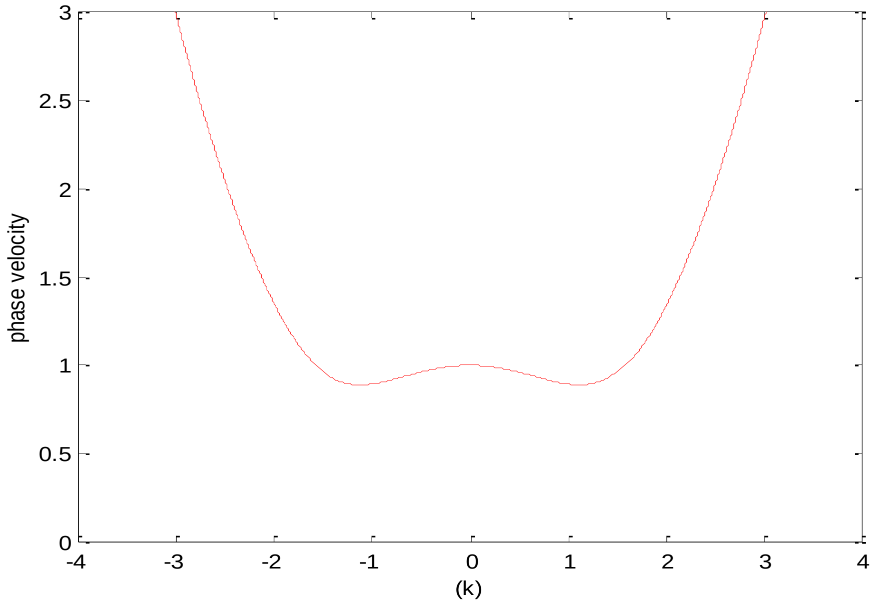

Fig. 1. Phase velocity

IV. Conclusion

Waves on the surface of an ideal fluid are governed by the Navier-Stokes equations in the canal from which a few number of mathematical models for irrotational, viscous and incompressible flow are created. Expanding velocity potential in power series with respect to the vertical coordinates, odd terms have been rejected. Using scaled horizontal velocity at the bottom of the channel, linear wave equation is formulated. Also using linearized term in the higher order generalization of Navier-Stokes equations and positing a solution of the form $e^{i(kx-wt)}$, the wave frequency has been derived. Then phase speed for small wave number is also established where the first three terms defines the full linearized dispersion relation and other terms are due to viscous force.

REFERENCES

[1] J. L Bona, M. Chen and J. –C. Saut, "Boussinesq equations and other systems for small-amplitude long waves in nonlinear Diespersive Media. I: derivation and linear theory", *J. Nonlinear Sci*., vol. 12, pp. 283-318, 2002.

[2] J. L Bona, M. Chen and J. –C. Saut, "Boussinesq equations and other systems for small-amplitude long waves in nonlinear diespersive media. II: The nolinear theory", *Institute of Physics Publication, Nonlinearity*, vol. 17, pp. 925-952, 2004.

[3] J. L. Bona and M. Chen, "A Boussinesq system for two way propagation of nonlinear dispersive waves", *Physica D*., vol. 116, pp. 191-224, 1998.

[4] A. Ouahsine, F. Sergent and S. Hadji, "Modeling on nonlinear waves by an extended Boussinesq model", *Engineering Applications of Computational Mechanics*, vol. 2, no. 1, pp. 11-21, 2008.

[5] F. Marche, "Derivation of new two dimensional viscous shallow water model with varying topography, bottom friction and capillary effects", *European Journal of Mechanics-B/Fluids*, vol. 26, pp. 49-63, 2007.

[6] D. Dutykh and F. Dias, "Viscous potential free surface flows ia a fluid layer of finite depth", *C. R. Math. Acad. Sci. Paris*, vol. 345, issue 2, pp. 113-118, 2007.

[7] Denys Dutykh, "Visco-Potential Free Surface Flows and Long Wave Modeling", *Eur. J. Mech. B/ Fluids*., vol. 28, no. 3, pp. 430-443, 2009.

[8] R. Barros, S. L. Gavrilyuk and V. M. Teshukov, "Dispersive nonlinear waves in two layer flows with free surface, I. model derivation and general properties", *Stud. Appl. Math*., vol. 119, pp. 191-211, 2007.

[9] J. L. Bona, T. Colin and D. Lannes, "Long waves approximations for water waves", *Arch. Rational Mech. Anal*., vol. 178, pp. 373-410, 2005.

[10] T .H. C. Herbers, S. Elgar, N. A. Sarap and R. T. Guza, "Nonlinear dispersion of gravity waves in shallow water", Americam Meteorological Siciety, 2002.

[11] Y. A. Li, J. M. Hyman and W. Choi, "A numerical study of the exact evolution equations for surface waves in water of finite depth", *Stud. Appl. Math.*, vol. 113, pp. 303-324, 2004.

[12] P. L. –F. Liu and A. Orfila, "Viscous effects on transient long wave propagation", *J. Fluid Mech*., vol. 520, pp. 83-92, 2004.

[13] V. Duchene, "Asymptotic shallow water models for internal waves in a two fluid system with a free surface", *Siam J. Math. Anal*., vol. 42, no. 5, pp. 2229-2260, 2010.

[14] D. Lannes, "Well Posedness of the water waves equations", *J. Amer. Math. Soc*., vol. 18, no. 3, pp. 605-654, 2005.

[15] D. Bresch and B. Desjardins, "On the construction of approximate solutions for the two dimensional viscous shallow water model and for compressible navier-stokes models", *J. Math. Pure Appl*., vol. 86, issue 4, pp. 362-368, 2006.

[16] M. Chen, S. Dumont, L. Dupaigue and O. Goubet, "Decay of solutions to a water wave model with a nonlocal viscous dispersive term", *Discret Conten. Dyn. Syst*., vol. 27, no. 4, pp. 1473-1492, 2010.

[17] J. L. Bona and H. Chen, "Comparison of model equations for small amplitude long waves", *Nonlinear Anal*., vol. 38, pp. 625-647, 1999.