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Fuzzy Possibility Clustering Algorithm Based on Complete Mahalanobis Distances

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Abstract**—***The two known fuzzy partition clustering algorithms, FCM and FPCM are based on Euclidean distance function, which can only be used to detect spherical structural clusters. GK clustering algorithm and GG clustering algorithm, were developed to detect non-spherical structural clusters, but both of them need additional prior information. In our previous studies, we developed four improved algorithms, FCM-M, FPCM-M, FCM-CM and FPCM-CM based on unsupervised Mahalanobis distance without any additional prior information. In first two algorithms, only the local covariance matrix of each cluster was considered, In last two algorithms, not only the local covariance matrix of each cluster but also the overall covariance matrix was considered, and FPCM-CM is the better one. In this paper, a more information about "separable criterion" is considered, and the further improved new algorithm, "fuzzy possibility c-mean based on complete Mahalanobis distance and separable criterion, (FPCM-CMS)" is proposed. It can get more information and higher accuracy by considering the additional separable criterion than FPCM-CM. A real data set was applied to prove that the performance of the FPCM-CMS algorithm is better than those of above six algorithms*.

Keywords**—***Fuzzy partition Custering Algorithms, Fuzzy Possibility Clustering Algorithm, Complete Mahalanobis distances*.

I. Introduction

The clustering analysis plays an important role in data analysis and interpretation. It groups the data into classes or clusters so that the data objects within a cluster have high similarity in comparison to one another, but are very dissimilar to those data objects in other clusters.

Fuzzy partition clustering is a branch in cluster analysis, it is widely used in pattern recognition field. The well known fuzzy Possibility partition clustering algorithms, PCM [3], and FPCM [4] are proposed to improve the problems of outlier and noise in FCM [1], [2], but the above three algorithms were based on Euclidean distance function, which can only be used to detect spherical structural clusters.

Extending Euclidean distance to Mahalanobis distance, Gustafson-Kessel (GK) clustering algorithm [5] and Gath-Geva (GG) clustering algorithm [6], are developed to detect non-spherical structural clusters, but both of them needed additional prior information. In our previous studies, we developed four improved algorithms, FCM-M [7], [8], FPCM-M [9], FCM-CM [10] and FPCM-CM [11], [12] based on unsupervised Mahalanobis distance without any additional prior information. In first two algorithms, only the local covariance matrix of each cluster was considered, In last two algorithms, not only the local covariance matrix of each cluster but also the overall covariance matrix was considered, and FPCM-CMS is the better one.

In this paper, a more information about "separable criterion" is considered [13], and the further improved new algorithm, "fuzzy possibility c-mean based on complete Mahalanobis distance and separable criterion, (FPCM-CMS)" is proposed [14]. It can get more information and higher accuracy by considering the additional separable criterion than FPCM-CM. A real data set was applied to prove that the performance of the FPCM-CMS algorithm is better than those of above six algorithms.

A real data set was applied to prove that the performance of the FPCM-CMS algorithm is better than those of the previous six algorithms.

This paper is organized as followings: Fuzzy c-mean algorithm is introduced in section II(A), Fuzzy possibility c-mean algorithm is introduced in section II(B), FPCM-M algorithm is introduced in section II(C). FCM-CMS algorithm is described in section II(D). FPCM-CMS algorithm is described in section II(E), Experiment and result are described in section III and final section is for conclusions and future works [15].

II. Literature Review

Extending Euclidean distance to Mahalanobis distance, the well known fuzzy partition clustering algorithms, Gustafson-Kessel (GK) clustering algorithm and Gath-Geva (GG) clustering algorithm were developed to detect non- spherical structural clusters, but these two algorithms fail to consider the relationships between cluster centers in the objective function, GK algorithm must have prior information of shape volume in each data class, otherwise, it can only be considered to detect the data classes with same volume. GG algorithm must have prior probabilities of the clusters.

A. Fuzzy c-Mean Algorithm

The objective function used in FCM is given by the following Equation.

$$J_{FCM}^{m}\left(U,A,X\right)=\sum_{i=1}^{c}\sum_{j=1}^{n}\mu_{ij}^{m}d_{ij}^{2}=\sum_{i=1}^{c}\sum_{j=1}^{n}\mu_{ij}^{m}\left\|\underline{x}_j-\underline{a}_i\right\|^2 \tag{1}$$

$\mu_{ij}\in\left[0,1\right]$ is the membership degree of data object $\underline{x}_j$ in cluster C_i, and it satisfies the following constraint given by Equation (2)

$$\sum_{i=1}^{c}\mu_{ij}=1,\ \forall j=1,2,...,n \tag{2}$$

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C is the number of clusters, m is the fuzzifier, m>1,which controls the fuzziness of the method. They are both parameters and need to be specified before running the algorithm. $d_{ij}^2 = \|\underline{x}_j - \underline{a}_i\|^2$ is the square Euclidean distance between data object $\underline{x}_j$ to center $\underline{a}_i$.

Minimizing objective function (1) with constraint (2), the updating function for $\underline{a}_i$ and μ_{ij} is obtained as (3) and (4),

$$\underline{a}_i = \frac{\sum_{j=1}^{n} \mu_{ij}{}^m \underline{x}_j}{\sum_{j=1}^{n} \mu_{ij}{}^m} \quad i = 1, 2, \ldots, c \tag{3}$$

$$\mu_{ij} = \left[\sum_{l=1}^{c} \left[\frac{(\underline{x}_j - \underline{a}_i)'(\underline{x}_j - \underline{a}_i)}{(\underline{x}_j - \underline{a}_l)'(\underline{x}_j - \underline{a}_l)} \right]^{\frac{1}{m-1}} \right]^{-1} \tag{4}$$

B. Fuzzy Possibility C-Mean Algorithm

The improved fuzzy partition clustering algorithms "Fuzzy Possibility C-Mean (FPCM)" is given by Equation (5)

$$J_{FPCM}^m(U, T, A, X) = \sum_{i=1}^{c}\sum_{j=1}^{n}\left(\mu_{ij}^m + t_{ij}^{\delta}\right)\|\underline{x}_j - \underline{a}_i\|^2 \tag{5}$$

constraints：membership

$$\sum_{i=1}^{c} \mu_{ij} = 1, \ \forall j = 1, 2, \ldots, n, \tag{6}$$

typicality $\sum_{j=1}^{n} t_{ij} = 1, \ \forall i = 1, 2, \ldots, c$ (7)

Minimizing objective function (5) with constraint (6) and (7), the updating function for $\underline{a}_i$, μ_{ij} and t_{ij} is obtained as (8), (9) and (10)

$$\underline{a}_i = \frac{\sum_{j=1}^{n}(\mu_{ij}^m + t_{ij}^{\delta})\underline{x}_j}{\sum_{j=1}^{n}(\mu_{ij}^m + t_{ij}^{\delta})}, \quad i = 1, 2, \ldots, c \tag{8}$$

$$\mu_{ij} = \left[\sum_{l=1}^{c} \left[\frac{(\underline{x}_j - \underline{a}_i)'(\underline{x}_j - \underline{a}_i)}{(\underline{x}_j - \underline{a}_l)'(\underline{x}_j - \underline{a}_l)} \right]^{\frac{1}{m-1}} \right]^{-1}, \tag{9}$$

$i = 1, 2, \ldots, c$, $j = 1, 2, \ldots, n$

$$t_{ij} = \left[\sum_{l=1}^{n} \left[\frac{(\underline{x}_j - \underline{a}_i)'(\underline{x}_j - \underline{a}_i)}{(\underline{x}_l - \underline{a}_i)'(\underline{x}_l - \underline{a}_i)} \right]^{\frac{1}{\delta-1}} \right]^{-1} \tag{10}$$

$i = 1, 2, \ldots, c$. $j = 1, 2, \ldots, n$

C. FPCM-M Algorithm

The improved fuzzy partition clustering algorithms "Fuzzy Possibility C-Mean (FPCM)" is given by Equation (5)

For improving the FPCM algorithm, we added the class covariance matrix and a regulating factor of covariance matrix, $-\ln\left|{}_{+}\Sigma_i^{-1}\right|$, to each class in objective function (5). The improved new algorithm, "Fuzzy Possibility C-Mean based on Mahalanobis distance (FPCM-M)", is obtained, and the objective function of FPCM-M is given as (11) and constraints (12);

$$J_{FPCM-M}^m(U, T, A, \Sigma, X) = \sum_{i=1}^{c}\sum_{j=1}^{n}\left(\mu_{ij}^m + t_{ij}^{\delta}\right)\left[(\underline{x}_j - \underline{a}_i)'\,\Sigma_i^{-1}(\underline{x}_j - \underline{a}_i) - \ln\left|\Sigma_i^{-1}\right|\right] \tag{11}$$

constraints $\sum_{i=1}^{c} \mu_{ij} = 1, \ \forall j = 1, 2, \ldots, n$ (12)

$$\sum_{j=1}^{n} t_{ij} = 1, \ \forall i = 1, 2, \ldots, c \tag{13}$$

Minimizing objective function (11) with constraint (12), (13) the updating function for $\underline{a}_i$, μ_{ij} , t_{ij} and Σ_i is obtained as (14), (15), (16) and (17)

$$\underline{a}_i = \left[\sum_{j=1}^{n} \mu_{ij}^m \Sigma_i^{-1}\right]^{-1} \sum_{j=1}^{n} \mu_{ij}^m \Sigma_i^{-1} \underline{x}_j \tag{14}$$

$i = 1, 2, \ldots, c$

$$\mu_{ij} = \left[\sum_{s=1}^{c} \left[\frac{(\underline{x}_j - \underline{a}_i)'\,\Sigma_i^{-1}(\underline{x}_j - \underline{a}_i) - \ln\left|\Sigma_i^{-1}\right|}{(\underline{x}_j - \underline{a}_s)'\,\Sigma_i^{-1}(\underline{x}_j - \underline{a}_s) - \ln\left|\Sigma_i^{-1}\right|} \right]^{\frac{1}{m-1}} \right] \tag{15}$$

$$t_{ij} = \left[\sum_{s=1}^{n} \left[\frac{(\underline{x}_j - \underline{a}_i)'\Sigma_i^{-1}(\underline{x}_j - \underline{a}_i) - \ln\left|\Sigma_i^{-1}\right| -}{(\underline{x}_s - \underline{a}_i)'\Sigma_i^{-1}(\underline{x}_s - \underline{a}_i) - \ln\left|\Sigma_s^{-1}\right|} \right]^{\frac{1}{\delta-1}} \right]^{-1} \tag{16}$$

$$\Sigma_i = \frac{\sum_{j=1}^{n}\left(\mu_{ij}^m + t_{ij}^{\delta}\right)(\underline{x}_j - \underline{a}_i)(\underline{x}_j - \underline{a}_i)'}{\sum_{j=1}^{n}\left(\mu_{ij}^m + t_{ij}^{\delta}\right)} \tag{17}$$

D. FCM-CMS Algorithm

Now, for improving the above two problems, we added a regulating factor of covariance matrix,

$$-\ln\left|{}_{+}\Sigma_i^{-1}\right| - (\underline{a}_i - \underline{a}_t)'\,\Sigma_t^{-1}(\underline{a}_i - \underline{a}_t)$$
$$-\frac{1}{c(c-1)}\sum_{i=1}^{c}\sum_{l=1}^{c} v_{il}^m (\underline{a}_i - \underline{a}_l)'(\underline{a}_i - \underline{a}_l),$$

to each class in objective function, and deleted the constraint of the determinant of covariance matrices, $|M_i| = \rho_i$,in GK Algorithm as the objective function (18). The improved new algorithm, "Fuzzy C-Mean based on Mahalanobis Complete distance (FCM-CMS)", is obtained as following.

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$$J^{m}_{FCM-CMS}\left(U,A,\Sigma,X\right)$$
$$=\sum_{i=1}^{c}\sum_{j=1}^{n}\mu_{ij}^{m}\left[\left(\underline{x}_j-\underline{a}_i\right)'\Sigma_i^{-1}\left(\underline{x}_j-\underline{a}_i\right)-\ln\left|\Sigma_i^{-1}\right|-\left(\underline{a}_i-\underline{a}_t\right)'\Sigma_t^{-1}\left(\underline{a}_i-\underline{a}_t\right)\right]$$
$$-\frac{1}{c\left(c-1\right)}\sum_{i=1}^{c}\sum_{l=1}^{c}v_{il}^{m}\left(\underline{a}_i-\underline{a}_l\right)'\left(\underline{a}_i-\underline{a}_l\right) \tag{18}$$

Constraints: membership,

$$\sum_{i=1}^{c}\mu_{ij}=1,\ \forall\ j=1,2,...,n, \tag{19}$$

where

$$v_{il}^{(k)}=\frac{w_{il}^{k-1}-\min\limits_{1\le r,s\le c}w_{rs}^{k-1}}{\max\limits_{1\le r,s\le c}w_{rs}^{k-1}-\min\limits_{1\le r,s\le c}w_{rs}^{k-1}},$$

and

$$w_{rs}^{k-1}=\left\|\underline{a}-\underline{a}_r^{(k-1)}\right\|^2+\left\|\underline{a}-\underline{a}_s^{(k-1)}\right\|^2$$

Using the Lagrange multiplier method, to minimize the objective function (22) with constraint (23) respect to parameters $\underline{a}_i$, μ_{ij}, Σ_i, we can obtain the solutions as (25). To avoid the singular problem and to select the better initial membership matrix , the updating functions for $\underline{a}_i$, μ_{ij}, and Σ_i are obtained as (24) ~ (25).

$$\underline{a}_i=F^{-1}\left[\sum_{j=1}^{n}\mu_{ij}^{m}\left(\Sigma_i^{-1}\underline{x}_j-\Sigma_t^{-1}\underline{a}_t\right)-\frac{1}{c\left(c-1\right)}\sum_{l=1}^{c}v_{il}^{m}\underline{a}_l\right] \tag{20}$$

where $F=\left[\sum_{j=1}^{n}\mu_{ij}^{m}\left(\Sigma_i^{-1}-\Sigma_t^{-1}\right)-\frac{1}{c\left(c-1\right)}\sum_{l=1}^{c}v_{il}^{m}I\right]$

$$\mu_{ij}=\left[\sum_{s=1}^{c}\left[\frac{\left(\underline{x}_j-\underline{a}_i\right)'\Sigma_i^{-1}\left(\underline{x}_j-\underline{a}_i\right)-\ln\left|\Sigma_i^{-1}\right|-\left(\underline{a}_i-\underline{a}_t\right)'\Sigma_t^{-1}\left(\underline{a}_i-\underline{a}_t\right)}{\left(\underline{x}_j-\underline{a}_s\right)'\Sigma_s^{-1}\left(\underline{x}_j-\underline{a}_s\right)-\ln\left|\Sigma_s^{-1}\right|-\left(\underline{a}_s-\underline{a}_t\right)'\Sigma_t^{-1}\left(\underline{a}_s-\underline{a}_t\right)}\right]^{\frac{1}{m-1}}\right]^{-1} \tag{21}$$

where

$$\underline{a}_t=\frac{1}{n}\sum_{j=1}^{n}\underline{x}_j,\ \Sigma_t=\frac{1}{n}\sum_{j=1}^{n}\left(\underline{x}_j-\underline{a}_t\right)\left(\underline{x}_j-\underline{a}_t\right)'$$

$$\Sigma_i=\frac{\sum_{j=1}^{n}\mu_{ij}^{m}\left(\underline{x}_j-\underline{a}_i\right)\left(\underline{x}_j-\underline{a}_i\right)'}{\sum_{j=1}^{n}\mu_{ij}^{m}}$$

$$\Sigma_i=\sum_{s=1}^{p}\lambda_{si}\Gamma_{si}\Gamma'_{si},\ i=1,2,...,c$$

$$\Rightarrow {}_{+}\Sigma_i^{-1}=\sum_{s=1}^{p}\left(\lambda_{si}^{-1}\right)^{+}\Gamma_{si}\Gamma'_{si},$$

$$\left(\lambda_{si}^{-1}\right)^{+}=\begin{cases}\lambda_{si}^{-1} & if\ \lambda_{si}>0\\ 0 & if\ \lambda_{si}\le 0\end{cases}$$

$$\left|{}_{+}\Sigma_i^{-1}\right|=\prod_{1\le s\le p,\lambda_{si}>0}\lambda_{si}^{-1}$$

The new fuzzy clustering algorithm (FCM-CMS) can be summarized in the following steps:

Step 1: Determining the number of cluster; c and m-value (let m=2), given converging error, $\varepsilon>0$ (such as $\varepsilon=0.001$).

Method 1: choose the result membership matrix of FCM algorithm as the initial one.

Method 2: let $\underline{a}_i^{(0)}, i=1,2,...,c$ be the result centers of k-mean algorithm, and $d_{ij}=\left\|\underline{x}_j-\underline{a}_i^{(0)}\right\|$ be distances between data object $\underline{x}_j$ to center $\underline{a}_i^{(0)}$.

$$d_M=\max_{1\le i\le c,1\le j\le n}d_{ij}, \tag{22}$$

$$\mu_{ij}^{(0)}=\frac{\left(d_M-d_{ij}\right)}{\sum_{s=1}^{c}\left(d_M-d_{sj}\right)},\ i=1,2,...,c,\ j=1,2,...,n \tag{23}$$

$$\underline{a}_i^{(0)}=\left(\sum_{j=1}^{n}\left[\mu_{ij}^{(0)}\right]\underline{x}_j\right)\left(\sum_{j=1}^{n}\left[\mu_{ij}^{(0)}\right]\right)^{-1},\quad i=1,2,...,c$$
$$\Rightarrow\Sigma_i^{(0)}=\frac{\sum_{j=1}^{n}\left(\mu_{ij}^{(0)}\right)^{m}\left(\underline{x}_j-\underline{a}_i^{(0)}\right)\left(\underline{x}_j-\underline{a}_i^{(0)}\right)'}{\sum_{j=1}^{n}\left(\mu_{ij}^{(0)}\right)^{m}} \tag{24}$$

$$U^{(0)}=\begin{bmatrix}\mu_{11}^{(0)}\ \mu_{12}^{(0)}\ldots\mu_{1n}^{(0)}\\ \mu_{21}^{(0)}\ \mu_{22}^{(0)}\ldots\mu_{2n}^{(0)}\\ \ldots\ \ \ldots\ \ \ldots\\ \mu_{c1}^{(0)}\ \mu_{c2}^{(0)}\ldots\mu_{cn}^{(0)}\end{bmatrix}$$
$$=\begin{bmatrix}\mu_1^{(0)}\left(\underline{x}_1\right)\ \mu_1^{(0)}\left(\underline{x}_2\right)\ldots\mu_1^{(0)}\left(\underline{x}_n\right)\\ \mu_2^{(0)}\left(\underline{x}_1\right)\ \mu_2^{(0)}\left(\underline{x}_2\right)\ldots\mu_2^{(0)}\left(\underline{x}_n\right)\\ \ldots\ \ \ldots\ \ \ldots\\ \mu_c^{(0)}\left(\underline{x}_1\right)\ \mu_c^{(0)}\left(\underline{x}_2\right)\ldots\mu_c^{(0)}\left(\underline{x}_n\right)\end{bmatrix} \tag{25}$$

$$T^{(0)}=\begin{bmatrix}t_{11}^{(0)}\ t_{12}^{(0)}\ldots t_{1n}^{(0)}\\ t_{21}^{(0)}\ t_{22}^{(0)}\ldots t_{2n}^{(0)}\\ \ldots\ \ \ldots\ \ \ldots\\ t_{c1}^{(0)}\ t_{c2}^{(0)}\ldots t_{cn}^{(0)}\end{bmatrix}$$
$$=\begin{bmatrix}t_1^{(0)}\left(\underline{x}_1\right)\ t_1^{(0)}\left(\underline{x}_2\right)\ldots t_1^{(0)}\left(\underline{x}_n\right)\\ t_2^{(0)}\left(\underline{x}_1\right)\ t_2^{(0)}\left(\underline{x}_2\right)\ldots t_2^{(0)}\left(\underline{x}_n\right)\\ \ldots\ \ \ldots\ \ \ldots\\ t_c^{(0)}\left(\underline{x}_1\right)\ t_c^{(0)}\left(\underline{x}_2\right)\ldots t_c^{(0)}\left(\underline{x}_n\right)\end{bmatrix}$$

$$A^{(0)}=\left[\underline{a}_1^{(0)}\ \underline{a}_2^{(0)}\ldots\underline{a}_c^{(0)}\right] \tag{26}$$

$$\Sigma^{(0)}=\left[\Sigma_1^{(0)},\Sigma_2^{(0)},...,\Sigma_c^{(0)}\right] \tag{27}$$

$$w_{rs}^{(0)}=\left\|\underline{a}-\underline{a}_r^{(0)}\right\|^2+\left\|\underline{a}-\underline{a}_s^{(0)}\right\|^2$$

Step 2: Find

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$$v_{il}^{(k)} = \frac{w_{il}^{k-1} - \min\limits_{1\le r,s\le c} w_{rs}^{k-1}}{\max\limits_{1\le r,s\le c} w_{rs}^{k-1} - \min\limits_{1\le r,s\le c} w_{rs}^{k-1}},$$

where

$$w_{rs}^{k-1} = \left\| \underline{a} - \underline{a}_r^{(k-1)} \right\|^2 + \left\| \underline{a} - \underline{a}_s^{(k-1)} \right\|^2$$

$$\mu_{ij}^{(k)} = \left[\sum_{s=1}^{c} \left[\frac{\left(\underline{x}_j - \underline{a}_i^{(k)}\right)' \left[\Sigma_i^{(k)}\right]^{-1} \left(\underline{x}_j - \underline{a}_i^{(k)}\right) - \ln\left|\Sigma_i^{(k)}\right|^{-1} - \left(\underline{a}_i^{(k)} - \underline{a}_t\right)' \Sigma_t^{-1} \left(\underline{a}_i^{(k)} - \underline{a}_t\right)}{\left(\underline{x}_j - \underline{a}_s^{(k)}\right)' \left[\Sigma_s^{(k)}\right]^{-1} \left(\underline{x}_j - \underline{a}_s^{(k)}\right) - \ln\left|\Sigma_s^{(k)}\right|^{-1} - \left(\underline{a}_s^{(k)} - \underline{a}_t\right)' \Sigma_t^{-1} \left(\underline{a}_s^{(k)} - \underline{a}_t\right)} \right]^{\frac{1}{m-1}} \right]^{-1} \tag{28}$$

$$a_i^{(k)} = \left[F^{(k)}\right]^{-1} \left[\sum_{j=1}^{n} \left[\mu_{ij}^{(k-1)}\right]^m \left[\left[\Sigma_i^{(k-1)}\right]^{-1} x_j - \Sigma_t^{-1} \underline{a}_t \right] - \frac{1}{c(c-1)} \sum_{l=1}^{c} \left[v_{il}^{(k)}\right]^m \underline{a}_l^{(k-1)} \right] \tag{29}$$

$$, i = 1, 2, \ldots, c$$

$$F^{(k)} = \sum_{j=1}^{n} \left[\mu_{ij}^{(k-1)}\right]^m \left[\left[\Sigma_i^{(k-1)}\right]^{-1} - \Sigma_t^{-1} \right] - \frac{1}{c(c-1)} \sum_{l=1}^{c} \left[v_{il}^{(k)}\right]^m I \tag{30}$$

$$\Rightarrow \Sigma_i^{(k)} = \frac{\sum_{j=1}^{n} \mu_{ij}^m \left(\underline{x}_j - \underline{a}_i^{(k)}\right)\left(\underline{x}_j - \underline{a}_i^{(k)}\right)'}{\sum_{j=1}^{n} \mu_{ij}^m}$$

where

$$\Sigma_i^{(k)} = \sum_{s=1}^{p} \lambda_{si}^{(k)} \Gamma_{si}^{(k)} \left(\Gamma_{si}^{(k)}\right)',$$

$$\left[\lambda_{si}^{-1}\right]^{(k)} = \begin{cases} \left[\lambda_{si}^{(k)}\right]^{-1} & if\ \lambda_{si}^{(k)} > 0 \\ 0 & if\ \lambda_{si}^{(k)} = 0 \end{cases} \tag{31}$$

$$\left[\Sigma_i^{(k)}\right]^{-1} = \sum_{s=1}^{p} \left[\lambda_{si}^{(-1)}\right]^{(k)} \Gamma_{si}^{(k)} \left(\Gamma_{si}^{(k)}\right)'$$

$$\left|\Sigma_i^{(k)}\right|^{-1} = \prod_{1\le s\le p, \lambda_{si}^{(k)}>0} \left[\lambda_{si}^{(k)}\right]^{-1}$$

Step 3: Increment k until $\max\limits_{1\le i\le c} \left\| \underline{a}_i^{(k)} - \underline{a}_i^{(k-1)} \right\| < \varepsilon$.

E. New Algorithm - FPCM-CMS

The clustering optimization was based on objective functions. The choice of an appropriate objective function is the point to the success of the cluster analysis. In FCM-M algorithm, it didn't consider the relationships between cluster centers in the objective function, now, we proposed an improved Fuzzy C-Mean algorithm, FPCM-CMS, which is not only based on unsupervised Mahalanobis distance, but also considering the relationships between cluster centers, and the relationships between the center of all points and the cluster centers in the objective function, the singular and the initial values problems were also solved. Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a set of n data points represented by p-dimensional feature vectors $x_j = (x_{1j}, x_{2j}, \ldots, x_{pj})' \in R^p$. The p×n data matrix Z has the cluster center matrix $A=[a_1, \ldots, a_c]$, $1<c<n$ and the membership matrix $U = [\mu_{ij}]_{cxn}$, where μ_{ij} is the membership value of x_j belonging to a_i. $V = [v_{ik}]_{cxc}$ express the weighting matrix, and v_{ik} is the weighting value between v_i and v_k. The fuzzy exponent m is greater than 1. Thus, we can obtained the objective function of Fuzzy Possibility c-Mean based on Complete Mahalanobis distance and separable criterion (FPCM-CMS) as following.

$$J_{FPCM-CMS}^m (U, T, A, \Sigma, X) =$$

$$\sum_{i=1}^{c}\sum_{j=1}^{n} \left(\mu_{ij}^m + t_{ij}^\delta\right) \left[\left(\underline{x}_j - \underline{a}_i\right)' \Sigma_i^{-1} \left(\underline{x}_j - \underline{a}_i\right) - \ln\left|\Sigma_i^{-1}\right| - \left(\underline{a}_i - \underline{a}_t\right)' \Sigma_t^{-1} \left(\underline{a}_i - \underline{a}_t\right) \right] \tag{32}$$

$$- \frac{1}{c(c-1)} \sum_{i=1}^{c}\sum_{l=1}^{c} v_{il}^m \left(\underline{a}_i - \underline{a}_l\right)' \left(\underline{a}_i - \underline{a}_l\right)$$

Constraints: membership

$$\sum_{i=1}^{c} \mu_{ij} = 1, \ \forall j = 1, 2, \ldots, n$$

typicality

$$\sum_{j=1}^{n} t_{ij} = 1, \ \forall i = 1, 2, \ldots, c$$

where

$$\underline{a}_t = \frac{1}{n} \sum_{j=1}^{n} \underline{x}_j,$$

$$\Sigma_t = \frac{1}{n} \sum_{j=1}^{n} \left(\underline{x}_j - \underline{a}_t\right)\left(\underline{x}_j - \underline{a}_t\right)'$$

$$v_{il}^{(k)} = \frac{w_{il}^{k-1} - \min\limits_{1\le r,s\le c} w_{rs}^{k-1}}{\max\limits_{1\le r,s\le c} w_{rs}^{k-1} - \min\limits_{1\le r,s\le c} w_{rs}^{k-1}}, \tag{33}$$

where

$$w_{rs}^{k-1} = \left\| \underline{a} - \underline{a}_r^{(k-1)} \right\|^2 + \left\| \underline{a} - \underline{a}_s^{(k-1)} \right\|^2$$

$$\underline{a}_i = F^{-1} \left[\sum_{j=1}^{n} \left(\mu_{ij}^m + t_{ij}^\delta\right) \left[\Sigma_i^{-1} \underline{x}_j - \Sigma_t^{-1} \underline{a}_t\right] \right] - F^{-1} \frac{1}{c(c-1)} \sum_{l=1}^{c} v_{il}^m \underline{a}_l$$

$$F = \sum_{j=1}^{n} \left(\mu_{ij}^m + t_{ij}^\delta\right) \left[\Sigma_i^{-1} - \Sigma_t^{-1}\right] - \frac{1}{c(c-1)} \sum_{l=1}^{c} v_{il}^m I$$

$$\mu_{ij} = \left[\sum_{s=1}^{c} \left[\frac{\left(\underline{x}_j - \underline{a}_i\right)' \Sigma_i^{-1} \left(\underline{x}_j - \underline{a}_i\right) - \ln\left|\Sigma_i^{-1}\right| - \left(\underline{a}_i - \underline{a}_t\right)' \Sigma_t^{-1} \left(\underline{a}_i - \underline{a}_t\right)}{\left(\underline{x}_j - \underline{a}_s\right)' \Sigma_s^{-1} \left(\underline{x}_j - \underline{a}_s\right) - \ln\left|\Sigma_s^{-1}\right| - \left(\underline{a}_s - \underline{a}_t\right)' \Sigma_t^{-1} \left(\underline{a}_s - \underline{a}_t\right)} \right]^{\frac{1}{m-1}} \right]^{-1} \tag{34}$$

$$(i, j) \rightarrow (s, j) \Rightarrow \mu_{ij}$$

$$t_{ij} = \left[\sum_{s=1}^{n} \left[\frac{\left(\underline{x}_j - \underline{a}_i\right)' \Sigma_i^{-1} \left(\underline{x}_j - \underline{a}_i\right) - \ln\left|\Sigma_i^{-1}\right| - \left(\underline{a}_i - \underline{a}_t\right)' \Sigma_t^{-1} \left(\underline{a}_i - \underline{a}_t\right)}{\left(\underline{x}_s - \underline{a}_i\right)' \Sigma_i^{-1} \left(\underline{x}_s - \underline{a}_i\right) - \ln\left|\Sigma_i^{-1}\right| - \left(\underline{a}_i - \underline{a}_t\right)' \Sigma_t^{-1} \left(\underline{a}_i - \underline{a}_t\right)} \right]^{\frac{1}{\delta-1}} \right]^{-1} \tag{35}$$

$$\Sigma_i = \frac{\sum_{j=1}^{n} \left(\mu_{ij}^m + t_{ij}^\delta\right) \left(\underline{x}_j - \underline{a}_i\right)\left(\underline{x}_j - \underline{a}_i\right)'}{\sum_{j=1}^{n} \left(\mu_{ij}^m + t_{ij}^\delta\right)} \tag{36}$$

$$\Sigma_i = \sum_{s=1}^{p} \lambda_{si} \Gamma_{si} \Gamma'_{si} \quad i = 1, 2, \ldots, c$$

$$\Rightarrow {}_{+}\Sigma_i^{-1} = \sum_{s=1}^{p} \left(\lambda_{si}^{-1}\right)^{+} \Gamma_{si} \Gamma'_{si}, \ \left(\lambda_{si}^{-1}\right)^{+} = \begin{cases} \lambda_{si}^{-1} & if\ \lambda_{si} > 0 \\ 0 & if\ \lambda_{si} \le 0 \end{cases}$$

$$\left|{}_{+}\Sigma_i^{-1}\right| = \prod_{1\le s\le p, \lambda_{si}>0} \lambda_{si}^{-1}$$

The new fuzzy clustering algorithm (FPCM-CMS) can be summarized in the following steps:

Step 1: Determining the number of cluster; c, let m=2, $\delta = 3$, Given converging error $\varepsilon > 0$ (such as $\varepsilon = 0.001$) choose the result membership matrix of

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FPCM-CM algorithm as the initial one and the normalized result typicality matrix of FPCM-CM algorithm as the initial one respectively;

let $\underline{a}_i^{(0)}, i=1,2,\ldots,c$ be the result centers of k-mean algorithm, and $d_{ij}=\left\|\underline{x}_j-\underline{a}_i^{(0)}\right\|$ be distances between data object $\underline{x}_j$ to center $\underline{a}_i^{(0)}$.

$$\mu_{ij}^{(0)}=\frac{\left(d_M-d_{ij}\right)}{\sum_{s=1}^{c}\left(d_M-d_{sj}\right)}, \quad (37)$$

$i=1,2,\ldots,c$, $j=1,2,\ldots,n$

$$t_{ij}^{(0)}=\frac{\left(d_M-d_{ij}\right)}{\sum_{s=1}^{n}\left(d_M-d_{is}\right)}, \quad (38)$$

$i=1,2,\ldots,c$, $j=1,2,\ldots,n$

$$\underline{a}_i^{(0)}=(\sum_{j=1}^{n}\left[\mu_{ij}^{(0)}\right]\underline{x}_j)(\sum_{j=1}^{n}\left[\mu_{ij}^{(0)}\right])^{-1}, \quad i=1,2,\ldots,c \quad (39)$$

$$\Sigma_i^{(0)}=$$

$$(\sum_{j=1}^{n}\left[\mu_{ij}^{(0)}\right]^m\left(\underline{x}_j-\underline{a}_i^{(0)}\right)\left(\underline{x}_j-\underline{a}_i^{(0)}\right)')(\sum_{j=1}^{n}\left[\mu_{ij}^{(0)}\right]^m)^{-1} \quad (40)$$

$$\Sigma_i^{(0)}=\frac{\sum_{j=1}^{n}\left[\left[\mu_{ij}^{(0)}\right]^m+\left[t_{ij}^{(0)}\right]^\delta\right]\left(\underline{x}_j-\underline{a}_i^{(0)}\right)\left(\underline{x}_j-\underline{a}_i^{(0)}\right)'}{\sum_{j=1}^{n}\left[\left[\mu_{ij}^{(0)}\right]^m+\left[t_{ij}^{(0)}\right]^\delta\right]} \quad (41)$$

$$U^{(0)}=\begin{bmatrix}\mu_{11}^{(0)} & \mu_{12}^{(0)} & \ldots\mu_{1n}^{(0)}\\ \mu_{21}^{(0)} & \mu_{22}^{(0)} & \ldots\mu_{2n}^{(0)}\\ \ldots & \ldots & \ldots\\ \mu_{c1}^{(0)} & \mu_{c2}^{(0)} & \ldots\mu_{cn}^{(0)}\end{bmatrix} \quad (42)$$

$$T^{(0)}=\begin{bmatrix}t_{11}^{(0)} & t_{12}^{(0)} & \ldots t_{1n}^{(0)}\\ t_{21}^{(0)} & t_{22}^{(0)} & \ldots t_{2n}^{(0)}\\ \ldots & \ldots & \ldots\\ t_{c1}^{(0)} & t_{c2}^{(0)} & \ldots t_{cn}^{(0)}\end{bmatrix} \quad (43)$$

$$A^{(0)}=\left[\underline{a}_1^{(0)}\ \underline{a}_2^{(0)}\ \ldots\underline{a}_c^{(0)}\right] \quad (44)$$

$$\Sigma^{(0)}=\left[\Sigma_1^{(0)},\Sigma_2^{(0)},\ldots,\Sigma_c^{(0)}\right] \quad (45)$$

Step 2: Find

$$\underline{a}_i^{(k)}=$$

$$\left[F^{(k-1)}\right]^{-1}\left[\sum_{j=1}^{n}\left(\left(\mu_{ij}^{(k-1)}\right)^m+\left(t_{ij}^{(k-1)}\right)^\delta\right)\left[\left(\Sigma_i^{(k-1)}\right)^{-1}\underline{x}_j-\Sigma_t^{-1}\underline{a}_t\right]-\frac{1}{c(c-1)}\sum_{l=1}^{c}\left(v_{il}^{(k-1)}\right)^m\underline{a}_l^{(k-1)}\right]$$

$$F^{(k-1)}=\left[\sum_{j=1}^{n}\left(\left(\mu_{ij}^{(k-1)}\right)^m+\left(t_{ij}^{(k-1)}\right)^\delta\right)\left[\left(\Sigma_i^{(k-1)}\right)^{-1}-\left(\Sigma_t^{(k-1)}\right)^{-1}\right]-\frac{1}{c(c-1)}\sum_{l=1}^{c}\left(v_{il}^{(k-1)}\right)^m I\right] \quad (46)$$

$$\Sigma_i^{(k)}=\frac{\sum_{j=1}^{n}\left(\left[\mu_{ij}^{(k-1)}\right]^m+\left[t_{ij}^{(k-1)}\right]^\delta\right)\left(\underline{x}_j-[\underline{a}_i]^{(k)}\right)\left(\underline{x}_j-[\underline{a}_i]^{(k)}\right)'}{\sum_{j=1}^{n}\left(\left[\mu_{ij}^{(k-1)}\right]^m+\left[t_{ij}^{(k-1)}\right]^\delta\right)} \quad (47)$$

$$\mu_{ij}^{(k)}=$$

$$\left[\sum_{s=1}^{c}\left[\frac{\left(\underline{x}_j-[\underline{a}_i]^{(k)}\right)'\left[\Sigma_i^{(k)}\right]^{-1}\left(\underline{x}_j-[\underline{a}_i]^{(k)}\right)-\ln\left|\left[\Sigma_i^{(k)}\right]^{-1}\right|-\left([\underline{a}_i]^{(k)}-\underline{a}_t\right)'\Sigma_t^{-1}\left([\underline{a}_i]^{(k)}-\underline{a}_t\right)}{\left(\underline{x}_j-[\underline{a}_s]^{(k)}\right)'\left[\Sigma_s^{(k)}\right]^{-1}\left(\underline{x}_j-[\underline{a}_s]^{(k)}\right)-\ln\left|\left[\Sigma_s^{(k)}\right]^{-1}\right|-\left([\underline{a}_s]^{(k)}-\underline{a}_t\right)'\Sigma_t^{-1}\left([\underline{a}_s]^{(k)}-\underline{a}_t\right)}\right]^{\frac{1}{m-1}}\right]^{-1} \quad (48)$$

$$t_{ij}^{(k)}=$$

$$\left[\sum_{s=1}^{n}\left[\frac{\left(\underline{x}_j-[\underline{a}_i]^{(k)}\right)'\left[\Sigma_i^{(k)}\right]^{-1}\left(\underline{x}_j-[\underline{a}_i]^{(k)}\right)-\ln\left|\left[\Sigma_i^{(k)}\right]^{-1}\right|-\left([\underline{a}_i]^{(k)}-\underline{a}_t\right)'\Sigma_t^{-1}\left([\underline{a}_i]^{(k)}-\underline{a}_t\right)}{\left(\underline{x}_s-[\underline{a}_i]^{(k)}\right)'\left[\Sigma_i^{(k)}\right]^{-1}\left(\underline{x}_s-[\underline{a}_i]^{(k)}\right)-\ln\left|\left[\Sigma_i^{(k)}\right]^{-1}\right|-\left([\underline{a}_i]^{(k)}-\underline{a}_t\right)'\Sigma_t^{-1}\left([\underline{a}_i]^{(k)}-\underline{a}_t\right)}\right]^{\frac{1}{\delta-1}}\right]^{-1} \quad (49)$$

where

$$\Sigma_i^{(k)}=\sum_{s=1}^{p}\lambda_{si}^{(k)}\Gamma_{si}^{(k)}\left(\Gamma_{si}^{(k)}\right)',$$

$$\left[\lambda_{si}^{-1}\right]^{(k)}=\begin{cases}\left[\lambda_{si}^{(k)}\right]^{-1} & if\ \lambda_{si}^{(k)}>0\\ 0 & if\ \lambda_{si}^{(k)}=0\end{cases} \quad (50)$$

$$\left[\Sigma_i^{(k)}\right]^{-1}=\sum_{s=1}^{p}\left[\lambda_{si}^{(-1)}\right]^{(k)}\Gamma_{si}^{(k)}\left(\Gamma_{si}^{(k)}\right)'$$

$$\left|\Sigma_i^{(k)}\right|^{-1}=\prod_{1\le s\le p,\lambda_{si}^{(k)}>0}\left[\lambda_{si}^{(k)}\right]^{-1} \quad (51)$$

Step 3: Increment k; until $\max_{1\le i\le c}\left\|\underline{a}_i^{(k)}-\underline{a}_i^{(k-1)}\right\|<\varepsilon$.

III. EXPERIMENT

A real data set of students with sample size 146 from elementary schools was selected. The main factors of the data were calculated by using factor analysis. According to the main factors, the samples were assigned to 4 clusters based on the clustering analysis. The results were shown in table I.

TABLE I. The characteristics of 4 clusters.

Cluster	Sample size	Mathematics Concepts	Average Distance of the points from center of Cluster
1	36	Partition	-.14984
2	89	Unit	.21161
3	16	Fraction	-.30416
4	5	Unknown unit	-.74490

Each 15 sample points were randomly drawn from Cluster 1, cluster 2, and cluster 3, respectively, and 5 from cluster 4.

The classification accuracies of testing samples were shown in table II.

TABLE II. Classification accuracies of testing samples.

Algorithm	Classification Accuracies (%)
FCM	56
FPCM	68
FPCM-M	70
FCM-CMS	79
FPCM-CMS	86

Accuracies (%) of Using different Fuzzy Clustering Algorithm

From the data of table II, we found that the FPCM-CMS algorithms could obtain the best results, up to 86%. Next, the FCM-CMS algorithms could obtain the second better results, up to 79%.

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IV. Conclusion

An improved new fuzzy clustering algorithm, FPCM-CMS, is developed to obtain better quality of fuzzy clustering results. The objective function includes a fuzzy within-cluster scatter matrix, a new between-prototypes scatter matrix, the regulating terms about the covariance matrices, and the regulating terms about the relationships between cluster centers, the relationships between the center of all points and the cluster centers. The update equations for the memberships and the cluster centers and the covariance matrices are directly derived from the Lagrange's method. Finally, a numerical example shows that FPCM-CMS gives more accurate clustering results than others.

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