Associated Bitopological Spaces of (i,j)- $g^*b$ –Closed Sets

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Abstract— The concept of gb-closed sets for general topology was introduced and studied analytically by M.Ganster & M. Steiner and which motivated the author to coin the concept of (i,j)-gb-closed as well as (i,j)-$g^*b$ –closed sets in bitopological spaces.

The exploration of these classes of (i,j)-gb-closed as well as (i,j)-$g^*b$ –closed sets provide the relation with the class of $i$-$b$-closed sets in the manner that the class of (i,j)-$g^*b$ –closed sets properly exists in between the class of $j$-$b$-closed sets and the class of (i,j)-gb-closed sets.

Of course, being the main object of the present paper, two new bitopological (i,j)-$T^*b_{1/2}$ & (i,j)-$Tb_{1/2}$ are introduced as an application of (i,j)-$g^*b$-closed sets.

No doubt in order to serve the purpose, the basic properties of (i,j)-$g^*b$-closed as well as (i,j)-gb-closed sets are investigated and analysed in this paper.

Keywords— (i,j)-$g^*b$-closed sets, (i,j)-$T^*b_{1/2}$ & (i,j)-$Tb_{1/2}$ spaces.

I. INTRODUCTION

A triple (X, $\tau_1$, $\tau_2$) consisting of a non-empty set X together with a pair of topologies $\tau_1$ & $\tau_2$ on X, is defined to be a bitopological space by J.C.Kelley as introduced through the famous mathematical paper entitled as “Bitopological spaces” and published in proceedings of London Mathematical Society, 13(71-89),1963[1]. Such a space, equipped with two arbitrary topologies, is beyond any doubt an original and fundamental work.

As a recall, in 1963, a semi-open set $A$ was defined as the subset $A$ of a topological space (X, $\tau$) iff there exists $O \in \tau$ such that $O \subseteq A \subseteq \text{Cl}(O)$ and denoted by s-open or equivalently iff $A \subseteq \text{Cl}(\text{int}(A))$.

In 1970, generalized closed sets were introduced and studied by N. Levine. He conceptualized a generalized closed set as a subset $A$ of a topological space (X, $\tau$) for which $A \subseteq O \in \tau \Rightarrow \text{Cl}(A) \subseteq O$ and denoted by g-closed.

In 1978, an $\alpha$-set was introduced as a subset $A$ of a topological space (X, $\tau$) for which $A \subseteq \text{int} (\text{Cl}(\text{int}(A)))$ and denoted by $\alpha$ – open set.

Then in 1981, a pre-open set was investigated as a subset $A$ of a topological space (X, $\tau$) in manner that $A \subseteq \text{int}(\text{Cl}(A))$ and denoted by p- open.

In 1983, M.E. Abd El – Monsef etc introduced a $\beta$-open set as subset $A$ of a topological space (X, $\tau$) in the manner that $A \subseteq \text{Cl}(\text{int}(A))$ and D. Andrijewic named $\beta$-open sets as semi-pre open sets.

C. Kuratowski mentioned that a subset $A$ of a topological space (X, $\tau$) is said to be regular open (resp. regular closed) or an open domain(resp. closed domain) if it is the interior of its closure(resp. if it is the closure of its interior) i.e. $A = \text{int} (\text{Cl}(A)).($resp. $A = \text{Cl}(\text{int}(A))$).

D. Andrijevic investigated a new class of generalized open sets in a topological space, the so called $b$-open sets, which is contained in the class of semi-pre-open sets and contains all semi-open sets and pre-open sets. Also, such a class of $b$-open sets generates the same topology as the class of pre-open sets.

Thus, a subset $A$ of a topological space (X, $\tau$) is known to be a $b$-open or $\gamma$-open set iff $A \subseteq \text{Cl}(\text{int}(A)) \cup \text{int}(\text{Cl}(A))$.

The compliments of the above mentioned open/closed sets are their respective closed/open sets. The smallest $\kappa$-closed set containing $A$ is called the $\kappa$-closure of $A$ and denoted by $\text{Cl}(A)$ where as the largest $\kappa$-open set contained in $A$ is called $\kappa$-interior of $A$ and denoted by $\text{int}(A)$ where $\kappa=\alpha, \text{semi}(s), \text{pre}(p), \text{regular}(r), b$ & $\beta$.

Extensive research on generalizing of closedness was done in the recent years and in the year 2007, M.Ganster and M.Steiner presented the concept & the study of generalized b-closed (briefly, gb-closed) sets in general topology.


In 1997, The concepts of pre-open sets and regular open sets have been extended to bitopological spaces symbolized by (i,j)-p-open and (i,j)-r-open sets respectively as mentioned in the mathematical paper ‘On Decomposition Of Pair wise Continuity ’ by S. Sampath Kumar, published in Bull. Cal. Math. Soc. 89 (1997), pp- 441-446 [8].

The aim of this paper is to extend the concept of gb-closed sets in general topology for bitopology. So, we introduce (i,j)-$g^*b$-closed sets and (i,j)-gb-closed sets and their fundamental interesting properties.
Again, the bitopological spaces (i,j) - T^1 intersection T^2 and (i,j) - T^1 intersection T^2 associated with (i,j)-g*b-closed sets are natural out come as an application. Such spaces are analysed to some extent in this paper.

The following abbreviations are adopted throughout this paper.
(a) All through this paper, the spaces X and Y, always represent non-empty bitopological spaces or (X, T_i, T_j) and (Y, σ_1, σ_2) respectively on which no separation axioms are assumed unless explicitly stated.
(b) The interior (resp. closure) of a subset A of a bitopological space (X, T_i, T_j) w.r.t topology T_i (i = 1, 2) is denoted by T_i - int(A) (resp. T_i - cl(A)).
(c) The set of all closed (resp. open) sets w.r.t. the topology T_i is denoted by i-C(X) (resp. i-O(X)). And by (i,j) we mean the pair of topologies (T_i, T_j) & (i,j) ∈ {1,2} such that i ≠ j.
(d) As usual the family of all K-open (K = a, pre, semi, b, β, r, and g) sets is denoted by the symbol i-L(O)(X) (L = a, P, S, B, g, β, RG) accordingly.

The following definitions are cited for the reference of all the self-explanatory arguments of the content projected in this paper.

Definition (1.1): A subset A of a bitopological space (X, T_1, T_2) is called
(a) (i,j)-g-closed [2&9] if T_j-cl(A) ⊆ U whenever A ⊆ U and U ∈ i-O(X).
(b) (i,j)-g*-closed [3] if T_j-cl(A) ⊆ U whenever A ⊆ U and U ∈ i-GO(X).
(c) (i,j)-rg-closed [4] if T_j-cl(A) ⊆ U whenever A ⊆ U and U ∈ i-RO(X).
(d) (i,j)-w-closed [5] if T_j-cl(A) ⊆ U whenever A ⊆ U and U ∈ i-SO(X).
(e) (i,j)-wg-closed [6] if T_j-cl(T_j-int A) ⊆ U whenever A ⊆ U and U ∈ i-O(X).
(f) (i,j)-p-closed [8&10] if U ⊆ A and U ∈ j-O(X) ⇒ ∃F ∈ i-C(X) such that U ⊆ F ⊆ A.
(g) (i,j)-r-closed [8&9] if A = T_i-cl(T_j-int(A)).

The family of all (i,j)-g-closed (resp. (i,j)-g*-closed, (i,j)-rg-closed, (i,j)-w-closed, (i,j)-wg-closed, (i,j)-p-closed and (i,j)-r-closed) subsets of a bitopological space (X, T_1, T_2) is denoted by D(i,j)(resp. D^*(i,j), D(i,j), C(i,j), W(i,j), P(i,j)&R(i,j))[11].

Definition (1.2):
(a) A subset A of a bitopological space (X, T_1, T_2) is said to be(i,j)-T^1 intersection T^2 space[2],(resp. (i,j)-T^1 intersection T^2 space) if every (i,j)-g-closed(resp. (i,j)-g*-closed) set is T_j-closed (resp. T_j-b-closed).
(b) A bitopological space (X, T_1, T_2) is said to be strongly pairwise T^1 intersection T^2 (resp. strongly pairwise T^1 intersection T^2 space [2]) if it is both (i,j)-T^1 intersection T^2 space and (i,j)-T^1 intersection T^2 space (resp. (i,j)-T^1 intersection T^2 space and (i,j)-T^1 intersection T^2 space), where i,j ∈ [1,2] & i ≠ j.

Definition (1.3): A mapping f : (X, T_1, T_2) → (Y, σ_1, σ_2) is called
(a) T_j - σ_k continuous [7] if f^{-1}(V) ∈ T_j - for every V ∈ σ_k.
(b) L(i,j) - σ_k continuous if the inverse image of every σ_k closed set is (i,j)-t-closed where L = D, D^*, D, C, W, P & R and t = g, g*, r, w, w*, w & r accordingly.[4,5,6] & [7].
• Any other notion & symbol, not defined here, may be found in the appropriate reference.

II. (i,j)-g*b-closed & (i,j)-gb-closed sets

This section carries the introduction & discussion of the concepts of (i,j)-g*b-closed sets as well as (i,j)-gb-closed sets in a bitopological space. Also, it includes the related fundamental properties with suitable examples.

Definition (2.1): A subset A of a bitopological space (X, T_1, T_2) is said to be
(a) an (i,j)-g*b-closed set if T_j-bcl(A) ⊆ U whenever A ⊆ U and U ∈ i-GO(X).
(b) an (i,j)-gb-closed set if T_j-bcl(A) ⊆ U whenever A ⊆ U and U ∈ i-O(X).

The family of all (i,j)-g*b-closed sets(resp. (i,j) gb-closed sets) is denoted by D_b*(i,j)(resp. D_b(i,j)).

Proposition (2.2): In a bitopological space (X, T_1, T_2)
(a) Every T_j-closed set is (i,j)-g*b-closed.
(b) Every (i,j)-g*b-closed set is (i,j)-gb-closed.
(c) Every (i,j)-g*b-closed set is (i,j)-gb-closed.
(d) Every (i,j)-g*b-closed set is (i,j)-gb-closed.
(e) Every (i,j)-g*b-closed set is (i,j)-gb-closed.
(f) Every (i,j)-gb-closed set is (i,j)-gb-closed.

Proof:
(a) Suppose that a T_j-closed subset of a bitopological space (X, T_1, T_2) is A which is to be shewn to be an (i,j)-g*b-closed set.

Let U be any i-g-open set in (X, T_1) such that A ⊆ U. Since, every T_j-closed set is a T_j-b-closed set, hence A is a T_j-b-closed. i.e. A = T_j - bcl(A). Consequently, it follows that A ⊆ U & U ∈ i-GO(X) ⇒ T_j - bcl(A) ⊆ U. This means that A is an (i,j)-g*b-closed set.

(b) Let an open set U and an (i,j)-g*b-closed set A in a space(X, T_1, T_2) be such that A ⊆ U. Now, (i,j)-g*b-closed
closedness of A provides that \( \tau_j - \text{cl}(A) \subseteq V \) whenever A \( \subseteq V \) & V is an i-g-open set ………….. (I)

Since, every i-open set is an i-g-open set , hence, (I) turns to be \( \tau_j - \text{cl}(A) \subseteq U \) whenever A \( \subseteq U \) & U is i-g-open. But U \( \subseteq \text{i-O}(X) \), so \( \tau_j - \text{cl}(A) \subseteq U \) whenever A \( \subseteq U \) & U \( \subseteq \text{i-O}(X) \) holds good. Consequently, A is an (i,j)-gb-open set.

(c) Let A be an (i,j)-g-closed set in a bitopological space \((X, \tau_1, \tau_2)\). Let V \( \subseteq \text{i-O}(X) \) be such that A \( \subseteq V \). Then, (i,j)-g-closedness of A provides that \( \tau_j - \text{cl}(A) \subseteq V \) whenever A \( \subseteq V \) & V \( \subseteq \text{i-O}(X) \). Since, \( \tau_j \)-closed set is a \( \tau_j \)-b-closed set, hence, \( \tau_j - \text{cl}(A) \subseteq \tau_j - \text{cl}(A) \). Combining all these facts, we have, \( \tau_j - \text{cl}(A) \subseteq V \) whenever A \( \subseteq V \) & \( V \subseteq \text{i-O}(X) \) holds good. Consequently, A is an (i,j)-gb-closed set.

(d) Let A be an (i,j)-g*-closed set in a bitopological space \((X, \tau_1, \tau_2)\). Let V \( \subseteq \text{i-O}(X) \) be such that A \( \subseteq V \). Then, (i,j)-g*-closedness of A provides that \( \tau_j - \text{cl}(A) \subseteq V \) whenever A \( \subseteq V \) & V \( \subseteq \text{i-O}(X) \). Since, \( \tau_j \)-closed set is a \( \tau_j \)-b-closed set, hence, \( \tau_j - \text{cl}(A) \subseteq \tau_j - \text{cl}(A) \). Combining all these facts, we have, \( \tau_j - \text{cl}(A) \subseteq V \) whenever A \( \subseteq V \) & \( V \subseteq \text{i-O}(X) \) which means that A is an (i,j)-gb*-closed set.

(e) Let A be an (i,j)-g*-closed set in a bitopological space \((X, \tau_1, \tau_2)\). Let U \( \subseteq \text{i-O}(X) \) be such that A \( \subseteq U \). Then, (i,j)-g*-closedness of A provides that \( \tau_j - \text{cl}(A) \subseteq U \) whenever A \( \subseteq U \) & U \( \subseteq \text{i-O}(X) \) holds good. Since, every i-open set is an i-g-open set , hence, (I) turns to be \( \tau_j - \text{cl}(A) \subseteq U \) whenever A \( \subseteq U \) & U \( \subseteq \text{i-O}(X) \). Consequently, A is (i,j)-g-closed.

(f) Let A be an (i,j)-g*-closed set in a bitopological space \((X, \tau_1, \tau_2)\). Let U \( \subseteq \text{i-O}(X) \) be such that A \( \subseteq U \). Then, (i,j)-g*-closedness of A provides that \( \tau_j - \text{cl}(A) \subseteq U \) whenever A \( \subseteq U \) & U \( \subseteq \text{i-O}(X) \) holds good. Consequently, A is (i,j)-g-closed.

Observation (2.3): (A) Of course, the class of (i,j) -g*-b-closed sets properly fits between the class of j-b-closed sets and the class of(i,j) -gb-closed sets in a bitopological \((X, \tau_1, \tau_2)\) and i,j \( \in \{1,2\}, i \neq j\). Also, j-BC(X) \( \subseteq D^b(i,j) \subseteq D_{\text{gb}}(i,j) \).

(I) In the space \((X, \tau_1, \tau_2)\) where X = \{a,b,c\}, \( \tau_1 = \{\varnothing, X, \{a\}, \{a,c\}\} \) and \( \tau_2 = \{\varnothing, X, \{a\}, \{a,b\}\} \), the subset \{b\} and \{a\} are \((1,2)\)-gb*-closed set but neither \( \tau_2 \)-closed nor \( \tau_2 \)-b-closed.

(II) In the space \((X, \tau_1, \tau_2)\) where X = \{a,b,c\}, \( \tau_1 = \{\varnothing, X, \{a\}, \{a,b\}\} \) and \( \tau_2 = \{\varnothing, X, \{a\}, \{a,b\}\} \), the subset \{c\} is \((1,2)\)-gb*-closed but not \((1,2)\) - gb*-closed.

(B) The following diagram summarized the relationship of (i,j)gb*-b-closed & (i,j)-gb-closed sets with other existing different closed sets:

\[ \text{Implication diagram} \]

A \( \rightarrow \) B (resp. A \( \leftrightarrow \) B) represents A implies B but not conversely (resp. A & B are independent.)

The following few examples stand as a support of the arrows/ independent relations in the above implication diagram.

Example (2.4):

(I) If X = \{a,b,c\}, \( \tau_1 = \{\varnothing, X, \{a\}\} \) and \( \tau_2 = \{\varnothing, X, \{a\}, \{a,b\}\} \), then

(a) the subsets \{b\}, \{c\}, \{a\}, \{a,b\} are \((1,2)\)-gb*-closed but not \((1,2)\) - gb*-closed.

(b) the subset\{c\} is \((1,2)\)-gb*-closed but not \((1,2)\)-gb*-closed.

(II) If X = \{a,b,c\}, \( \tau_1 = \{\varnothing, X, \{a\}, \{a,b\}\} \) and \( \tau_2 = \{\varnothing, X, \{a\}, \{a,b\}\} \), then

(a) the subset \{b\} is a \((1,2)\)-rg-closed set but not \((1,2)\) - gb*-closed set.

(b) the subset \{b\} is a \((1,2)\)-wg-closed set but not \((1,2)\)-gb*-closed set.

(III) (a) If X = \{a,b,c,d,e\}, \( \tau_1 = \{\varnothing, X, \{a\}, \{a,b\}, \{a,b,c\}\} \) and \( \tau_2 = \{\varnothing, X, \{a\}, \{a,b\}, \{a,b,c\}\} \), then

the subsets \{c,d\} is \((1,2)\)-w-closed but not \((1,2)\) - gb*-closed.

(b) If X = \{a,b,c\}, \( \tau_1 = \{\varnothing, X, \{a\}, \{a,b\}\} \) and \( \tau_2 = \{\varnothing, X, \{a\}, \{a,b\}\} \), then

the subsets \{a,c\} is \((1,2)\)-gb*-closed but not \((1,2)\)-w-closed. These means that \((1,2)\) - w-closedness & \((1,2)\) - gb*-closedness are independent.

(IV) (a) If X = \{a,b,c\}, \( \tau_1 = \{\varnothing, X, \{b\}, \{b,c\}\} \) and \( \tau_2 = \{\varnothing, X, \{a\}, \{a,b\}\} \), then

the subsets \{a\} is \((1,2)\)-gb*-closed but not \( \tau_2 \)-g-closed.

\[ \text{Diagram} \]
(b) If \( X = \{a,b,c\}, \tau_1 = \{\emptyset, X, \{a\}, \{a,b\}\} \) and \( \tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\} \), the subset \( \{c\} \) is \( \tau_2 \)-g-closed but not \( (1,2) \)-g*b-closed. This means that \( \tau_2 \)-g-closedness & \( (1,2) \)-g*b-closedness are independent.

**Proposition (2.5):**

The intersection of two \((i,j)\)-g*b-closed sets is an \((i,j)\)-g*b-closed set, but their union need not be an \((i,j)\)-g*b-closed set.

**Proof:** Suppose that A and B are any two \((i,j)\)-g*b-closed sets in a bitopological space \( (X, \tau_1, \tau_2) \). Let \( U \in i\text{-GO}(X) \) be such that \( A \subseteq U \) & \( B \subseteq U \) so that \( A \bigcap B \subseteq U \) , then the \( (i,j) \)-g*b-closedness of \( A \) & \( B \) provides that \( \tau_j - bcl(A) \subseteq U \) & \( \tau_j - bcl(B) \subseteq U \) . So \( \tau_j - bcl(A) \bigcap \tau_j - bcl(B) \subseteq U \) . Again, Since, \( \tau_j - bcl(A \bigcap B) \subseteq \tau_j - bcl(A) \bigcap \tau_j - bcl(B) \) , hence, \( \tau_j - bcl(A \bigcap B) \subseteq U \).

Hence, it establishes that \( \tau_j - bcl(A \bigcap B) \subseteq U \) whenever \( A \bigcap B \subseteq U \) & \( U \in i\text{-GO}(X) \) i.e. \( A \bigcap B \) is an \((i,j)\)-g*b-closed set.

But the union of two \((i,j)\)-g*b-closed sets is not necessarily an \((i,j)\)-g*b-closed set as illustrated by the following example:

In the Let \( X = \{a,b,c\} \), \( \tau_1 = \{\emptyset, X, \{a\}, \{a,b\}\} \) and \( \tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\} \), then for the bitopological space \( (X, \tau_1, \tau_2) \) the subsets \( \{a\} \& \{b\} \) are \( (1,2)-g*b\)-closed but \( \{a\} \bigcup \{b\} = \{a,b\} \) is not \( (i,j)-g*b\)-closed where \( i = 1 \& j = 2 \).

**Corollary (2.6):** The intersection of two \((i,j)\)-gb-closed sets is an \((i,j)\)-gb-closed set whereas their union is not necessarily an \((i,j)\)-gb-closed set.

Because, \( D^*_{\delta}(i,j) \subseteq D_0(i,j) \) & \( A,B \subseteq D^*_{\delta}(i,j) \Rightarrow A,B \subseteq D_0(i,j) \) again, \( A,B \subseteq D^*_{\delta}(i,j) \Rightarrow A \bigcap B \subseteq D_0(i,j) \) & \( A \bigcap B \subseteq D_0(i,j) \).

So, \( A \bigcap B \subseteq D_0(i,j) \) & \( D_0(i,j) \subseteq D_0(i,j) \).

**Proposition (2.7):** In a bitopological space \( (X, \tau_1, \tau_2) \), \( i\text{-GO}(X) \subseteq j\text{-BC}(X) \) if and only if every subset of \( X \) is an \((i,j)\)-g*b-closed set.

**Proof:** Let \( (X, \tau_1, \tau_2) \) be a bitopological space in which \( i\text{-GO}(X) \subseteq j\text{-BC}(X) \). Let \( A \) be a subset of \( X \) such that \( A \subseteq U \) & \( U \in i\text{-GO}(X) \). Then \( A \subseteq U \) implies that \( \tau_j - bcl(A) \subseteq \tau_j - bcl(U) \) as \( U \in j\text{-BC}(X) \) . Thus we, have \( \tau_j - bcl(A) \subseteq U \) whenever \( A \subseteq U \) & \( U \in i\text{-GO}(X) \) i.e. \( A \) is an \((i,j)\)-g*b-closed set.

Conversely, suppose that every subset of \( X \) is an \((i,j)\)-g*b-closed set. Let \( U \subseteq i\text{-GO}(X) \). Since, \( U \) is \( ij\)-g*b-closed, hence, \( \tau_j - bcl(U) \subseteq U \) whenever \( A \subseteq U \) & \( U \in i\text{-GO}(X) \). Now, \( \tau_j - bcl(U) \subseteq U \Rightarrow U = \tau_j - bcl(U) \) i.e. \( U \subseteq j\text{-BC}(X) \) i.e. \( i\text{-GO}(X) \subseteq j\text{-BC}(X) \).

**Proposition (2.8):** If \( A \) is an \((i,j)\)-g*b-closed set in a bitopological space \( (X, \tau_1, \tau_2) \), and \( B \) is any set such that \( A \subseteq bcl(\{x\}) \), then \( B \) is also an \((i,j)\)-g*b-closed set.

**Proof:** In a bitopological space \( (X, \tau_1, \tau_2) \), let \( B \subseteq U \) where \( U \subseteq i\text{-GO}(X) \). Since, \( A \) is an \((i,j)\)-g*b-closed set, hence, \( \tau_j - bcl(A) \subseteq U \) whenever \( A \subseteq U \& U \in i\text{-GO}(X) \). Again, \( B \subseteq \tau_j - bcl(A) \Rightarrow \tau_j - bcl(B) \subseteq \tau_j - bcl(A) \) & \( \tau_j - bcl(A) \subseteq U \).

So, we have, \( \tau_j - bcl(B) \subseteq U \) whenever \( A \subseteq U \& U \in i\text{-GO}(X) \), which claims that \( B \) is an \((i,j)\)-g*b-closed set.

**Proposition (2.9):** If in a bitopological space \( (X, \tau_1, \tau_2) \), a subset \( A \) of \( X \) is \( i\)-g-open as well as \( ij\)-g*b-closed, then \( A \) is \( j\)-b-closed.

**Proof:** Let a subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \) be such that \( A \subseteq i\text{-GO}(X) \& A \subseteq D^*_{\delta}(i,j) \).

Now, the \((i,j)\)-g*b-closedness of \( A \) gives that \( \tau_j - bcl(A) \subseteq U \) whenever \( A \subseteq U \& U \subseteq i\text{-GO}(X) \).

Since, \( A \subseteq i\text{-GO}(X) \), hence, \( (i) \) also holds good as \( \tau_j - bcl(A) \subseteq U \).

Again, \( \tau_j - bcl(A) \subseteq A \Rightarrow A = \tau_j - bcl(A) \) i.e. \( A \) is \( j\)-b-closed and so \( A \subseteq j\text{-BC}(X) \).

**Proposition (2.10):** For each element \( x \) of \( (X, \tau_1, \tau_2) \), \( \{x\} \) is either g-closed or \((i,j)\)-g*b-open.

**Proof:** Let \( x \in X \) where \( (X, \tau_1, \tau_2) \) is bitopological space.

(a) Suppose that \( \{x\} \) is not g-closed set. This implies that \( \{x\} \subseteq \{x\} \), which is not a g-open set. So, the \( X \) is the only i-g-open set such that \( \{x\} \subseteq X \) . Now, \( \tau_j - bcl((\{x\}) = \tau_j - bcl(X) = X \).

Thus, \( \{x\} \subseteq X \Rightarrow X \subseteq \tau_j - bcl((\{x\}) = X \).

holds good, which in turns establishes that \( \{x\} \) is an \((i,j)\)-g*b-closed set.

**Proposition (2.11):** For an \((i,j)\)-g*b-closed subset \( A \) of \( (X, \tau_1, \tau_2) \), \( \{\tau_j - bcl(A) - A\} \) does not contain any non-empty \((i,j)\)-g*b-closed set.

**Proof:** Let \( A \) be an \((i,j)\)-g*b-closed subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \). Let \( F \) be an \( i\)-g-closed set such that \( F \subseteq \{\tau_j - bcl(A) - A\} \) i.e. \( F \subseteq \{\tau_j - bcl(A) \bigcap A\} \) & \( F \subseteq \{x\} \) i.e. \( A \subseteq F \).

Again, \( F \) is \( i\)-g-open set & \( F \subseteq \{x\} \) as well as \( F \) is an \((i,j)\)-g*b-closed set. So, \( \tau_j - bcl(A) \subseteq F \). This implies that \( F \) is \( \{\tau_j - bcl(A)\} \subseteq F \). Now, \( F \subseteq \{\tau_j - bcl(A) \bigcap A\} \) i.e. \( F \subseteq \emptyset \) & so \( F = \emptyset \).
Corollary (2.12): In a bitopological space $(X, \tau_1, \tau_2)$, an $(i,j)$-g*-b-closed set $A$ is $j$-b-closed iff $\{ \tau_j - \text{bcl}(A) - A \}$ is $i$-g-closed.

Proof: Let $A$ be $(i,j)$-g*-b-closed. If $A$ is $j$-b-closed, then $\{ \tau_j - \text{bcl}(A) - A \} = \varnothing$ which is $i$-g-closed.

Conversely, let $\{ \tau_j - \text{bcl}(A) - A \}$ be $i$-g-closed then by proposition (2.10) $\{ \tau_j - \text{bcl}(A) - A \}$ does not contain any non-empty $i$-g-closed subset and since $\{ \tau_j - \text{bcl}(A) - A \}$ is $i$-g-closed subset of itself, then $\{ \tau_j - \text{bcl}(A) - A \} = \varnothing$. This means that $A = \{ \tau_j - \text{bcl}(A) \}$ and so $A$ is $j$-b-closed.

Proposition (2.13): A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be an $(i,j)$-g*-b-open set if $F \subseteq \tau_j\text{-bint}(A)$ whenever $F$ is $i$-g-closed and $F \subseteq A$.

Proof: Let $A$ be an $(i,j)$-g*-b-open set and $F$ be an $i$-g-closed set such that $F \subseteq A$ in a bitopological space $(X, \tau_1, \tau_2)$. Then $A^c$ is an $(i,j)$-g*-b-closed set contained in the $i$-g-closed set $F^c$. Hence, $\tau_j\text{-bcl}(A^c) \subseteq F^c$. This means that $[\tau_j\text{-bint}(A)]^c \subseteq F^c$ i.e. $F \subseteq \tau_j\text{-bint}(A)$.

Conversely, if $F$ is $i$-g-closed with $F \subseteq \tau_j\text{-bint}(A)$ & $F \subseteq A$, then $[\tau_j\text{-bint}(A)]^c \subseteq F^c$. This means that $\tau_j\text{-bcl}(A^c) \subseteq F^c$ i.e. $A^c$ is $(i,j)$-g*-b-closed. Consequently, $A$ is a $(i,j)$-g*-b-open set.

III. $i-T^*b_{1/2}$ AND $j-T^*b_{1/2}$ SPACES

The bitopological spaces $i-j T^*b_{1/2}$ & $i-j T^*b_{1/2}$ are introduced in this section, being an application of $(i,j)$-g*-b–closed sets.

Definition (3.1): A bitopological space $(X, \tau_1, \tau_2)$ is said to be an $i-j T^*b_{1/2}$ space if every $(i,j)$-g*-b–closed set is $\tau_j$-b-closed.

Definition (3.2): A bitopological space $(X, \tau_1, \tau_2)$ is said to be an $i-j T^*b_{1/2}$ space if every $(i,j)$-gb-closed set is $(i,j)$-g*-b-closed.

Proposition (3.3): A bitopological space $(X, \tau_1, \tau_2)$ is an $(i,j)$-T*-$b_{1/2}$ space if and only if it is both an $i-j T^*b_{1/2}$ and $i-j T^*b_{1/2}$ spaces.

Proof: Suppose that a bitopological space $(X, \tau_1, \tau_2)$ is an $(i,j)$-g*-b-space, then every $(i,j)$-gb-closed set is $T^*b$-closed. We, however, know that every $\tau_j$-b-closed is $(i,j)$g*-b-closed[prop.(2.2)(a)]. This means that in $(X, \tau_1, \tau_2)$, every $(i,j)$-gb-closed set is $(i,j)$g*-b-closed and consequently, $(X, \tau_1, \tau_2)$ is an $i-j T^*b_{1/2}$ space.

Also, an $(i,j)$-g*-b-closed set is $(i,j)$gb-closed i.e. $D^g_{s(i,j)} \subseteq D_{s(i,j)}$ which is a property of $(X, \tau_1, \tau_2)$ every $(i,j)$-g*-b-closed set is $\tau_j$-b-closed. Hence, $(X, \tau_1, \tau_2)$ is an $i-j T^*b_{1/2}$ space.

Conversely, let $(X, \tau_1, \tau_2)$ be both an $i-j T^*b_{1/2}$ and $i-j T^*b_{1/2}$ spaces.

For the space $(X, \tau_1, \tau_2)$ to be $i-j T^*b_{1/2}$, it provides that every $(i,j)$-gb-closed set is $(i,j)$g*-b-closed and for space$(X, \tau_1, \tau_2)$ to be $i-j T^*b_{1/2}$ if every $(i,j)$-g*-b-closed set is $\tau_j$-b-closed.

So, combining these two conditions, it happens to be the fact that, in the space $(X, \tau_1, \tau_2)$ every $(i,j)$ -gb-closed set is $\tau_j$-b-closed. Consequently, $(X, \tau_1, \tau_2)$ is an $(i,j)$-$T^*b_{1/2}$ space.

Corollary (3.4): If a bitopological space $(X, \tau_1, \tau_2)$ is both an $i-j T^*b_{1/2}$ and $i-j T^*b_{1/2}$ spaces, then it is $(i,j)$-$T^*b_{1/2}$ space.

Proof: Let $A$ a bitopological space $(X, \tau_1, \tau_2)$ be both an $i-j T^*b_{1/2}$ and $i-j T^*b_{1/2}$ spaces. Then, clearly, $(X, \tau_1, \tau_2)$ is $(i,j)$-$T^*b_{1/2}$ space. Since, $D(ij) \subseteq D(i,j)$ & $j-c(X) \subseteq j-BC(X)$, hence, due to concepts of $(i,j)$-$T^*b_{1/2}$ & $(i,j)$-$T^*b_{1/2}$ spaces, an $(i,j)$-$T^*b_{1/2}$ is also an $(i,j)$-$T^*b_{1/2}$ space, therefore, a bitopological space $(X, \tau_1, \tau_2)$, being both an $i-j T^*b_{1/2}$ and $i-j T^*b_{1/2}$ spaces is $(i,j)$-$T^*b_{1/2}$ space.

Proposition (3.5): A bitopological space $(X, \tau_1, \tau_2)$ is an $(i,j)$ - $T^*b_{1/2}$ space iff $(x)$ is $j$-open or $i$-g-closed for each $x \in X$.

Proof: Let $x \in X$ where $(X, \tau_1, \tau_2)$ is a bitopological space which is also an $(i,j)$ - $T^*b_{1/2}$ space. Suppose that $(x)$ is not $i$-g-closed. Then, $(x)$ is $(i,j)$-g*-b-open by proposition (2.10). i.e. $(x)$ is $(i,j)$-g*-b-closed.

Now, $(X, \tau_1, \tau_2)$ is an $(i,j)$ - $T^*b_{1/2}$ space, so $(x)^*\tau_j$ is $\tau_j$-b-closed . This means that $(x)$ is $j$-open or $i$-g-closed.

Conversely, let $F$ be an $(i,j)$-g*-b-closed set and $x \in \tau_j\text{-bcl}(F)$. By assumption, $(x)$ is either $j$-b-open or $i$-g-closed for any $x \in \tau_j\text{-bcl}(F)$.

Case I: Suppose that $(x)$ is $j$-b-open. Since,$(x) \cap F \neq \varnothing$, we have $x \in \mathcal{E}$.

Case II: Suppose that $(x)$ is $i$-g-closed. If $x \notin \mathcal{E}$, then $\{x\} \subseteq \tau_j\text{-bcl}(F)$, which is a contradiction to proposition (2.10). Therefore, $x \in \mathcal{E}$.

Thus, in both the cases, $x \in \tau_j\text{-bcl}(F) \Rightarrow x \in \mathcal{E}$ i.e. $\tau_j\text{-bcl}(F) \subseteq \mathcal{E}$. i.e. $F \subseteq \tau_j\text{-bcl}(F)$ i.e. $F$ is $(i,j)$-$T^*b_{1/2}$ closed.

Consequently, in the bitopological space $(X, \tau_1, \tau_2)$, every $(i,j)$-g*-b-closed set is $\tau_j$-b-closed. This provides that $(X, \tau_1, \tau_2)$ is an $(i,j)$ - $T^*b_{1/2}$ space.

IV. CONCLUSION

The concepts of $(i,j)$-g*-b-closed & $(i,j)$-gb-closed sets in a bitopological space $(X, \tau_1, \tau_2)$ have been introduced and their important properties have been also investigated & analyzed. As applications two new bitopological spaces $(i,j)$-$T^*b_{1/2}$ & $(i,j)$-$T^*b_{1/2}$ have been explored. The fundamental characteristics of such spaces have also been analyzed and studied.

The content of this paper opens the new horizon for the further study of such spaces and related existing spaces.
REFERENCES
