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Stability of a Cubic Functional Equation

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Abstract *– In this paper, the authors are interested in investigating the stability of a new type Cubic functional equation of the form*

$$f(2x-y)+f(x+2y)+f(x-y)=f(x-2y)+12\,f(x)+5\,f(y)-5\,f(x+y)$$

In the sense of Hyers-Ulam, Generalized Hyers-Ulam, Hyers - Ulam - Rassias.

Keywords *– Hyers-Ulam stability, Hyers - Ulam - Rassias stability, Cubic functional equation.*

I. INTRODUCTION

The different types of functional equations like Additive, Quadratic are introduced by Cauchy and D' Alembert. These functional equations and its stability were discussed vividly in many research papers in the middle years of 20th century. Later, Cubic, Quartic and mixed type functional equations were introduced and its stability and many other properties were investigated by many authors [12, 13, 14, 19, 21, 24, 25, 26, 27].

The stability problem of functional equations originated from a question of Ulam [33] in 1940, concerning the stability of group homomorphisms. Let $(G_1,.)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(.\,,\,.)$. Given $\varepsilon>0$, does there exist a $\delta>0$, such that if a mapping $h: G_1\to G_2$ satisfies the inequality $d\big(h(x.y),h(x)*h(y)\big)<\delta\ for\ all\ x,y\in G_1$, then there exists a homomorphism $H: G_1\to G_2$ with $d\big(h(x),H(x)\big)<\varepsilon\ for\ all\ x\in G_1$?. In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [16] gave first affirmative answer to the question of Ulam for Banach spaces. Let $f: E\to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\|\le\delta$$

for all $x,y\in E$, and for some $\delta>0$. Then there exists a unique additive mapping $A: E\to E'$ such that

$$\|f(x)-A(x)\|\le\delta$$

for all $x\in E$, Moreover if $f(tx)$ is continuous in t for each fixed $x\in E$, then A is linear. Finally in 1978, Th. M. Rassias [30] proved the following theorem.

Theorem 1.1. Let $f: E\to E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon\,(\|x\|^p+\|y\|^p)\qquad(1.1)$$

for all $x,y\in E$, where $\varepsilon\ and\ p$ are constants with $\varepsilon>0\ and\ p<1$. Then there exists a unique additive mapping $A: E\to E'$ such that

$$\|f(x)-A(x)\|\le\frac{2\varepsilon}{2-2^p}\,\|x\|^p\qquad(1.2)$$

$for\ all\ x\in E$. If $p<0$, then inequality (1. 1) holds $for\ all\ x,y\ne 0$, and (1. 2) $for\ x\ne 0$. Also, if the function $t\to f(tx)$ form $\mathbb{R}\ into\ E'$ is continuous for each fixed $x\in E$, the A is Linear.

Also in 1978, Th. M. Rassias [30] provided a generalization of the Hyers theorem which allows the Cauchy difference to be unbounded.

A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [30]. Since then, the stability problems of various functional equations have been extensively investigated by a number of authors [2, 4, 5, 7, 8, 10 - 17, 20, 21 - 32]. In particular, one of the important functional equations studied is the following functional equation [1, 2, 4, 13, 18, 25, 27]

$$f(x+y)+f(x-y)=2\,f(x)+2\,f(y)\qquad(1.3)$$

The quadratic function $f(x)=x^2$ is a solution of this functional equation, and so one usually is said to be the above functional equation to be quadratic.

In 1991, Z. Gajda [8] answered the question for the case p > 1, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations [1, 4, 5, 7 - 10, 15 – 17, 28, 29, 31].

In [23], W.-G. Park and J. H. Bae, considered the following functional equation

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$$f(2x+y)+f(2x-y)=4\left(f(x+y)+f(x-y)\right)+24\,f(x)-6\,f(y) \qquad (1.4)$$

In fact they proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function

$$B : X\times X\times X\times X \to Y$$

such that $f(x)=B\,(x,x,x,x)\ \ for\ all\ x$ [3, 5, 6, 19, 20, 22, 23, 28]. It is easy to show that the function $f(x)=\ a\,x^4$ satisfies the functional equation (1.3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

In this paper, we investigate the Hyers – Ulam – Rassias stability of a new type Cubic functional equation of the form

$$\boldsymbol{f(2x-y)+f(x+2y)\ +\ f(x-y)=f(x-2y)+\ 12\,f(x)+5\,f(y)-5\,f(x+y)} \qquad (1.5)$$

It easy to check that the function $f(x)=a\,x^3$ is a solution of the functional equation (1. 5). That is, it satisfies solution means, it is Cubic functional equation. Now in the present paper we would like to investigate the generalized Hyers – Ulam – Rassias stability of the equation (1. 5).

II. General Solution

In this section we establish the general solutions of the functional equation (1.5). Through this section, X and Y be the real vector spaces.

Theorem 2.1. A function $f : X \to Y$ is a solution of the cubic functional equation (1.5) if and only if it is of the form $f(x)=A^3(x)$ for all $x,y\ \in X$.

Proof. Assume that f satisfies the functional equation (1.5). Replacing $x=0\,,y=0$ in (1.5), we get, $f(0)=0$. Replacing $x=0\,,y=x$ in (1. 5), we obtain that $f(-2x)=-f(2x)$. Since f is a cubic functional equation, applying some simplification on (1.5), then we arrive that $f(2x)=\ 2^3\,f(x)$. On the other hand, it is obvious that, the converse part is also true. Hence, we can conclude that the Theorem is proved.

III. Hyers-Ulam-Rassias Stability

In this section, we prove the Hyers-Ulam-Rassias stability of the cubic functional equation (1.5). Throughout this section, X and Y will be a real normed space and a real Banach space, respectively. Also, we investigate the generalized Hyers – Ulam – Rassias stability of the given Cubic functional equation (1. 5). Let $f : X \to Y$ be a function then we define $D_f : X\times X \to Y$ by

$$D_f\,(x\,,y)=\boldsymbol{f(2x-y)+f(x+2y)\ +\ f(x-y)-f(x-2y)-\ 12\,f(x)-5\,f(y)+5\,f(x+y)}$$

for all $x,y\in X$.

Theorem 3. 1. Let $\varphi : X\times X \to [0,\infty)$ be a function satisfies

$$\sum_{i=0}^{\infty}\frac{\varphi(2^i x,\ 0)}{2^i}<\infty$$

for all $x\in X$, and

$$\lim_{n\to\infty}\frac{(2^n x\,,2^n y)}{2^n}=0$$

for all $x\,,y\ \in\ X$. If $f : X \to Y$ is an even function such that $f(0)=0$, and that

$$\left\|D_f(x,y)\right\|<\varphi(x,y) \qquad (3.1)$$

for all $x\,,y\in\ X$, then there exists a unique quartic function $Q : X \to Y$ satisfying (1. 5) and

$$\|f(x)-Q(x)\|\le\frac{1}{6}\sum_{i=0}^{\infty}\frac{\varphi(2^i x\,,0)}{2^i} \qquad (3.2)$$

for all $x\in X$.

Proof. Putting $y=0$ in (3.1), then we have,

$$\|\,f(2x)-6\,f(x)\|\ \le\ \varphi(x,0) \qquad (3.3)$$

Then dividing by 8 on both sides in (3. 3), to obtain,

$$\left\|\frac{f(2x)}{3.2}-f(x)\right\|\le\frac{1}{6}\varphi(x,0) \qquad (3.4)$$

for all $x\in\ X$. Replacing x by $2x$ in (3.4), we get

$$\left\|\frac{f(4x)}{3.2}-f(2x)\right\|\le\frac{1}{6}\varphi(2x,0) \qquad (3.5)$$

Combine (3.4) and (3.5) by use of the triangle inequality to get

$$\left\|\frac{f(4x)}{3.2^2}-f(x)\right\|\le\frac{1}{6}\left(\frac{\varphi(2x,0)}{2}+\varphi(x,0)\right) \qquad (3.6)$$

By induction on $n\ \in N$, we can prove,

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$$\left\|\frac{f(2^n x)}{3.2^n} - f(x)\right\| \leq \frac{1}{6}\sum_{i=0}^{n-1} \frac{\varphi(2^i x, 0)}{2^i} \qquad (3.7)$$

Dividing (3. 7) by 2^m and replacing x by $2^m x$ we get,

$$\left\|\frac{f(2^{m+n}x)}{2^{m+n}} - \frac{f(2^m x)}{2^m}\right\| = \frac{1}{2^m}\|f(2^n 2^m x) - f(2^m x)\|$$

$$\leq \frac{1}{6 \times 2^m}\sum_{1=0}^{n-1}\frac{\varphi(2^i x, 0)}{2^i}$$

$$\leq \frac{1}{6}\sum_{i=0}^{\infty}\frac{\varphi(2^i 2^m x, 0)}{2^{m+i}}$$

for all $x \in X$. This shows that $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence in Y, by taking the $\lim m \to \infty$. Since Y is a Banach space, then the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ converges. We define $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

for all $x \in X$. Since f is even function, then Q is even. On the other hand we have

$$\|D_Q(x,y)\| = \lim_{n\to\infty}\frac{1}{2^n}\|D_f(2^n x, 2^n y)\|$$

$$\leq \lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{2^n} = 0.$$

for all $x, y \in X$. Hence by our assumption, we conclude that Q is a quartic function. Now we have to show Q is unique. Suppose that there exists another quartic function $\bar{Q} : X \to Y$ which satisfies (1.5) and (3.2). We have, $Q(2^n x) = 2^n Q(x)$ and $\bar{Q}(2^n x) = 2^n \bar{Q}(x)$ for all $x \in X$. It follows that

$$\|\bar{Q}(x) - Q(x)\| = \frac{1}{2^n}\|\bar{Q}(2^n x) - Q(2^n x)\|$$

$$\leq (\|\bar{Q}(2^n x) - f(2^n x)\| + \|f(2^n x) - Q(2^n x)\|)$$

$$\leq \frac{1}{3}\sum_{i=0}^{\infty}\frac{\varphi(2^{n+i}x, 0)}{2^{n+i}}$$

for all $x \in X$. By taking $n \to \infty$ in this inequality we have,

$$\lim_{n\to\infty}\|\bar{Q}(x) - Q(x)\| \to 0$$

$$\bar{Q}(x) = Q(x).$$

Theorem 3. 2. Let $\varphi : X \times X \to [0, \infty)$ be a function satisfies

$$\sum_{i=0}^{\infty} 2^i \varphi(2^{-i-1}x, 0) < \infty$$

for all $x, y \in X$, and $\lim_{n\to\infty} 2^n \varphi(2^{-n}x, 2^{-n}y) = 0$ for all $x, y \in X$. Suppose that an even function $f : X \to Y$ satisfies $f(0) = 0$, and (3. 1). Then the limit $Q(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ exists for all $x \in X$ and $Q : X \to Y$ is a unique quartic function satisfies (1. 5) and

$$\|f(x) - Q(x)\| \leq \frac{1}{3}\sum_{i=0}^{\infty} 2^i \varphi(2^{-i-1}x, 0) \qquad (3.8)$$

for all $x, y \in X$.

Proof. Put $x = 0$ in (3. 1), we get,

$$\| f(x) - f(2x)\| \leq \varphi(x, 0) \qquad (3.9)$$

Replacing x by $\frac{x}{2}$ in (3. 9) and dividing by 4, we get,

$$\left\| f\left(\frac{x}{2}\right) - f(x)\right\| \leq \frac{1}{3}\varphi\left(\frac{x}{2}, 0\right)$$

$$\| f(2^{-1}x) - f(x)\| \leq \frac{1}{3}\varphi(2^{-1}x, 0) \qquad (3.10)$$

for all $x \in X$, replacing x by $\frac{x}{2}$ in (3. 10), we obtain,

$$\| f(4^{-1}x) - f(2^{-1}x)\| \leq \frac{1}{3}\varphi(2^{-2}x, 0)$$

$$\| f(2^{-2}x) - f(2^{-1}x)\| \leq \frac{1}{3}\varphi(2^{-2}x, 0) \qquad (3.11)$$

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Multiplying 2 on both sides of (3. 11), we get,

$$\|2^2 f(2^{-2} x) - 2 f(2^{-1}x)\| \le \frac{1}{3}\left(2 \ \varphi(2^{-2}x, 0)\right) \tag{3.12}$$

Combining (3. 10) and (3. 12) by use of triangle inequality to obtain,

$$\|2^2 f(2^{-2} x) - f(x)\| \le \frac{1}{3} (2 \ \varphi(2^{-2}x, 0) + \varphi(2^{-2}x, 0)\ ) \tag{3.13}$$

By induction on $n \in N$, we have,

$$\|2^n f(2^{-n} x) - f(x)\| \le \frac{1}{3} \sum_{i=0}^{n-1} 2^i \ \varphi(2^{-i-1}x, 0) \tag{3.14}$$

Multiplying (3. 14) by 2^m and replacing $x \; by \; 2^{-m} x$ to obtain,

$$\begin{aligned}\|2^{m+n} f(2^{-m-n} x) - 2^m f(2^{-m}x)\| &= 2^m \|2^n f(2^{-m}2^{-n} x) - f(2^{-m}x)\| \\ &\le 2^m \frac{1}{3} \sum_{i=0}^{n-1} 2^i \ \varphi(2^{-i-1}\, 2^{-m} x, 0) \\ &\le \frac{1}{3} \sum_{i=0}^{n-1} 2^{m+i} \ \varphi(2^{-i-1}\, 2^{-m} x, 0)\end{aligned}$$

for all $x, y \in X$. By taking the $\lim m \to \infty$, it follows that $\{ 2^n f(2^{-n}x) \}$ is a Cauchy sequence in Y. Since Y is a Banach space, then the sequence $\{ 2^n f(2^{-n}x) \}$ converges. Now we will define a function, $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$$

for all $x \in X$. Then the rest of the proof is similar to the Theorem 3. 1, that is,

Since f is even function, then Q is even. On the other hand we have

$$\begin{aligned}\|D_Q(x, y)\| &= \lim_n 2^n \|D_f(2^{-n}x, 2^{-n}y)\| \\ &\le \lim_n 2^n \ \varphi\,(2^{-n}x, 2^{-n}y) = 0.\end{aligned}$$

for all $x, y \in X$. Hence by our assumption, we conclude that Q is a quartic function. Now we have to show Q is unique. Suppose that there exists another quartic function $\bar{Q} : X \to Y$ which satisfies (1.5) and (3. 8). We have, $Q(2^{-n}x) = 2^n Q(x)$ and $\bar{Q}(2^{-n}x) = 2^n \bar{Q}(x)$ for all $x \in X$. It follows that

$$\begin{aligned}\|\bar{Q}(x) - Q(x)\| &= 2^n \|\bar{Q}(2^{-n}x) - Q(2^{-n}x)\| \\ &\le (\|\bar{Q}(2^{-n}x) - f(2^{-n}x)\| + \|f(2^{-n}x) - Q(2^{-n}x)\|) \\ &\le \frac{2}{3} \sum_{i=0}^{\infty} 2^{n+i} \varphi(2^{-i-1}x, 0)\end{aligned}$$

for all $x \in X$. By taking $n \to \infty$ in this inequality we have,

$$\lim_{n\to\infty} \|\bar{Q}(x) - Q(x)\| \to 0$$

$$\bar{Q}(x) = Q(x).$$

This completes the proof of the Theorem.

Theorem 3. 3. Let $\varphi : X \times X \to [0, \infty)$ be a function such that

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, \ 0)}{2^i} < \infty$$

and

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0 \tag{3.15}$$

for all $x, y \in X$. If $f : X \to Y$ in an odd function such that

$$\|D_f(x, y)\| < \varphi(x, y) \tag{3.16}$$

for all $x, y \in X$, then there exists a unique additive function $A : X \to Y$ satisfying (1. 5) and

$$\|f(x) - A(x)\| \le \frac{1}{3} \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 0)}{2^i} \tag{3.17}$$

for all $x \in X$.

Proof. Setting $x = 0$ in (3. 16) to get,

$$\| f(2x) - 2 f(x)\| \le \frac{1}{3} \ \varphi(x, 0) \tag{3.18}$$

Then dividing by 2 on both sides in (3. 18), to obtain,

$$\left\|\frac{f(2x)}{2} - f(x)\right\| \le \frac{1}{6}\,\varphi(x,0) \qquad (3.19)$$

for all $x \in X$. Replacing x by $2x$ in (3.19), we get

$$\left\|\frac{f(4x)}{2} - f(2x)\right\| \le \frac{1}{6}\,\varphi(2x,0) \qquad (3.20)$$

Combine (3.19) and (3.20) by use of the triangle inequality to get

$$\left\|\frac{f(4x)}{2^2} - f(x)\right\| \le \frac{1}{6}\left(\frac{\varphi(2x,0)}{2} + \varphi(x,0)\right) \qquad (3.21)$$

Now we using iterative method and induction on $n \in N$, we can prove our next relation.

$$\left\|\frac{f(2^n x)}{2^n} - f(x)\right\| \le \frac{1}{6}\sum_{i=0}^{n-1} \frac{\varphi(2^i x,0)}{2^i} \qquad (3.22)$$

Dividing (3. 22) by 2^m and replacing x by $2^m x$ we get,

$$\begin{aligned}\left\|\frac{f(2^{m+n}x)}{2^{m+n}} - \frac{f(2^m x)}{2^m}\right\| &= \frac{1}{2^m}\|f(2^n 2^m x) - f(2^m x)\| \\ &\le \frac{1}{6\times 2^m}\sum_{1=0}^{n-1}\frac{\varphi(2^i x,0)}{2^i} \\ &\le \frac{1}{6}\sum_{i=0}^{\infty}\frac{\varphi(2^i 2^m x,0)}{2^{m+i}} \qquad (3.23)\end{aligned}$$

for all $x \in X$. Taking as $\lim m \to \infty$ in (3. 23), then the right hand side of the inequality tends to zero. Since Y is a Banach space, then

$$A(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

exists for all $x \in X$. Since f is an odd function, then A is odd. On the other hand by (3. 15) we have,

$$\begin{aligned}\|D_A(x,y)\| &= \lim_{n\to\infty}\frac{1}{2^n}\|D_f(2^n x, 2^n y)\| \\ &\le \lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{2^n} = 0.\end{aligned}$$

for all $x, y \in X$. Hence by our assumption, we conclude that A is an additive function. Now we have to show A is unique. Suppose that there exists another quartic function $\bar{A} : X \to Y$ which satisfies (1.5) and (3. 17). We have, $A(2^n x) = 2^n A(x)$ and $\bar{A}(2^n x) = 2^n \bar{A}(x)$ for all $x \in X$. It follows that

$$\begin{aligned}\|\bar{A}(x) - A(x)\| &= \frac{1}{2^n}\|\bar{A}(2^n x) - A(2^n x)\| \\ &\le (\|\bar{A}(2^n x) - f(2^n x)\| + \|f(2^n x) - A(2^n x)\|) \\ &\le \frac{1}{3}\sum_{i=0}^{\infty}\frac{\varphi(2^{n+i}x,0)}{2^{n+i}}\end{aligned}$$

for all $x \in X$. By taking $n \to \infty$ in this inequality we have,

$$\lim_{n\to\infty}\|\bar{A}(x) - A(x)\| \to 0$$
$$\bar{A}(x) = A(x).$$

This completes the proof.

Theorem 3. 4. Let $\varphi : X \times X \to [0,\infty)$ be a function satisfies

$$\sum_{i=0}^{\infty} 2^i\,\varphi(2^{-i-1}x,0) < \infty$$

for all $x, y \in X$, and $\lim_{n\to\infty} 2^n\,\varphi(2^{-n}x, 2^{-n}y) = 0$ for all $x, y \in X$. Suppose that an odd function $f : X \to Y$ satisfies $f(0) = 0$, and (3. 1). Then the limit $A(x) = \lim_{n\to\infty} 2^n\,f(2^{-n}x)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfing (1. 5) and

$$\|f(x) - A(x)\| \le \frac{1}{4}\sum_{i=0}^{\infty} 2^i\,\varphi(2^{-i-1}x,0)$$

for all $x, y \in X$.

Proof. The proof is similar to the proof of the Theorem 3. 3.

Theorem 3. 5. Let $\varphi : X \times X \to [0,\infty)$ be a function such that

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$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x,\ 0)}{2^i} < \infty \quad and \quad \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0$$

for all $x \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality $\|D_f(x,y)\| < \varphi(x,y)$ for all $x, y \in X$, and $f(0) = 0$. Then there exists a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1. 5) and

$$\|f(x) - Q(x) - A\,(x)\| \le \frac{1}{6} \sum_{i=0}^{\infty} \left(\frac{\varphi(2^i x\,, 0) + \varphi(-2^i x\,, 0)}{2 \times 2^i} + \frac{4\left(\varphi(2^i x\,, 0) - \varphi(-2^i x\,, 0)\right)}{2 \times 2^i} \right)$$

(3. 23) for all $x, y \in X$.

Proof. We have

$$\|D_{f_e}(x,y)\| < \frac{2}{3}\ (\varphi(x,y) + \varphi(-x,-y))$$

for all $x, y \in X$. Since $f_e(0) = 0$ and f_e is an even function, then by Theorem 3. 1, there exists a unique quartic function $Q : X \to Y$ satisfying,

$$\|\ f_e(x) - Q(x)\| \ \le\ \frac{1}{6} \sum_{i=0}^{\infty} \frac{\varphi(2^i x\,, 0) + \varphi(-2^i x\,, 0)}{2 \times 2^i} \qquad (3.24)$$

for all $x, y \in X$. On the other hand f_o is odd function and we have,

$$\|D_{f_o}(x,y)\| < \frac{2}{3}\ (\varphi(x,y) - \varphi(-x,-y))$$

for all $x, y \in X$. Since $f_o(0) = 0$ and f_e is an even function, then by Theorem 3. 3, there exists a unique additive function $A : X \to Y$ satisfying,

$$\|\ f_o(x) - A(x)\| \ \le\ \frac{2}{3} \sum_{i=0}^{\infty} \frac{\varphi(2^i x\,, 0) - \varphi(-2^i x\,, 0)}{2 \times 2^i} \qquad (3.25)$$

for all $x, y \in X$. Combining (3. 24) and (3. 25) we obtain (3. 23). This concludes the proof of the Theorem.

By Theorem 3. 5, we are going to investigate the Hyers – Ulam – Rassias stability problem for functional equation (1. 5).

Corollary 3. 6. Let $\theta \ge 0, P < 1$ suppose $f : X \to Y$ satisfies the inequality

$$\|D_f(x,y)\| \ \le\ \theta\ (\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Since $f(0) = 0$. Then there exists a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1. 5), and

$$\|f(x) - Q(x) - A\,(x)\| \le \frac{\theta}{6}\ \ \|x\|^p \left(\frac{2}{2 -\ 2^p} + \frac{4}{1 -\ 2^{p-1}}\right)$$

for all $x, y \in X$.

By corollary 3. 6, we solve the following Hyers – Ulam stability problem for the functional equation (1. 5).

Corollary 3. 7. Let ε be the positive real number, and let $f : X \to Y$ be a function satisfies

$$\|D_f(x,y)\| \ \le\ \varepsilon$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1. 5), and we have,

$$\|f(x) - Q(x) - A\,(x)\| \le \frac{\varepsilon}{3}$$

for all $x, y \in X$.

By applying Theorem 3. 2 and 3. 4, we have the following theorem.

Theorem 3. 8. Let $\varphi : X \times X \to [0, \infty)$ be a function such that

$$\sum_{i=0}^{\infty} 2^i\ \varphi(2^{-i-1}x, 0) < \infty\ and\ \lim_{n} 2^n\ \ \varphi\ (2^n x, 2^n y)\ = 0$$

for all $x, y \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$\|D_f(x,y)\| < \varphi(x,y)$$

for all $x, y \in X$, and $f(0) = 0$. Then there exists a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1. 5) and

$$\|f(x) - Q(x) - A\,(x)\| \le \sum_{i=0}^{\infty} \left(\frac{2^i}{3} + 2^i\right) \left(\frac{\varphi(2^{-i-1}x\,, 0) + \varphi(-2^{-i-1}x\,, 0)}{2}\right)$$

for all $x, y \in X$.

Corollary 3. 9. Let $\theta \ge 0,\ P > 4$. Suppose that $f : X \to Y$ satisfies the inequality

$$\left\|D_f(x,y)\right\| \leq \theta\,(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Since $f(0) = 0$. Then there exists a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1. 5), and

$$\|f(x) - Q(x) - A\,(x)\| \leq \frac{\theta}{3 \times 2^p}\ \|x\|^p \left(\frac{1}{1 - 2^{4-p}} + \frac{1}{1 - 2^{1-p}}\right)$$

for all $x, y \in X$.

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