IJSES

International Journal of Scientific Engineering and Science

Volume 1, Issue 7, pp. 65-69, 2017. *ISSN (Online): 2456-7361*

Stability of Quintic Functional Equation in Matrix Normed Spaces: A Fixed Point Approach

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Abstract— *In this paper, we prove the Hyers-Ulam stability for a quintic functional equation* $f(3u+v)-5f(2u+v)+f(2u+v)+10f(u+v)-5f(u-v)=10f(v)+f(3u)-3f(2u)-27f(u)$ *in Matrix Normed Space using Fixed point method.*

Keywords— *Hyers-Ulam stability, Quintic functional equation, Matrix Normed space. Mathematics Subject Classification: 39B82, 34K20, 26D10.*

I. Introduction

The first stability problem of functional equations created from a question of Ulam [21] relating to group homomorphisms and solved by Hyers [8]. The result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [19] for linear mappings. The result of Rassias has furnished a lot of influence during the past years in the development of the Hyers-Ulam cocepts. This new concept is called the Hyers-Ulam-Rassias stability. A generalization of Rassias's theorem was obtained by Gavruta [6] by replacing the difference cauchy equation by a general control function. In 2010, Xu et al., [22] obtained the general solution and investigated the Ulam stability problem for the following quintic functional equation

$$f(u+3v)-5f(u+2v)+10f(u+v)-10f(u)+5f(u-v)-f(u-2v)=120f(v) \tag{1}$$

In 2013, Park et al., [15] introduced the following new form of quintic functional equation

$$f(3u+v)-5f(2u+v)+f(2u+v)+10f(u+v)-5f(u-v)=10f(v)+f(3u)-3f(2u)-27f(u) \tag{2}$$

It is easily verified that that the function $f(u)=au^5$ satisfies the above functional equations. In other words, every solution of the quintic functional equation is called a quintic mapping. The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of matricially normed spaces [20] implies that quotients, mapping spaces, and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory.

The proof given in [20] appealed to the theory of ordered operator spaces [2] . Effros and Ruan [5] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [18] and Haagerup [7].

We will use the following notations:

$M_n(X)$ is the set of all $n\times n$-matrices in X ;

$e_j \in M_{1,n}(\mathbb{C})$ is that jth component is 1 and the other components are zero ;

$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;

$E_{ij}\otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero. For $x\in M_n(X), y\in M_k(X)$.

$$x\oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X,\{\|.\|_n\})$ is a matrix normed space if and only if $(M_n(X),\|.\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \le \|A\|\|B\|\|x\|_n$ holds for $A\in M_{k,n}(\mathbb{C})$, $B\in M_{n,k}(\mathbb{C})$ and $x=(x_{ij})\in M_n(X)$, and that $(X,\{\|.\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X,\{\|.\|_n\})$ is a matrix normed space. A matrix normed space $(X,\{\|.\|_n\})$ is called an L^∞ -matrix normed space if $\|x\oplus y\|_{n+k} = max\{\|x\|_n,\|y\|_k\}$ holds for all $x\in M_n(X)$ and all $y\in M_k(X)$.

International Journal of Scientific Engineering and Science
Volume 1, Issue 7, pp. 65-69, 2017. ISSN (Online): 2456-7361

Let E,F be vector spaces. For a given mapping $h:E\to F$ and a given positive integer n, define $h_n:M_n(E)\to M_n(F)$ by, $h_n([x_{ij}])=[h(x_{ij})]$ for all $[x_{ij}]\in M_n(E)$.

Throughout this paper, let $(X,\|.\|_n)$ be a matrix normed space, $(Y,\|.\|_n)$ be a matrix Banach space and let n be a fixed positive integer.

Lemma 1 *[4, 10, 11] Let* $(X,\{\|.\|_n\})$ *be a matrix normed space.*

1. $\|E_{kl}\otimes x\|_n=\|x\|$ for all $x\in X$.
2. $\|x_{kl}\|\le\|[x_{ij}]\|_n\le\sum_{i,j=1}^{n}\|x_{ij}\|$, for $[x_{ij}]\in M_n(X)$.
3. $\lim_{n\to\infty}x_n=x$ if and only if $\lim_{n\to\infty}x_{nij}=x_{ij}$ for $x_n=[x_{nij}]$, $x=[x_{ij}]\in M_n(X)$.

Proof. (1) Since $E_{kl}\otimes x=e_k^*xe_l$ and $\|e_k^*\|=\|e_l\|=1$,$\|E_{kl}\otimes x\|_n\le\|x\|$.

Since $e_k(E_{kl}\otimes x)e_l^*=x$, $\|x\|\le\|E_{kl}\otimes x\|_n$. So, $\|E_{kl}\otimes x\|_n=\|x\|$.

(2) $e_kxe_l^*=x_{kl}$ and $\|e_k\|=\|e_l^*\|=1$,$\|x_{kl}\|\le\|[x_{ij}]\|_n$. Since $[x_{ij}]=\sum_{i,j=1}^{n}E_{ij}\otimes x_{ij}$, $\|[x_{ij}]\|_n=\left\|\sum_{i,j=1}^{n}E_{ij}\otimes x_{ij}\right\|_n\le\sum_{i,j=1}^{n}\|E_{ij}\otimes x_{ij}\|_n=\sum_{i,j=1}^{n}\|x_{ij}\|$

(3) By (2)$\Rightarrow$ $\|x_{nkl}-x_{kl}\|\le\|[x_{nij}-x_{ij}]\|_n=\|[x_{nij}]-[x_{ij}]\|_n\le\sum_{i,j=1}^{n}\|x_{nij}-x_{ij}\|$. So we get the result.

Definition 2 *[3] Let X be a set. A function $\rho:X\times X\to[0,\infty]$ is called a generalized metric on X if ρ satisfies*

1. $\rho(u,v)=0$ if and only if $u=v$;
2. $\rho(u,v)=\rho(v,u)$ for all $u,v\in X$;
3. $\rho(u,w)\le\rho(u,v)+\rho(v,w)$ for all $u,v,w\in X$.

Theorem 3 *[3] Let (X,ρ) be a complete generalized metric space and $L:X\to Y$ be a strictly contractive mapping with a Lipschitz constant $\eta<1$. Then, for each given element $u\in U$, either* $\rho(L^ku,L^{k+1}u)=\infty$ for all nonnegative integer k or there exists a positive integer k_0 such that

1. $\rho(L^ku,L^{k+1}u)<\infty$, for all $k\ge k_0$;
2. the sequence $\{L^ku\}$ converges to a fixed point v^* of L;
3. v^* is the unique fixed point of L in the set $Y=\{v\in X\mid\rho(L^{k_0}u,v)<\infty\}$;
4. $\rho(v,v^*)\le\dfrac{1}{1-\eta}\rho(v,Lv)$ for all $v\in Y$.

In 1996, Isac and Rassias [9] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. A number of mathematicians were attracted to the relevent stability results of Rassias and stimulated to investigate the stability problems of functional equations.

In 2013, Choonkil park et.al.[16, 17], were first to provide the Hyers -Ulam stability of functional equations in matrix normed spaces by using direct method and an additive functional inequality in matrix normed space by using fixed point method. In 2015, Murali and Vithya [14] were first to proved the Hyers- Ulam stability of additive and quadrative functional equations in matrix normed space by using fixed point method.

In recently, a number of authors have investigate the stability of several functional equations in different spaces via fixed point approach. In this paper, we prove the stability of a quintic functional equation (2) in matrix normed spaces by using fixed point method.

II. Stability of Quintic Functional Equation in Matrix Normed Spaces

In this section, we prove the stability of the quintic functional equation (2) in matrix normed spaces by using fixed point method.

For a mapping $f:X\to Y$, define $Df:X^2\to Y$ and $Df_n:M_n(X^2)\to M_n(Y)$ by,

$Df(u+v) = f(3u+v) - 5f(2u+v) + f(2u+v) + 10f(u+v) - 5f(u-v) - 10f(v) - f(3u) + 3f(2u) + 27f(u)$
for all $u, v \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 4 *Let* $\phi : X^2 \to [0, \infty)$ *be a function such that there exists a* $\eta < 5$ *with*

$$\phi(u,v) \le 2^5 \eta \phi(\frac{u}{2}, \frac{v}{2}) \tag{3}$$

for all $u, v \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\| Df_n([x_{ij}],[y_{ij}]) \right\| \le \sum_{i,j=1}^{n} \phi(x_{ij}, y_{ij}) \tag{4}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\left\| f_n([x_{ij}]) - Q_n([x_{ij}]) \right\|_n \le \sum_{i,j=1}^{n} \frac{1}{2^5(1-\eta)} \phi(x_{ij}, 0) \qquad \forall x = [x_{ij}] \in M_n(X). \tag{5}$$

Proof. Let $n = 1$. Then (4) is equivalent to

$$\left\| f(3u+v) - 5f(2u+v) + f(2u+v) + 10f(u+v) - 5f(u-v) - 10f(v) - f(3u) + 3f(2u) + 27f(u) \right\| \le \phi(u,v) \qquad \forall u, v \in X. \tag{6}$$

Letting $v = 0$ in (6), we get

$$\left\| f(2u) - 2^5 f(u) \right\| \le \phi(u,0) \qquad \forall u \in X. \tag{7}$$

So $$\left\| f(u) - \frac{1}{2^5} f(2u) \right\| \le \frac{1}{2^5} \phi(u,0) \qquad \forall u \in X. \tag{8}$$

Consider the set $N = \{ f : X \to Y \}$ and introduce the generalized metric on N

$$\rho(f,g) = \inf \{ \mu \in R_+ : \| f(u) - g(u) \| \le \phi(u,0), \forall u \in X \},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (N, ρ) is complete (see the proof of [[12], Lemma 2.1]).
Now we consider the linear mapping $L : N \to N$ such that

$$Lf(u) = \frac{1}{2^5} f(2u) \qquad \forall u \in X.$$

Let $f, g \in N$ be given such that $d(f,g) = \nu$. This means that

$$\| f(u) - g(u) \| \le \phi(u,0) \qquad \forall u \in X.$$

Hence $\| Lf(u) - Lg(u) \| = \left\| \frac{1}{2^5} f(2u) - \frac{1}{2^5} g(2u) \right\| \le \eta \phi(u,0)$ for all $u \in X$.

Let $f, g \in N$ be given such that $\rho(f,g) = \lambda$ implies that $\rho(Lf, Lg) \le \eta \lambda$.
This means that

$$\rho(Lf, Lg) \le \eta \rho(f,g) \qquad \forall f, g \in N.$$

It follows from (8) that $\rho(f, Lf) \le \frac{1}{2^5}$.

By Theorem 3, there exists a mapping $Q : X \to Y$ satisfying the following:
(1) Q is a fixed point of L, i.e.,

$$Q(2u) = 2^5 Q(u) \qquad \forall u \in X. \tag{9}$$

The mapping Q is a unique fixed point of L in the set $S = \{ g \in N : d(f,g) < \infty \}$. The mapping Q is a unique mapping satisfies (9) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\| f(u) - g(u) \| \le \phi(u,0) \forall u \in X.$$

(2) $\rho(L^k f, Q) \to 0$ as $k \to \infty$. This implies the equality

$$\lim_{k\to\infty}\frac{1}{2^{5k}}f(2^k u)=\mathrm{Q}(u) \qquad \forall \mathrm{u}\in \mathrm{X}.$$

(3) $\rho(f,\mathrm{Q})\le\frac{1}{1-\eta}\rho(f,Lf)$, which implies the inequality $\rho(f,\mathrm{Q})\le\frac{1}{2^5(1-\eta)}$.

So $$\|f(u)-\mathrm{Q}(u)\|\le\frac{1}{2^5(1-\eta)}\phi(u,0) \qquad \forall \mathrm{u}\in \mathrm{X}. \tag{10}$$

By (6),

$$\lim_{k\to\infty}\frac{1}{2^{5k}}\Big\|f(2^k(3u+v))-5f(2^k(2u+v))+f(2^k(2u+v))+10f(2^k(u+v))$$
$$-\ 5f(2^k(u-v))-10f(2^k(v))-f(2^k(3u))+3f(2^k(2u))+27f(2^k(u))\Big\|\le\lim_{k\to\infty}\frac{1}{2^5K}\phi(2^k u,2^k v)$$

$\mathrm{Q}(3u+v)-5\mathrm{Q}(2u+v)+\mathrm{Q}(2u+v)+10\mathrm{Q}(u+v)-5\mathrm{Q}(u-v)-10\mathrm{Q}(v)-\mathrm{Q}(3u)+3\mathrm{Q}(2u)+27\mathrm{Q}(u)\to 0$ $\forall$ $u,v\in X$. Thus

$$\mathrm{Q}(3u+v)-5\mathrm{Q}(2u+v)+\mathrm{Q}(2u+v)+10\mathrm{Q}(u+v)-5\mathrm{Q}(u-v)=10\mathrm{Q}(v)+\mathrm{Q}(3u)-3\mathrm{Q}(2u)-27\mathrm{Q}(u).$$

So, the mapping $\mathrm{Q}:X\to Y$ is quintic. By Lemma 1 and (10),

$$\|f_n([x_{ij}])-\mathrm{Q}_n([x_{ij}])\|\le\sum_{i,j=1}^{n}\|f(x_{ij})-C(x_{ij})\|\le\sum_{i,j=1}^{n}\frac{1}{2^5(1-\eta)}\phi(x_{ij},0)$$

for all $x=[x_{ij}]\in M_n(x)$. Thus $\mathrm{Q}:X\to Y$ is a unique quintic mapping satisfying (5).

Corollary 1 *Let q,σ be positive real numbers with $q<5$. Let $f:X\to Y$ be a mapping such that*

$$\|Df_n([x_{ij}],[y_{ij}])\|_n\le\sum_{i,j=1}^{n}\sigma(\|x_{ij}\|^q+\|y_{ij}\|^q) \tag{11}$$

for all $x=[x_{ij}],y=[y_{ij}]\in M_n(X)$. Then there exists a unique quintic mapping $\mathrm{Q}:X\to Y$ such that

$$\|f_n([x_{ij}])-\mathrm{Q}_n([x_{ij}])\|_n\le\sum_{i,j=1}^{n}\frac{\sigma}{2^5-2^q}\|x_{ij}\|^q \quad \forall\ x=[x_{ij}]\in M_n(X).$$

Proof. The proof follows from Theorem 4 by taking $\phi(u,v)=\sigma(\|u\|^q+\|v\|^q)$ for all $u,v\in X$ Then we can choose $\beta=2^{q-5}$, and we get the desired result.

Theorem 5 *Let $f:X\to Y$ be a mapping satisfying (4) for which there exists a function $\phi:X^2\to[0,\infty)$ such that there exists a $\eta<5$ with*

$$\phi(u,v)\le\frac{\eta}{2^5}\phi(tu,tv)$$

for all $u,v\in X$. Then there exists a unique quintic mapping $\mathrm{Q}:X\to Y$ such that

$$\|f_n([x_{ij}])-\mathrm{Q}_n([x_{ij}])\|_n\le\sum_{i,j=1}^{n}\frac{\eta}{2^5(1-\eta)}\phi(x_{ij},0) \qquad \forall \mathrm{x}=[\mathrm{x}_{\mathrm{ij}}]\in \mathrm{M}_\mathrm{n}(\mathrm{X}). \tag{12}$$

Proof. Let (N,ρ) be the generalized metric space defined in the proof of Theorem 4. Now we consider the linear mapping $L:N\to N$ such that $Lf(u)=2^5 f(\frac{u}{2})$ for all $u\in X$. It follows from (7) that $\rho(f,Lf)\le\frac{\eta}{2^5}$. So, $\rho(f,\mathrm{Q})\le\frac{\eta}{2^5(1-\eta)}$.

The rest of the proof is similar to the proof of Theorem 4.

Corollary 2 *Let q,σ be positive real numbers with $q>5$. Let $f:X\to Y$ be a mapping satisfying (11). Then there exists a unique quintic mapping $\mathrm{Q}:X\to Y$ such that*

$$\left\| f_n([x_{ij}]) - \mathrm{Q}_n([x_{ij}]) \right\|_n \le \sum_{i,j=1}^{n} \frac{\sigma}{2^q - 2^5} \left\| x_{ij} \right\|^q \qquad \text{x=[x\_ij] M\_n(X)}.$$

Proof. The proof follows from Theorem 5 by taking $\phi(u,v) = \sigma(\|u\|^q + \|v\|^q)$ for all $u, v \in X$ Then we can choose $\beta = 2^{5-q}$, and we get the desired result.

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