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Kumaraswamy SUSHILA Distribution

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Abstract— *In this paper, we present a new class of distributions called Kumaraswamy Sushila distribution. This class of distributions contains several distributions such as generalized Lindley distribution. The hazard function, moments and moment generating function are presented. Moreover, we discuss the maximum likelihood estimation of this distribution.*

Keywords—*Sushila distribution, Maximum likelihood estimation Moments.*

I. INTRODUCTION

In the last few years, new generated families of continuous distributions have attracted several statisticians to develop new models. These families are obtained by introducing one or more additional shape parameter(s) to the baseline distribution. Some of the genrated families are: the beta-G (Eugene et al., 2002), gamma-G (type 1) (Zografos and Balakrishanan, 2009), Kumaraswamy-G (Kw-G; Cordeiro and de Castro, 2011), gamma-G (type 2) (Risti´c and Balakrishanan, 2012), transformed-transformer (T-X; Alzaatreh et al., 2013), Weibull- G (Bourguignon et al. (2014), Garhy – G family is introduced by Elgarhy et al. (2016), exponentiated Weibull-G by Hassan and Elgarhy (2016 b), Hassan and Elgarhy (2016 a) introduced a new family called Kumaraswamy Weibull-generated (KwW-G), type II half logistic – G family intoduced by Hassan et al.(2017), Elgarhy et al. (2017) introduced exponentiated extended – G family. The cdf and pdf of Kumaraswamy-generated family is given by

$$F(x) = 1 - \{1 - G(x)^a\}^b \, , a, b > 0 \qquad (1)$$

$$f(x) = abg(x)G(x)^{a-1}\{1 - G(x)^a\}^{b-1}. \qquad (2)$$

Shanker et al. (2013) introduced Sushila distribution (SD) of which the Lindley distribution (LD) is a particular case. Quasi probability density function (pdf)

$$g(x) = \frac{\theta^2}{\alpha(\theta+1)}\left(1 + \frac{x}{\alpha}\right) e^{-\frac{\theta}{\alpha}x} \quad x, \theta, \alpha > 0. \qquad (3)$$

It can easily be seen that at $\alpha = 1$, the SD (3) reduces to the Lindley distribution (1958) with probability density function. The pdf (3) can be shown as a mixture of gamma$\left(1, \frac{\theta}{\alpha}\right) \sim \exp\left(\frac{\theta}{\alpha}\right)$ and gamma$\left(2, \frac{\theta}{\alpha}\right)$ distributions as follows

$$g(x, \theta, \alpha) = p g_1(x) + (1-p) g_2(x) \ ,$$

Wher $p = \frac{\theta}{\theta+1}$, $g_1(x) = \frac{\theta}{\alpha} e^{-\frac{\theta}{\alpha}x}$, $g_2(x) = \frac{\theta^2}{\alpha^2} x e^{-\frac{\theta}{\alpha}x}$.

The cumulative distribution function (cdf) of SD is obtained as

$$G(x) = 1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}. \qquad (4)$$

This paper offers new distribution with four parameters called Kumaraswamy Sushila distribution, this article is organized as follows. In Section 2, we define the Kumaraswamy Sushila distribution, the expansion for the density function of the ES distribution and some special cases. Quantile function, moments, moment generating function are discussed in Section 3. In Section 4 included Maximumlikelihood estimation. Finally, conclusion in Section 5.

II. KUMARASWAMY SUSHILA DISTRIBUTION

In this section, we introduce the four − parameter Kumaraswamy Sushila KwS distribution, the cdf and pdf of the KwS distribution can be written respectively as

$$F(x) = 1 - \left\{1 - \left\{1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}\right\}^a\right\}^b, \ \theta, \alpha, a, b > 0. \qquad (5)$$

$$f(x) = \frac{ab\theta^2}{\alpha(\theta+1)}\left(1 + \frac{x}{\alpha}\right) e^{-\frac{\theta}{\alpha}x} \left\{1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}\right\}^{a-1} \left\{1 - \left\{1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}\right\}^a\right\}^{b-1} \qquad (6)$$

The corresponding survival function, hazard function and reversed hazard rate function respectively,

$$R(x) = \left\{1 - \left\{1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}\right\}^a\right\}^b,$$

$$h(x) = \frac{f(x)}{R(x)} = \frac{\frac{ab\theta^2}{\alpha(\theta+1)}\left(1 + \frac{x}{\alpha}\right) e^{-\frac{\theta}{\alpha}x} \left\{1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}\right\}^{a-1}}{1 - \left\{1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}\right\}^a},$$

and

$$\tau(x) = \frac{f(x)}{F(x)} = \frac{\frac{ab\theta^2}{\alpha(\theta+1)}\left(1 + \frac{x}{\alpha}\right) e^{-\frac{\theta}{\alpha}x} \left\{1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}\right\}^{a-1} \left\{1 - \left\{1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}\right\}^a\right\}^{b-1}}{1 - \left\{1 - \left\{1 - \frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)} e^{-\frac{\theta}{\alpha}x}\right\}^a\right\}^b}.$$

Figures 1, 2, 3 and 4 illustrate some of the possible shapes of the pdf, cdf, survival function and hazard rate of the KwS distribution for selected values of the parameters θ, α, a and b respectively.

Special Cases of the KwS Distribution

The Kumaraswamy Sushila is very flexible model that approaches to different distributions when its parameters are changed. The ES distribution contains as special- models the following well known distributions. If X is a random variable with cdf (6), then we have the following cases.

- If a=*b=1*, then Equation (6) gives exponentiated Sushila distribution which is introduced by Elgarhy and Shawki

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- If a=*b=1*, then Equation (6) gives Sushila distribution which is introduced by Shanker et al. (2013).
- If $b = \alpha = 1$ we get the Generalized Lindley distribution which is introduced by Nadarajah et al . (2011).
- If $a = b = \alpha = 1$ we get the Lindley distribution Lindley (1958).
- If $\alpha = 1$ we get the Kumaraswamy Lindley distribution.
- If *b=1*, then Equation (6) gives Exponentiated Sushila distribution.

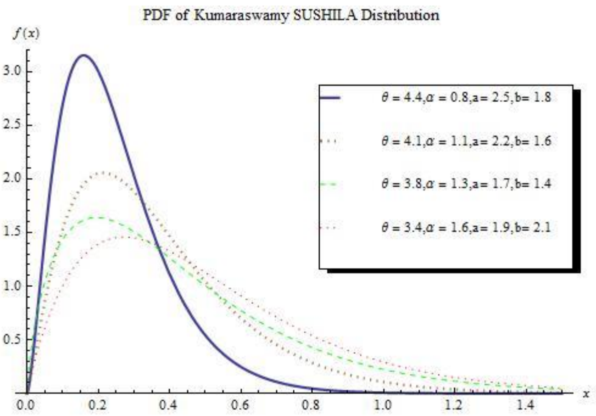

Fig. 1. The pdf of KwS distribution for various values of the parameters.

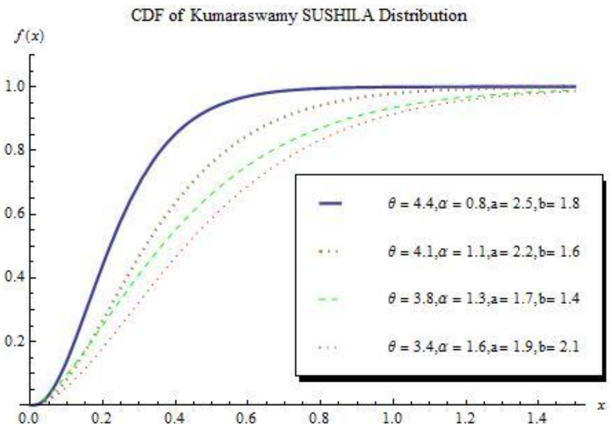

Fig. 2. The cdf of *KwS* distribution for various values of the parameters.

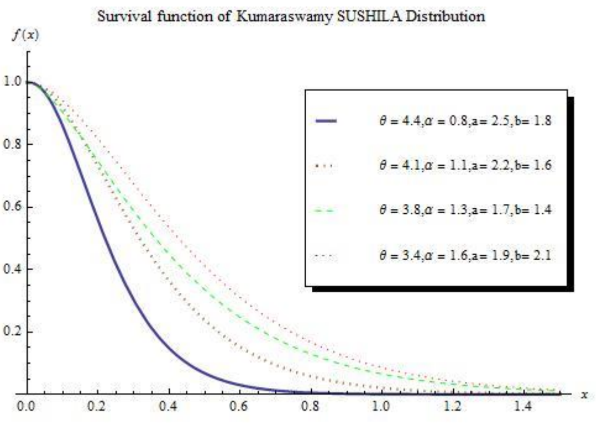

Fig. 3. The survival function of *KwS* distribution for various values of the parameters.

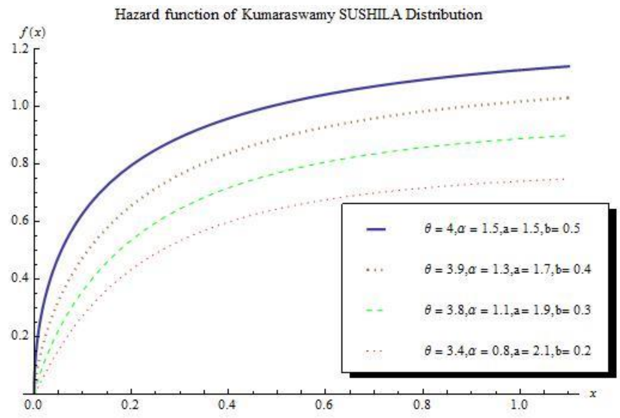

Fig. 4. hazard rate function of *KwS* distribution for various values of the parameters.

2.1. Expansion for the density function.

In this subsection we present some representations of pdf of Kumaraswamy Sushila distribution. The mathematical relation given below will be useful in this subsection. By using the generalized binomial theorem if β is a positive and $|z| < 1$, then

$$(1-z)^{\beta-1} = \sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j} z^j, \qquad (7)$$

the equation (6) become

$$f(x) = \frac{ab\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left\{1-\frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x}\right\}^{a(i+1)-1},$$

Again using binomial expansion we can write the last equation as follows:

$$f(x) = \frac{ab\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\left(1+\frac{\theta x}{\alpha(\theta+1)}\right)^j e^{-\frac{\theta(j+1)}{\alpha}x}, \qquad (8)$$

Then by using binomial theory

$$(1+z)^j = \sum_{j=0}^{\infty}\binom{j}{k}z^k \qquad (9)$$

Now using (9) in the last term of (8), we obtain

$$f(x) = \sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\left(x^k+\frac{x^{k+1}}{\alpha}\right)e^{-\frac{(j+1)\theta}{\alpha}x} \qquad (10)$$

where

$$\omega_{i,j,k} = ab\theta\left(\frac{\theta}{\alpha(\theta+1)}\right)^{k+1}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{j}{k}$$

III. Statistical Properties

This section is devoted to studying statistical properties of the *KwS* distribution, specifically quantile function, moments, and moment generating function.

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3.1. Quantile Function

The *KwS* quantile function, say $Q(U)=F^{-1}(U)$, is straightforward to be computed by inverting (5), we have

$$\left(1+\frac{\theta}{\alpha(\theta+1)}x_q\right)e^{-\frac{\theta}{\alpha}x_q}=1-\left(1-(1-u)^{\frac{1}{b}}\right)^{\frac{1}{a}}. \quad (11)$$

We can easily generatev X by taking U as a uniform random variable in (0,1) .

3.2. Moments

In this subsection we discuss the r_{th} non-central moment for *KwS* distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem (3.1).

If X has $KwS(x,\varphi),\emptyset=(\alpha,\theta,a,b)$ then the r_{th} non-central moment of X is given by the following

$$\mu_r^{\backslash}=\sum_{i,j,k=0}^{\infty}\omega_{i.j,k}\left[\frac{\Gamma(r+k+1)}{\left[\frac{(j+1)\theta}{\alpha}\right]^{r+k+1}}+\frac{\Gamma(r+k+2)}{\alpha\left[\frac{(j+1)\theta}{\alpha}\right]^{r+k+2}}\right]. \quad (12)$$

Proof:

Let X be a random variable with density function (6). The r_{th} non-central moment of the ES distribution is given by

$$\mu_r^{\backslash}=E(X^r)=\int_{-\infty}^{\infty}x^r f(x)dx$$

$$=\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\int_{0}^{\infty}\left(x^{r+k}+\frac{1}{\alpha}x^{r+k+1}\right)e^{-\frac{(j+1)\theta}{\alpha}x}dx.$$

Then

$$\mu_r^{\backslash}=\sum_{i,j,k=0}^{\infty}\omega_{i.j,k}\left[\frac{\Gamma(r+k+1)}{\left[\frac{(j+1)\theta}{\alpha}\right]^{r+k+1}}+\frac{\Gamma(r+k+2)}{\alpha\left[\frac{(j+1)\theta}{\alpha}\right]^{r+k+2}}\right].$$

Which completes the proof.

Substitution in the equation (12) by $r=1,2,3,4$ we get the first four moments of ESD as:

at $r=1,2,3$ and 4

$$\mu_1^{\backslash}=\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\left[\frac{\Gamma(k+2)}{\left[\frac{(j+1)\theta}{\alpha}\right]^{k+2}}+\frac{\Gamma(k+3)}{\alpha\left[\frac{(j+1)\theta}{\alpha}\right]^{k+3}}\right],$$

$$\mu_2^{\backslash}=\sum_{i,j=0}^{\infty}\omega_{i,j,k}\left[\frac{\Gamma(k+3)}{\left[\frac{(j+1)\theta}{\alpha}\right]^{k+3}}+\frac{\Gamma(k+4)}{\alpha\left[\frac{(j+1)\theta}{\alpha}\right]^{k+4}}\right],$$

$$\mu_3^{\backslash}=\sum_{i,j=0}^{\infty}\omega_{i,j,k}\left[\frac{\Gamma(k+4)}{\left[\frac{(j+1)\theta}{\alpha}\right]^{k+4}}+\frac{\Gamma(k+5)}{\alpha\left[\frac{(j+1)\theta}{\alpha}\right]^{k+5}}\right],$$

$$\mu_4^{\backslash}=\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\left[\frac{\Gamma(k+5)}{\left[\frac{(j+1)\theta}{\alpha}\right]^{k+5}}+\frac{\Gamma(k+6)}{\alpha\left[\frac{(j+1)\theta}{\alpha}\right]^{k+6}}\right].$$

Based on the first four moments of the ES distribution, the measures of skewness $A(\Phi)$ and kurtosis $K(\Phi)$ of the ES distribution can obtained as

$$A(\Phi)=\frac{\mu_3(\theta)-3\mu_1(\theta)\mu_2(\theta)+2\mu_1^3(\theta)}{[\mu_2(\theta)-\mu_1^2(\theta)]^{\frac{3}{2}}}$$

and

$$K(\Phi)=\frac{\mu_4(\theta)-4\mu_1(\theta)\mu_3(\theta)+6\mu_1^2(\theta)\mu_2(\theta)-3\mu_1^4(\theta)}{[\mu_2(\theta)-\mu_1^2(\theta)]^2}$$

3.3. Moment Generating Function

In this subsection we derived the moment generating function of *KwS* distribution.

Theorem (3.2):

If X has *KwS* distribution, then the moment generating function $\mathcal{M}_X(t)$ has the following form

$$\mathcal{M}_X(t)=\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\left[\frac{\Gamma(k+1)}{\left[\frac{(j+1)\theta}{\alpha}-t\right]^{k+1}}+\frac{\Gamma(k+2)}{\alpha\left[\frac{(j+1)\theta}{\alpha}-t\right]^{k+2}}\right]. \quad (13)$$

Proof.

We start with the well known definition of the moment generating function given by

$$\mathcal{M}_X(t)=E(e^{tX})=\int_{-\infty}^{\infty}e^{tx}f(x)dx$$

$$=\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\int_{0}^{\infty}\left(x^{k}+\frac{1}{\alpha}x^{k+1}\right)e^{-\left[\frac{(j+1)\theta}{\alpha}-t\right]x}dx.$$

Then

$$\mathcal{M}_X(t)=\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\left[\frac{\Gamma(k+1)}{\left[\frac{(j+1)\theta}{\alpha}-t\right]^{k+1}}+\frac{\Gamma(k+2)}{\alpha\left[\frac{(j+1)\theta}{\alpha}-t\right]^{k+2}}\right].$$

Which completes the proof.

In the same way, the characteristic function of the ES distribution becomes $\varphi_x(t)=\mathcal{M}_X(it)$ where $i=\sqrt{-1}$ the unit imaginary number is.

IV. ESTIMATION AND INFERENCE

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the *KwS* distribution from complete samples only. Let $X_1,X_2,\dots,X_n$ be a random sample of size n from $KwS(x,\varphi)$. The log-likelihood function for the vector of parameters $\emptyset=(\alpha,\theta,a,b)$ can be written as

$$\ln \mathcal{L} = n \ln a + n \ln b + 2n \ln \theta - n \ln \alpha - n \ln(\theta+1) - \frac{\theta}{\alpha}\sum x_i + \sum_{i=1}^{n} \ln\left(1+\frac{x_i}{\alpha}\right) + (a-1)\sum_{i=1}^{n} \ln\left\{1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right\} + (b-1)\sum_{i=1}^{n} \ln\left\{1-\left(1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right)^{a}\right\}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating. The components of the score vector are given by

$$\frac{\partial \ln \mathcal{L}}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln\left\{1-\left(1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right)^{a}\right\}, \quad (14)$$

$$\frac{\partial \ln \mathcal{L}}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \ln\left\{1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right\} - (b-1)\sum_{i=1}^{n}\frac{\left(1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right)^{a}\ln\left(1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right)}{1-\left(1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right)^{a}} \quad (15)$$

$$\frac{\partial \ln \mathcal{L}}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta+1} - \frac{1}{\alpha}\sum_{i=1}^{n} x_i - (a-1)\sum_{i=1}^{n}\frac{\left\{\frac{1}{(\theta+1)^2} - 1 - \frac{\theta x_i}{\alpha(\theta+1)}\right\}\frac{x_i}{\alpha}e^{-\frac{\theta}{\alpha}x_i}}{1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}} - a(b-1)\sum_{i=1}^{n}\frac{\left\{\frac{1}{(\theta+1)^2} - 1 - \frac{\theta x_i}{\alpha(\theta+1)}\right\}\frac{x_i}{\alpha}e^{-\frac{\theta}{\alpha}x_i}\left(1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right)^{a-1}}{1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}} \quad (16)$$

and

$$\frac{\partial \ln \mathcal{L}}{\partial \alpha} = \frac{-n}{\alpha} + \frac{\theta}{\alpha^2}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\frac{\frac{-x_i}{\alpha^2}}{1+\frac{x_i}{\alpha}} + (a-1)\sum_{i=1}^{n}\frac{\frac{\theta}{\alpha^2}\left(1+\frac{\theta x_i}{\alpha(\theta+1)}+\frac{1}{\theta+1}\right)x_i e^{-\frac{\theta}{\alpha}x_i}}{1-\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}} - a(b-1)\sum_{i=1}^{n}\frac{\frac{\theta}{\alpha^2}\left(1+\frac{\theta x_i}{\alpha(\theta+1)}+\frac{1}{\theta+1}\right)x_i e^{-\frac{\theta}{\alpha}x_i}\left(1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right)^{a-1}}{1-\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}}. \quad (17)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (14), (15), (16) and (17) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters.

V. Conclusion

We have introduced a new four-parameter Kumaraswamy Sushila distribution and study its different properties in this paper. It is observed that the proposed *KwS* distribution has several desirable properties. The *KwS* distribution covers some distributions.

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