Volume 1, Issue 7, pp. 9-12, 2017.
International Journal of Scientific Engineering and Science
ISSN (Online): 2456-7361

Exponentiated SUSHILA Distribution

M. Elgarhy[1], A. W. Shawki[2]
[1]Vice Presidency for Graduate Studies and Scientific Research, Jeddah University, Saudi Arabia
[2]Department of Mathematical Statistics, Institute of Statistical Studies and Research, Cairo University, Cairo, Egypt

Abstract*— *In this paper, we present a new class of distributions called Exponintiated Sushila distribution. This class of distributions contains several distributions such as generalized Lindley distribution. The hazard function, moments and moment generating function are presented. Moreover, we discuss the maximum likelihood estimation of this distribution.

Keywords*—*Sushila distribution, Maximum likelihood estimation Moment generating function

I. INTRODUCTION

Lindley distribution was proposed by Lindley (1958) in the context of Bayesian statistics, as a counter example of fiducially statistics. However, due to the popularity of the exponential distribution in statistics especially in reliability theory, Lindley distribution has been overlooked in the literature. Recently, many authors have paid great attention to the Lindley distribution as a lifetime model. From different point of view, Ghitany et al. (2008) showed that Lindley distribution is a better lifetime model than exponential distribution. More so, in practice, it has been observed that many real life system models have increasing failure rate with time. Mazucheli and Achcar (2011) studied a competing risk model when the causes of failures follow Lindley distribution. Krishna and Kumar (2011) estimated the parameter of Lindley distribution with progressive Type-II censoring scheme. They also showed that it may fit better than exponential, lognormal and gamma distributions in some real life situations.

Lindley (1958), introduced a one- parameter distribution known as Lindley distribution, given by its probability density function

$$g(x,\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\,;x>0\,,\ \theta>0 \qquad (1)$$

the cumulative distribution function (cdf) of Lindley distribution is obtained as

$$G(x,\theta)=1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\,;x>0\,,\theta>0 \qquad (2)$$

The Lindley distribution does not provide enough flexibilty for analysing different types of lifetime data. To increase the flexibilty for modelling purposes it will be useful to consider further alternatives of this distribution. Zakerzadeh and Dolati (2009) introduced a three parameter generalization of the Lindley distribution. They studied various properties of the new distribution and provide numerical examples to show the flexibilty of the model. Nadarajah et al. (2011) presented a two parameter generalization of the Lindley distribution. Bakouch et al. (2012) obtained an extended Lindley distribution and discussed its various properties and applications..Elbatal et al. (2013) presented anew generalized Lindley distribution. Elbatal and Elgarhy (2013a) presented transmuted quasi Lindley distributon. Elbatal and Elgarhy (2013b) presented Kumaraswamy quasi Lindley distributon. Elbatal et al. (2016) presented exponentiated quasi Lindley distributon. Elgarhy et al. (2016) presented transmuted generalized Lindley distributon.

Shanker et al. (2013) introduced Sushila distribution (SD) of which the Lindley distribution (LD) is a particular case. Quasi probability density function (pdf)

$$g(x)\ =\frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\quad x,\theta,\alpha>0. \qquad (3)$$

It can easily be seen that at $\alpha=1$, the SD (3) reduces to the Lindley distribution (1958) with probability density function. The pdf (3) can be shown as a mixture of gamma $\left(1,\frac{\theta}{\alpha}\right)\sim\exp\left(\frac{\theta}{\alpha}\right)$ and gamma $\left(2,\frac{\theta}{\alpha}\right)$ distributions as follows

$$g(x,\theta,\alpha)=pg_1(x)+(1-p)g_2(x)\ ,$$

$$Where\ p=\frac{\theta}{\theta+1}\,,g_1(x)=\frac{\theta}{\alpha}e^{-\frac{\theta}{\alpha}x}\,,g_2(x)=\frac{\theta^2}{\alpha^2}xe^{-\frac{\theta}{\alpha}x}.$$

The cumulative distribution function (cdf) of SD is obtained as

$$G(x)=1-\frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x}. \qquad (4)$$

This paper offers new distribution with three parameters called exponintiated Sushila distribution, this article is organized as follows. In Section 2, we define the Exponintiated Sushila distribution, the expansion for the density function of the ES distribution and some special cases. Quantile function, moments, moment generating function are discussed in Section 3. In Section 4 included Maximumlikelihood estimation. Finally, conclusion in Section 5.

II. EXPONINTIATED SUSHILA DISTRIBUTION

In this section, we introduce the three − parameter Exponintiated Sushila ES distribution, the cdf and pdf of the ES distribution can be written respectively as

$$F(x)=\left\{1-\frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x}\right\}^a\qquad \theta,\alpha,a>0. \qquad (5)$$

$$f(x)=\frac{a\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\left\{1-\frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x}\right\}^{a-1} \qquad (6)$$

The corresponding survival function, hazard function and reversed hazard rate function respectively,

$$R(x)=1-\left\{1-\frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x}\right\}^a,$$

$$h(x) = \frac{f(x)}{R(x)} = \frac{\frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}\left\{1-\frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x}\right\}^{a-1}}{1-\left\{1-\frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x}\right\}^{a}},$$

$$\tau(x) = \frac{f(x)}{F(x)} = \frac{\frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x}}{1-\frac{\alpha(\theta+1)+\theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x}}.$$

Figures 1, 2, 3, 4 and 5 illustrate some of the possible shapes of the pdf, cdf, survival function, hazard rate and reversed hazard rate of the ES distribution for selected values of the parameters θ, α and a respectively.

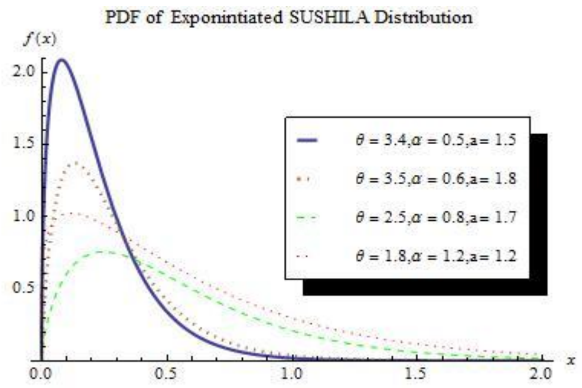

Fig 1. The pdf of ES distribution for various values of the parameters.

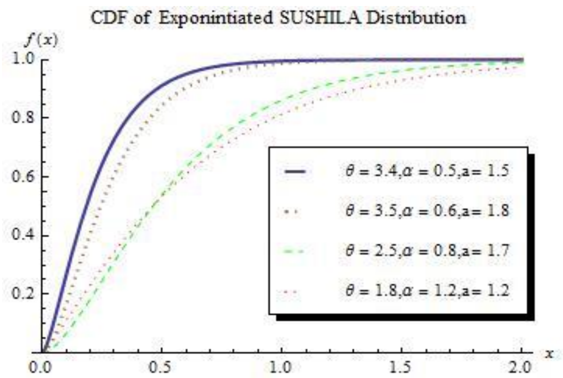

Fig. 2. The cdf of ES distribution for various values of the parameters.

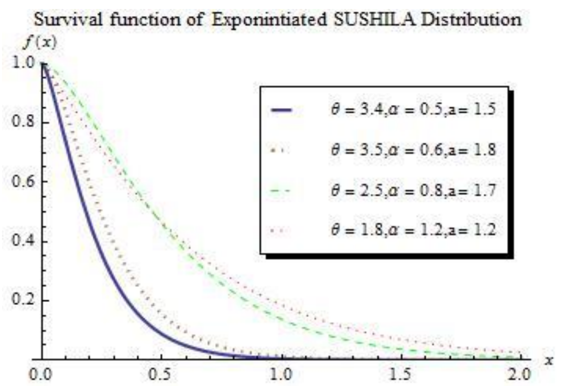

Fig. 3. The survival function of ES distribution for various values of the parameters.

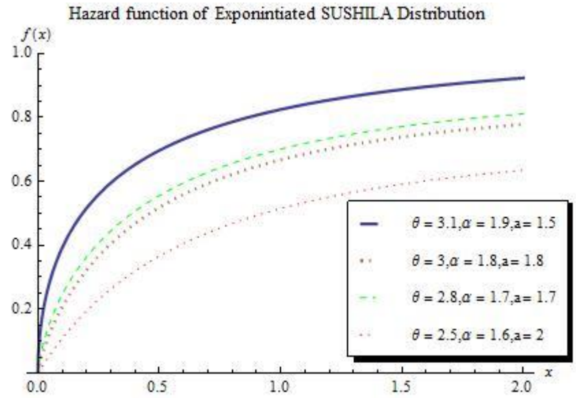

Fig. 4. Hazard rate function of ES distribution for various values of the parameters.

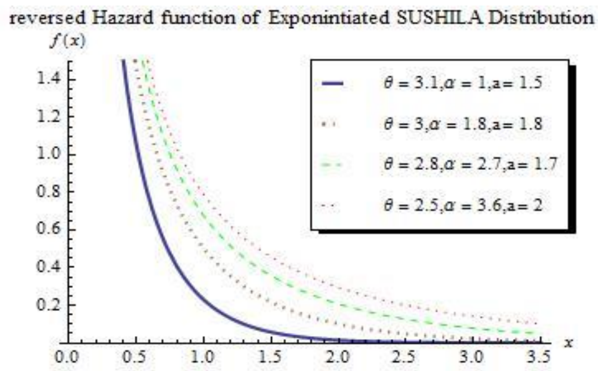

Fig. 5. Reversed hazard rate function of ES distribution for various values of the parameters

Special Cases of the ES Distribution

The Exponintiated Sushila is very flexible model that approaches to different distributions when its parameters are changed. The ES distribution contains as special- models the following well known distributions. If X is a random variable with cdf (6), then we have the following cases.

- If a=1, then Equation (3) gives Sushila distribution which is introduced by Shanker et al. (2013).
- If $\alpha = 1$ we get the Generalized Lindley distribution which is introduced by Nadarajah et al . (2011).
- If $a = \alpha = 1$ we get the Lindley distribution Lindley (1958).

2.1. Expansion for the cumulative and density functions.

In this subsection we present some representations of pdf of exponintiated Sushila distribution. The mathematical relation given below will be useful in this subsection. By using the generalized binomial theorem if β is a positive and $|z| < 1$, then

$$(1-z)^{\beta-1} = \sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}z^j, \quad (7)$$

the equation (6) become

$$f(x) = \frac{a\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}\left(1+\frac{\theta}{\alpha(\theta+1)}x\right)^i e^{-\frac{(i+1)\theta}{\alpha}x}, \quad (8)$$

Then by using binomial theory

$$(1+z)^j = \sum_{k=0}^{\infty}\binom{j}{k}z^k \quad (9)$$

Now using (9) in the last term of (8), we obtain

International Journal of Scientific Engineering and Science
Volume 1, Issue 7, pp. 9-12, 2017. ISSN (Online): 2456-7361

$$f(x) = \sum_{i,j=0}^{\infty} \omega_{i,j} \left(x^j + \frac{x^{j+1}}{\alpha}\right) e^{-\frac{(i+1)\theta}{\alpha}x} \qquad (10)$$

where

$$\omega_{i,j} = a\theta \left(\frac{\theta}{\alpha(\theta+1)}\right)^{j+1} (-1)^i \binom{a-1}{i}\binom{i}{j}$$

III. Statistical Properties

This section is devoted to studying statistical properties of the ES distribution, specifically quantile function, moments, and moment generating function.

3.1. Quantile Function

The ES quantile function, say $Q(U) = F^{-1}(U)$, is straightforward to be computed by inverting (5), we have

$$\left(1 + \frac{\theta}{\alpha(\theta+1)} x_q\right) e^{-\frac{\theta}{\alpha}x_q} = 1 - u^{\frac{1}{a}}. \qquad (11)$$

We can easily generatev X by taking U as a uniform random variable in (0,1) .

3.2. Moments

In this subsection we discuss the r_{th} non-central moment for ES distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem (3.1).

If X has $ES(x,\varphi), \emptyset = (\alpha,\theta,a)$ then the r_{th} non-central moment of X is given by the following

$$\mu_r^{\backslash} = \sum_{i,j=0}^{\infty} \omega_{i.j} \left[\frac{\Gamma(r+j+1)}{\left[\frac{(i+1)\theta}{\alpha}\right]^{r+j+1}} + \frac{\Gamma(r+j+2)}{\alpha\left[\frac{(i+1)\theta}{\alpha}\right]^{r+j+2}}\right]. \qquad (12)$$

Proof:

Let X be a random variable with density function (6). The r_{th} non-central moment of the ES distribution is given by

$$\mu_r^{\backslash} = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$$
$$= \sum_{i,j=0}^{\infty} \omega_{i,j} \int_0^{\infty} \left(x^{r+j} + \frac{1}{\alpha} x^{r+j+1}\right) e^{-\frac{(i+1)\theta}{\alpha}x} dx.$$

Then

$$\mu_r^{\backslash} = \sum_{i,j=0}^{\infty} \omega_{i,j} \left[\frac{\Gamma(r+j+1)}{\left[\frac{(i+1)\theta}{\alpha}\right]^{r+j+1}} + \frac{\Gamma(r+j+2)}{\alpha\left[\frac{(i+1)\theta}{\alpha}\right]^{r+j+2}}\right].$$

Which completes the proof.

Substitution in the equation (12) by $r = 1,2,3,4$ we get the first four moments of ESD as:

at $r = 1,2,3$ and 4

$$\mu_1^{\backslash} = \sum_{i,j=0}^{\infty} \omega_{i,j} \left[\frac{\Gamma(j+2)}{\left[\frac{(i+1)\theta}{\alpha}\right]^{j+2}} + \frac{\Gamma(j+3)}{\alpha\left[\frac{(i+1)\theta}{\alpha}\right]^{j+3}}\right],$$

$$\mu_2^{\backslash} = \sum_{i,j=0}^{\infty} \omega_{i,j} \left[\frac{\Gamma(j+3)}{\left[\frac{(i+1)\theta}{\alpha}\right]^{j+3}} + \frac{\Gamma(j+4)}{\alpha\left[\frac{(i+1)\theta}{\alpha}\right]^{j+4}}\right],$$

$$\mu_3^{\backslash} = \sum_{i,j=0}^{\infty} \omega_{i,j} \left[\frac{\Gamma(j+4)}{\left[\frac{(i+1)\theta}{\alpha}\right]^{j+4}} + \frac{\Gamma(j+5)}{\alpha\left[\frac{(i+1)\theta}{\alpha}\right]^{j+5}}\right],$$

and

$$\mu_4^{\backslash} = \sum_{i,j=0}^{\infty} \omega_{i,j} \left[\frac{\Gamma(j+5)}{\left[\frac{(i+1)\theta}{\alpha}\right]^{j+5}} + \frac{\Gamma(j+6)}{\alpha\left[\frac{(i+1)\theta}{\alpha}\right]^{j+6}}\right].$$

Based on the first four moments of the ES distribution, the measures of skewness $A(\Phi)$ and kurtosis $K(\Phi)$ of the ES distribution can obtained as

$$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^{\frac{3}{2}}}$$

and

$$K(\Phi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^2}$$

3.3. Moment Generating Function

In this subsection we derived the moment generating function of ES distribution.

Theorem (3.2):

If X has ES distribution, then the moment generating function $\mathcal{M}_X(t)$ has the following form

$$\mathcal{M}_X(t) = \sum_{i,j=0}^{\infty} \omega_{i,j} \left[\frac{\Gamma(j+1)}{\left[\frac{(i+1)\theta}{\alpha} - t\right]^{j+1}} + \frac{\Gamma(j+2)}{\alpha\left[\frac{(i+1)\theta}{\alpha} - t\right]^{j+2}}\right]. \qquad (13)$$

Proof.

We start with the well known definition of the moment generating function given by

$$\mathcal{M}_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$
$$= \sum_{i,j=0}^{\infty} \omega_{i,j} \int_0^{\infty} \left(x^j + \frac{1}{\alpha} x^{j+1}\right) e^{-\left[\frac{(i+1)\theta}{\alpha} - t\right]x} dx.$$

Then

$$\mathcal{M}_X(t) = \sum_{i,j=0}^{\infty} \omega_{i,j} \left[\frac{\Gamma(j+1)}{\left[\frac{(i+1)\theta}{\alpha} - t\right]^{j+1}} + \frac{\Gamma(j+2)}{\alpha\left[\frac{(i+1)\theta}{\alpha} - t\right]^{j+2}}\right].$$

Which completes the proof.

In the same way, the characteristic function of the ES distribution becomes $\varphi_x(t) = \mathcal{M}_X(it)$ where $i = \sqrt{-1}$ the unit imaginary number is.

IV. Estimation and Inference

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the ES distribution from complete samples only. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $ES(x, \varphi)$ .The log-likelihood function for the vector of parameters $\emptyset = (\alpha, \theta, a)$ can be written as

$$ln\,\mathcal{L} = n\,ln\,a + 2n\,ln\,\theta - n\,ln\,\alpha - n\,ln(\theta+1) - \frac{\theta}{\alpha}\sum x_i + \sum_{i=1}^{n} ln\left(1+\frac{x_i}{\alpha}\right) + (a-1)\sum_{i=1}^{n} ln\left\{1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right\}.$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating. The components of the score vector are given by

$$\frac{\partial\, ln\,\mathcal{L}}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} ln\left\{1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}\right\}, \quad (14)$$

$$\frac{\partial\, ln\,\mathcal{L}}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta+1} - \frac{1}{\alpha}\sum_{i=1}^{n} x_i - (a-1)\sum_{i=1}^{n}\frac{\left\{\frac{1}{(\theta+1)^2} - 1 - \frac{\theta x_i}{\alpha(\theta+1)}\right\}\frac{x_i}{\alpha}e^{-\frac{\theta}{\alpha}x_i}}{1-\left(1+\frac{\theta}{\alpha(\theta+1)}x_i\right)e^{-\frac{\theta}{\alpha}x_i}}, \quad (15)$$

and

$$\frac{\partial\, ln\,\mathcal{L}}{\partial \alpha} = \frac{-n}{\alpha} + \frac{\theta}{\alpha^2}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\frac{\frac{-x_i}{\alpha^2}}{1+\frac{x_i}{\alpha}} + (a-1)\sum_{i=1}^{n}\frac{\frac{\theta}{\alpha^2}\left(1+\frac{\theta x_i}{\alpha(\theta+1)}+\frac{1}{\theta+1}\right)x_i e^{-\frac{\theta}{\alpha}x_i}}{1-\left(1+\frac{\theta x_i}{\alpha(\theta+1)}\right)e^{-\frac{\theta}{\alpha}x_i}}. \quad (16)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (14), (15) and (16) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters.

V. Conclusion

We have introduced a new three-parameter exponentiated Sushila distribution and study its different properties in this paper. It is observed that the proposed ES distribution has several desirable properties. The ES distribution covers some distributions.

References

[1] H. S. Bakouch, B. M. Al-Zahrani, A. A. Al-Shomrani, V. A. Marchi, and F. Louzada, “An extended Lindley distribution,” *J. Korean Stat. Soc*, vol. 41 issue 1, pp. 75-85, 2012.
[2] I. Elbatal and M. Elgarhy, “Transmuted quasi Lindley distribution: a Generalization of the quasi Lindley distribution,” *International Journal of Pure and Applied Sciences and Technology*, vol. 18, issue 2, pp. 59-69, 2013a.
[3] I. Elbatal and M. Elgarhy, “Statistical Properties of Kumaraswamy Quasi Lindley Distribution,” *International Journal of Mathematics Trends and Technology*, vol. 4, issue 10, pp. 237-246, 2013b.
[4] I. Elbatal, L.S. Diab and M. Elgarhy, “Exponentiated Quasi Lindley Distribution,” *International Journal of Reliability and Applications*, vol. 17, issue 1, pp. 1-19, 2016.
[5] I. Elbatal, F. Merovci, and M. Elgarhy, “A new generalized Lindley distribution,” *Mathematical Theory and Modeling*, vol. 13, issue 3, pp. 30–47, 2013.
[6] M. Elgarhy, M. Rashed, and A. W. Shawki, “Transmuted generalized Lindley distribution.” *International Journal of Mathematics Trends and Technology*, vol. 29, issue 2, pp. 145-154, 2016.
[7] M. E. Ghitany, B. Atieh, and S. Nadarajah, “Lindley distribution and its Applications,” *Math. Comput. Simul*, vol. 78, issue 4, pp. 493-506, 2008.
[8] H. Krishna, and K. Kumar, “Reliability estimation in Lindley distribution with progressive type II right censored sample,” *Journal Mathematics and Computers in Simulation Archive*, vol. 82, issue 2, pp. 281-294, 2011.
[9] D.V. Lindley, “Fiducial distributions and Bayes' theorem,” *J. Royal Stat. Soc. Series B*, vol. 20, pp. 102-107, 1958.
[10] J. Mazucheli and J.A. Achcar, “The Lindley distribution applied to competing risks lifetime data,” *Comput. Methods Programs Biomed*, vol. 104, issue 2, pp.188-192, 2011.
[11] S. Nadarajah, H. S. Bakouch, and R. Tahmasbi, “A generalized Lindley distribution,” *Sankhya B*, vol. 73, pp. 331–359, 2011.
[12] R. Shanker, S. Sharma, U. Shanker, and R. Shanker, “SUSHILA distribution and its application to waiting times data,” *Opinion - International Journal of Business Management*, vol. 3, issue 2, pp. 1-11, 2013.
[13] H. Zakerzadah, and A. Dolati, “Generalized Lindley distribution,” *J. Math. Ext.*, vol. 3, issue 2, pp. 13-25, 2009.