

Transmuted Generalized Quasi Lindley Distribution

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Abstract—The Lindley distribution is one of the important for studying stress-strength reliability modeling. Besides, some researchers have proposed new classes of distributions based on modifications of the quasi Lindley distribution. In this paper, a new generalized version of this distribution which is called the transmuted generalized quasi Lindley (TGQL) distribution is introduced. A comprehensive mathematical treatment of the TGQL distribution is provided. We derive the r^{th} moment and moment generating function this distribution. Moreover, we discuss the maximum likelihood estimation of this distribution.

Keywords—Generalized Lindley distribution; Survival function; Moments. Maximum likelihood estimation.

I. INTRODUCTION AND MOTIVATION

Exponentiated quasi Lindley distribution with parameters a, θ and α was proposed by Elbatal et al. (2016). has probability density function (p.d.f) is given by

$$g(x; \theta, \alpha, a) = \frac{a\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta x} \times \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^{a-1}. \quad (1.1)$$

The cumulative distribution function (cdf) of **GLD** is obtained as

$$G(x; \theta, \alpha, a) = \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a, \quad x > 0, a, \theta > 0, \alpha > -1. \quad (1.2)$$

1.1. Transmutation Map

In this subsection we demonstrate transmuted probability distribution. Let F_1 and F_2 be the cumulative distribution functions, of two distributions with a common sample space. The general rank transmutation as given in (2007) is defined as

$$G_{R12}(u) = F_2(F_1^{-1}(u)) \quad \text{and} \quad G_{R21}(u) = F_1(F_2^{-1}(u)).$$

Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(y) = \inf_{x \in R} \{F(x) \geq y\} \quad \text{for } y \in [0, 1].$$

The functions $G_{R12}(u)$ and $G_{R21}(u)$ both map the unit interval $I = [0, 1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_{Rij}(0) = 0$ and $G_{Rij}(1) = 1$. A quadratic Rank Transmutation Map (QRTM) is defined as

$$G_{R12}(u) = u + \lambda u(1-u), \quad |\lambda| \leq 1, \quad (1.3)$$

from which it follows that the cdf's satisfy the relationship

$$F_2(x) = (1 + \lambda)F_1(x) - \lambda F_1(x)^2 \quad (1.4)$$

which on differentiation yields,

$$f_2(x) = f_1(x) [(1 + \lambda) - 2\lambda F_1(x)] \quad (1.5)$$

where $f_1(x)$ and $f_2(x)$ are the corresponding pdfs associated with cdf $F_1(x)$ and $F_2(x)$ respectively. An extensive

information about the quadratic rank transmutation map is given in Shaw and Buckely. (2007). Observe that at $\lambda = 0$ we have the distribution of the base random variable. The following Lemma proved that the function $f_2(x)$ in given (1.5) satisfies the property of probability density function.

Lemma: $f_2(x)$ given in (1.5) is a well-defined probability density function.

Proof: Rewriting $f_2(x)$ as $f_2(x) = f_1(x) [(1 - \lambda(2F_1(x) - 1))]$

we observe that $f_2(x)$ is nonnegative. We need to show that the integration over the support of the random variable is equal one. Consider the case when the support of $f_1(x)$ is $(-\infty, \infty)$. In this case we have

$$\begin{aligned} \int_{-\infty}^{\infty} f_2(x) dx &= \int_{-\infty}^{\infty} f_1(x) [(1 + \lambda) - 2\lambda F_1(x)] dx \\ &= (1 + \lambda) \int_{-\infty}^{\infty} f_1(x) dx - \lambda \int_{-\infty}^{\infty} 2f_1(x) F_1(x) dx \\ &= (1 + \lambda) - \lambda = 1 \end{aligned}$$

Similarly, other cases where the support of the random variable is a part of real line follows. Hence $f_2(x)$ is a well-defined probability density function. We call $f_2(x)$ the transmuted probability density of a random variable with base density $f_1(x)$. Also note that when $\lambda = 0$ then $f_2(x) = f_1(x)$. This proves the required result.

Many authors dealing with the generalization of some well-known distributions. Aryal and Tsokos (2009) defined the transmuted generalized extreme value distribution and they studied some basic mathematical characteristics of transmuted Gumbel probability distribution and it has been observed that the transmuted Gumbel can be used to model climate data. Elbatal and Elgarhy (2013) presented transmuted quasi Lindley distribution. Elgarhy et al. (2016) studied the transmuted generalized Lindley distribution. The rest of the paper is organized as follows. In Section 2 we demonstrate transmuted probability distribution, the hazard rate and reliability functions of TGQL distribution. In Section 3 we studied the statistical properties include quantile functions, expansion of density function, moments, moment generating

function. The distribution of order statistics is expressed in Section 4. Finally, In Section5, we demonstrate the maximum likelihood estimates of the unknown parameters.

II. TRANSMUTED GENERALIZED QUASI LINDLEY DISTRIBUTION

In this section we studied the transmuted generalized quasi Lindley ($TGQL$) distribution. Now substituting (1.2) in (1.4) we have the cdf of transmuted generalized quasi Lindley distribution

$$F_{TGQL}(x; a, \theta, \lambda, \alpha) = \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \times \left\{ 1 + \lambda - \lambda \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \right\}, \quad (2.1)$$

where θ and α scale parameters, a shape parameter and λ is the transmuted parameter. The probability density function (pdf) of the transmuted generalized Lindley distribution is given by

$$f_{TGQL}(x; a, \theta, \lambda, \alpha) = \frac{a\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta x} \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^{a-1} \times \left\{ 1 + \lambda - 2\lambda \left(1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right)^a \right\}. \quad (2.2)$$

The reliability function of the transmuted generalized quasi Lindley distribution is denoted by $R_{TGQL}(x)$ also known as the survival function and is defined as

$$R_{TGQL}(x) = 1 - F_{TGQL}(x) = 1 - \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \times \left\{ 1 + \lambda - \lambda \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \right\}.$$

It is important to note that $R_{TGQL}(x) + F_{TGQL}(x) = 1$. One of the characteristic in reliability analysis is the hazard rate and the reversed hazard rate functions defined by

$$h_{TGQL}(x) = \frac{f_{TGQL}(x)}{1 - F_{TGQL}(x)} = \frac{\frac{a\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta x} \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^{a-1}}{1 - \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \times \left\{ 1 + \lambda - 2\lambda \left(1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right)^a \right\}} \times \frac{\left\{ 1 + \lambda - \lambda \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \right\}}{\left\{ 1 + \lambda - \lambda \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \right\}}.$$

$$\tau_{TGQL}(x) = \frac{f_{TGQL}(x)}{F_{TGQL}(x)} = \frac{\frac{a\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta x}}{\left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a} \times \left\{ 1 + \lambda - 2\lambda \left(1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right)^a \right\} \times \frac{\left\{ 1 + \lambda - \lambda \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \right\}}{\left\{ 1 + \lambda - \lambda \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \right\}}.$$

respectively. It is important to note that the units for $h_{TGQL}(x)$ is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters. The cumulative hazard function of the transmuted generalized Lindley distribution is denoted by $H_{TGQL}(x)$ and is defined as

$$H_{TGQL}(x) = -\ln \left\{ \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \left\{ 1 + \lambda - \lambda \left[1 - e^{-\theta x} \left(1 + \frac{\theta x}{\alpha + 1} \right) \right]^a \right\} \right\}.$$

It is important to note that the units for $H_{TGQL}(x)$ is the cumulative probability of failure per unit of time, distance or cycles. We can show that. For all choice of parameters the distribution has the decreasing patterns of cumulative instantaneous failure rates.

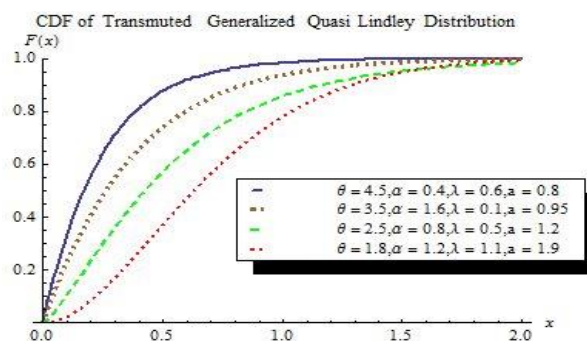


Fig. 1. Plot of the cdf of TGQL distribution for selected values of the parameters.

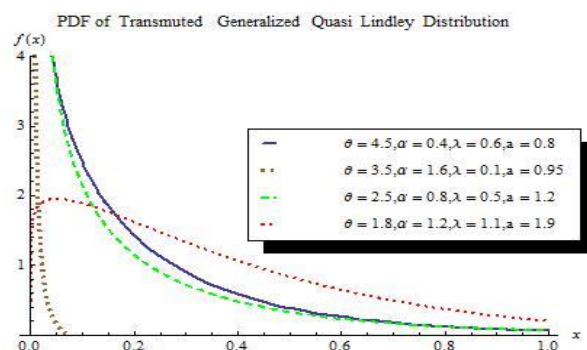


Fig. 2. Plot of the pdf of TGQL distribution for selected values of the parameters.

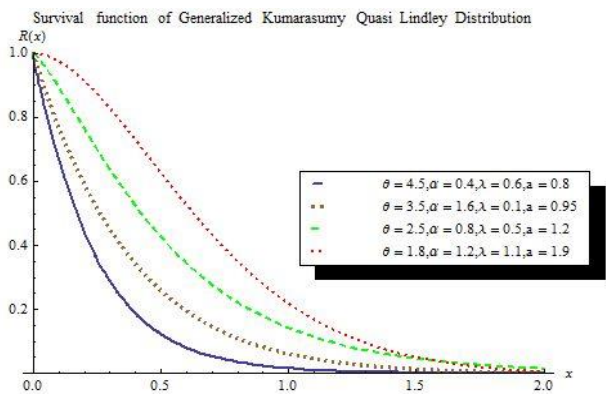


Fig. 3. Plot of the survival function of TGQL distribution for selected values of the parameters.

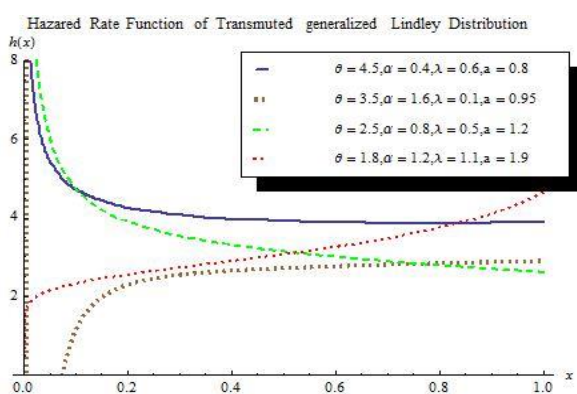


Fig. 4. Plot of the hazard rate function of TGQL distribution for selected values of the parameters.

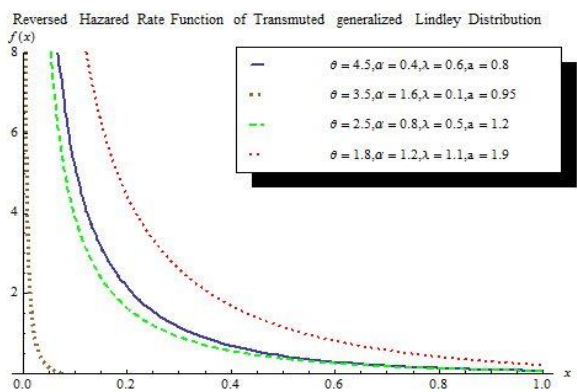


Fig. 5. Plot of thereversed hazard rate function of TGL distribution for selected values of the parameters.

Special Cases of the TGQL Distribution

The transmuted generalized quasi Lindley is very flexible model that approaches to different distributions when its parameters are changed. The TGQL distribution contains as special- models the following well known distributions. If X is a random variable with cdf (2.1), then we have the following cases:

- If $a=1$ then Equation (2.1) gives transmuted quasi Lindley distribution. That introduced by Elbatal and Elgarhy (2013).

- If $\alpha = \theta$ then Equation (2.1) gives transmuted generalized Lindley distribution. That introduced by Elgarhy et al. (2016).
- If $\lambda = 0$ then Equation (2.1) gives generalized quasi Lindley distribution. That introduced by Elbatal et al. (2016).
- If $\lambda = 0, a=1$ then Equation (2.1) gives quasi Lindley distribution. That introduced by Rama and Mishra (2013).
- If $\lambda = 0, \alpha = \theta$ then Equation (2.1) gives generalized Lindley distribution. That introduced by Nadrajah et al. (2011).
- If $a=1, \alpha = \theta$ then Equation (2.1) gives transmuted Lindley distribution. That introduced by Fatou (2013).
- If $\lambda = 0, a=1, \alpha = \theta$ then Equation (2.1) gives Lindley distribution. That introduced by Lindley (1958).Math.

III. STATISTICAL PROPERTIES

This section is devoted to studying statistical properties of the (TGQL) distribution, specifically quantile function, moments and moment generating function.

3.1. Quantile Function

The q th quantile x_q of the transmuted generalized quasi Lindley distribution can be obtained from (2.1) as

$$e^{-\theta x_q} \left(1 + \frac{\theta x_q}{\alpha + 1} \right) = 1 - \left\{ \frac{(\lambda + 1) + \sqrt{(\lambda - 1)^2 - 4\lambda q}}{2\lambda} \right\}^{\frac{1}{a}}, \quad (3.1)$$

We simulate the TGQL distribution by solving the nonlinear equation

$$e^{-\theta x_q} \left(1 + \frac{\theta x_q}{\alpha + 1} \right) = 1 - \left\{ \frac{(\lambda + 1) + \sqrt{(\lambda - 1)^2 - 4\lambda u}}{2\lambda} \right\}^{\frac{1}{a}}$$

where u has the uniform $U(0,1)$ distribution.

3.2. Expansion of Density Function

In this section representation of pdf for transmuted generalized quasi Lindley distribution will be presented. The mathematical relation given below will be useful in this subsection.

It is well-known that, if $\beta > 0$ is real non integer and $|z| < 1$, the generalized binomial theorem is written as follows

$$(1 - z)^{\beta - 1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta - 1}{i} z^i. \quad (3.2)$$

Then, by applying the binomial theorem (3.2) in (2.2), the probability density function n of TGL distribution becomes

$$f(x) = \frac{a\theta(1+\lambda)}{\alpha+1} (\alpha + \theta x) \sum_{i=0}^{\infty} (-1)^i \binom{a-1}{i} e^{-\theta(i+1)x} \left(1 + \frac{\theta x}{\theta+1} \right)^i - \frac{2a\theta\lambda}{\alpha+1} (\alpha + \theta x) \sum_{i=0}^{\infty} (-1)^i \binom{2a-1}{i} e^{-\theta(i+1)x} \left(1 + \frac{\theta x}{\theta+1} \right)^i,$$

by applying the binomial theorem again in the last equation the pdf become

$$f(x) = \frac{a\theta(1+\lambda)}{\alpha+1} (\alpha+\theta x) \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^i \binom{a-1}{i} \binom{i}{j} e^{-\theta(i+1)x} \left(\frac{\theta}{\alpha+1}\right)^j x^j - \frac{2a\theta\lambda}{\alpha+1} (\alpha+\theta x) \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^i \binom{2a-1}{i} \binom{i}{j} e^{-\theta(i+1)x} \left(\frac{\theta}{\alpha+1}\right)^j x^j, \\ f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^i M (\alpha x^j + \theta x^{j+1}) e^{-\theta(i+1)x} - \sum_{i=0}^{\infty} \sum_{j=0}^i N (\alpha x^j + \theta x^{j+1}) e^{-\theta(i+1)x}. \quad (3.3)$$

where

$$M = a(1+\lambda)(-1)^i \binom{a-1}{i} \binom{i}{j} \left(\frac{\theta}{\alpha+1}\right)^{j+1},$$

and

$$N = 2a\lambda(-1)^i \binom{2a-1}{i} \binom{i}{j} \left(\frac{\theta}{\alpha+1}\right)^{j+1}.$$

If β is an integer the index i in the previous sum stops at $\beta-1$.

3.3. Moments

In this subsection we discuss the r^{th} moment for $TGQL$ distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem (3.1).

If X has $TGQL(x; \theta, a, \lambda, \alpha)$, then the r^{th} moment of X is given by the following

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^i M \left[\frac{\alpha\Gamma(r+j+1)}{[\theta(i+1)]^{r+j+1}} + \frac{\theta\Gamma(r+j+2)}{[\theta(i+1)]^{r+j+2}} \right] - \sum_{i=0}^{\infty} \sum_{j=0}^i N \left[\frac{\alpha\Gamma(r+j+1)}{[\theta(i+1)]^{r+j+1}} + \frac{\theta\Gamma(r+j+2)}{[\theta(i+1)]^{r+j+2}} \right]. \quad (3.4)$$

Proof: Let X be a random variable with density function (3.3). The r^{th} ordinary moment of the ($TGQL$) distribution is given by

$$\mu_r(x) = E(X^r) = \int_0^{\infty} x^r f(x) dx \\ = \sum_{i=0}^{\infty} \sum_{j=0}^i \int_0^{\infty} M (\alpha x^{r+j} + \theta x^{r+j+1}) e^{-\theta(i+1)x} dx - \sum_{i=0}^{\infty} \sum_{j=0}^i \int_0^{\infty} N (\alpha x^{r+j} + \theta x^{r+j+1}) e^{-\theta(i+1)x} dx,$$

then

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^i M \left[\frac{\alpha\Gamma(r+j+1)}{[\theta(i+1)]^{r+j+1}} + \frac{\theta\Gamma(r+j+2)}{[\theta(i+1)]^{r+j+2}} \right] - \sum_{i=0}^{\infty} \sum_{j=0}^i N \left[\frac{\alpha\Gamma(r+j+1)}{[\theta(i+1)]^{r+j+1}} + \frac{\theta\Gamma(r+j+2)}{[\theta(i+1)]^{r+j+2}} \right].$$

Which completes the proof.

Based on the first four moments of the $TGQL$ distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the $TGQL$ distribution can obtained as

$$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^{\frac{3}{2}}}, \quad (3.5)$$

and

$$k(\Phi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^2}. \quad (3.6)$$

3.4. Moment generating function

In this subsection we derived the moment generating function of $TGQL$ distribution.

Theorem (3.2): If X has $TGQL$ distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = \sum_{i=0}^{\infty} \sum_{j=0}^i M \frac{\alpha\Gamma(j+1)}{[\theta(i+1)-t]^{j+1}} + \frac{\theta\Gamma(r+j+2)}{[\theta(j+1)-t]^{j+2}} - \sum_{i=0}^{\infty} \sum_{j=0}^i N \left[\frac{\alpha\Gamma(j+1)}{[\theta(i+1)-t]^{j+1}} + \frac{\theta\Gamma(r+j+2)}{[\theta(j+1)-t]^{j+2}} \right]. \quad (3.7)$$

Proof: We start with the well known definition of the moment generating function given by

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f_{TGQL}(x) dx \\ = \sum_{i=0}^{\infty} \sum_{j=0}^i \int_0^{\infty} M (\alpha x^j + \theta x^{j+1}) e^{-[t-\theta(i+1)]x} dx - \sum_{i=0}^{\infty} \sum_{j=0}^i \int_0^{\infty} N (\alpha x^j + \theta x^{j+1}) e^{-[t-\theta(i+1)]x} dx,$$

then,

$$M_X(t) = \sum_{i=0}^{\infty} \sum_{j=0}^i M \frac{\alpha\Gamma(j+1)}{[\theta(i+1)-t]^{j+1}} + \frac{\theta\Gamma(r+j+2)}{[\theta(j+1)-t]^{j+2}} - \sum_{i=0}^{\infty} \sum_{j=0}^i N \left[\frac{\alpha\Gamma(j+1)}{[\theta(i+1)-t]^{j+1}} + \frac{\theta\Gamma(r+j+2)}{[\theta(j+1)-t]^{j+2}} \right].$$

Which completes the proof.

IV. DISTRIBUTION OF THE ORDER STATISTICS

In this section, we derive closed form expressions for the pdfs of the r^{th} order statistic of the $TGQL$ distribution, also,

the measures of skewness and kurtosis of the distribution of the r^{th} order statistic in a sample of size n for different choices of $n; r$ are presented in this section. Let X_1, X_2, \dots, X_n be a simple random sample from $TGQL$ distribution with pdf and cdf given by (2.1) and (2.2), respectively.

Let X_1, X_2, \dots, X_n denote the order statistics obtained from this sample. We now give the probability density function of $X_{r:n}$, say $f_{r:n}(x; \theta, a, \lambda, \alpha)$ and the moments of $X_{r:n}$, $r=1, 2, \dots, n$. Therefore, the measures of skewness and kurtosis of the distribution of the $X_{r:n}$ are presented. The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x, \theta, a, \lambda, \alpha) = \frac{1}{B(r, n-r+1)} [F(x; \theta, a, \lambda, \alpha)]^{r-1} \times [1-F(x; \theta, a, \lambda, \alpha)]^{n-r} f(x; \theta, a, \lambda, \alpha). \quad (4.1)$$

where $F(x; \theta, a, \lambda, \alpha)$ and $f(x; \theta, a, \lambda, \alpha)$ are the cdf and pdf of the $TGQL$ distribution given by (2.1), (2.2), respectively, and $B(\dots)$ is the beta function, since $0 < F(x; \theta, a, \lambda, \alpha) < 1$, for $x > 0$, by using the binomial series expansion of $[1-F(x, \theta, a, \lambda, \alpha)]^{n-r}$, given by

$$[1-F(x, \theta, a, \lambda, \alpha)]^{n-r} = \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x, \theta, a, \lambda, \alpha)]^j, \quad (4.2)$$

we have

$$f_{r:n}(x, \theta, a, \lambda, \alpha) = \frac{1}{B(r, n-r+1)} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \times [F(x, \theta, a, \lambda, \alpha)]^{r+j-1} f(x, \theta, a, \lambda, \alpha), \quad (4.3)$$

substituting from (2.1) and (2.2) into (4.3), we can express the k^{th} ordinary moment of the r^{th} order statistics $X_{r:n}$ say $E(X_{r:n}^k)$ as a liner combination of the k^{th} moments of the $TGQL$ distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ can be calculated.

V. ESTIMATION AND INFERENCE

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the $TGQL$ distribution from complete samples only. Let X_1, X_2, \dots, X_n be a random sample of size n from $TGQL(x, \theta, a, \lambda, \alpha)$. The likelihood function for the vector of parameters $\Phi = (\theta, a, \lambda, \alpha)$ can be written as

$$L = f(x_i, \Phi) = \prod_{i=1}^n f(x_i, \Phi) = \left(\frac{a\theta}{\alpha+1}\right)^n \prod_{i=1}^n (\alpha + \theta x_i) e^{-\theta \sum_{i=1}^n x_i} \prod_{i=1}^n \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right]^{a-1} \prod_{i=1}^n \left\{ (1+\lambda) - 2\lambda \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right]^a \right\}. \quad (5.1)$$

Taking the log-likelihood function for the vector of parameters $\Phi = (\theta, a, \lambda, \alpha)$ we get

$$\ln L = n \ln a + n \ln \theta - n \ln(\alpha+1) + \sum_{i=1}^n \ln(\alpha + \theta x_i) - \theta \sum_{i=1}^n x_i + (a-1) \sum_{i=1}^n \ln \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right] + \sum_{i=1}^n \ln \left[(1+\lambda) - 2\lambda \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right]^a \right]. \quad (5.2)$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (5.2). The components of the score vector are given by

$$\frac{\partial \ln L}{\partial a} = \frac{n}{a} + \sum_{i=1}^n \ln \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right] - \sum_{i=1}^n \frac{2\lambda \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right]^a \ln \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right]}{(1+\lambda) - 2\lambda \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right]^a}, \quad (5.3)$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^n \frac{1 - 2 \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right]^a}{(1+\lambda) - 2\lambda \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right]^a}. \quad (5.4)$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{-n}{\alpha+1} + \sum_{i=1}^n \frac{1}{\alpha + \theta x_i} + (a-1) \sum_{i=1}^n \frac{e^{-\theta x_i} \frac{\theta x_i}{(\alpha+1)^2}}{\left(1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right)} + \sum_{i=1}^n \frac{2\lambda a \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right]^{a-1} e^{-\theta x_i} \frac{\theta x_i}{(\alpha+1)^2}}{1 + \lambda - 2\lambda \left(1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha+1}\right)\right)^a}, \quad (5.5)$$

and

$$\begin{aligned} \frac{\partial \ln L}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^n \frac{x_i}{\alpha + \theta x_i} - \sum_{i=1}^n x_i \\ &+ (a-1) \sum_{i=1}^n \frac{x_i e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha + 1}\right) - \frac{x_i e^{-\theta x_i}}{\alpha + 1}}{1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha + 1}\right)} \\ &- 2\lambda a \sum_{i=1}^n \frac{\left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha + 1}\right)\right]^{a-1}}{\left(1 + \lambda\right) - 2\lambda \left[1 - e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha + 1}\right)\right]^a} \\ &\times \left[x_i e^{-\theta x_i} \left(1 + \frac{\theta x_i}{\alpha + 1}\right) - \frac{x_i e^{-\theta x_i}}{\alpha + 1} \right]. \end{aligned} \tag{5.6}$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters. Also, all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix} \hat{\theta} \\ \hat{a} \\ \hat{\lambda} \\ \hat{\alpha} \end{pmatrix} \approx N \left[\begin{matrix} \hat{\theta} \\ \hat{a} \\ \hat{\lambda} \\ \hat{\alpha} \end{matrix} \begin{bmatrix} V_{\theta\theta} & V_{\theta a} & V_{\theta\lambda} & V_{\theta\alpha} \\ V_{a\theta} & V_{aa} & V_{a\lambda} & V_{a\alpha} \\ V_{\lambda\theta} & V_{\lambda a} & V_{\lambda\lambda} & V_{\lambda\alpha} \\ V_{\alpha\theta} & V_{\alpha a} & V_{\alpha\lambda} & V_{\alpha\alpha} \end{bmatrix} \right] \tag{5.7}$$

$$V^{-1} = -E \begin{bmatrix} V_{\theta\theta} & V_{\theta a} & V_{\theta\lambda} & V_{\theta\alpha} \\ V_{a\theta} & V_{aa} & V_{a\lambda} & V_{a\alpha} \\ V_{\lambda\theta} & V_{\lambda a} & V_{\lambda\lambda} & V_{\lambda\alpha} \\ V_{\alpha\theta} & V_{\alpha a} & V_{\alpha\lambda} & V_{\alpha\alpha} \end{bmatrix}$$

Where

$$\begin{aligned} V_{aa} &= \frac{\partial^2 L}{\partial a^2}, V_{\lambda\lambda} = \frac{\partial^2 L}{\partial \lambda^2}, V_{\theta\theta} = \frac{\partial^2 L}{\partial \theta^2}, V_{\theta\theta} = \frac{\partial^2 L}{\partial \alpha^2}, \\ V_{a\lambda} &= \frac{\partial^2 L}{\partial a \partial \lambda}, V_{a\theta} = \frac{\partial^2 L}{\partial a \partial \theta}, V_{a\alpha} = \frac{\partial^2 L}{\partial a \partial \alpha}, \\ V_{\lambda\theta} &= \frac{\partial^2 L}{\partial \theta \partial \lambda}, V_{\lambda\alpha} = \frac{\partial^2 L}{\partial \lambda \partial \alpha} \end{aligned}$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and covariance's of these ML estimators for $\hat{\theta}, \hat{a}, \hat{\alpha}$ and $\hat{\lambda}$. Using (5.6), we approximate 100(1- γ)% confidence intervals for a, λ, α and θ are determined respectively as

$$\hat{\theta} \pm z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{\theta\theta}}, \hat{a} \pm z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{aa}} \text{ and } \hat{\lambda} \pm z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{\lambda\lambda}}.$$

where z_{γ} is the upper 100 γ the percentile of the standard normal distribution.

VI. APPLICATIONS

In this section, two real data sets are provided to illustrate the usefulness of the TGQL distribution. The TGQL distribution will be fitted to a data sets and compare the results with transmuted Lindley (TL), quasi Lindley (QL) and Lindley (L) distributions.

Data set 1 represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by (Bjerkedal, 1960). The data is as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Data set 2 is obtained from (Hinkley, 1977). It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data is as follows:

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

The estimate of the unknown parameters of each distribution is obtained by the maximum-likelihood method. In order to compare the four distribution models, various criteria are used. Criteria like; $-2\ln L$, Akaike information criterion (AIC), Bayesian information criterion (BIC), the correct Akaike information criterion ($CAIC$), Hannan information criterion ($HQIC$), the Kolmogorov-Smirnov ($K-S$) and p -value statistics are considered for the data set. The formula for these criteria is as follows

$$\begin{aligned} AIC &= 2k - 2\ln L, \quad CAIC = AIC + \frac{2k(k+1)}{n-k-1}, \\ BIC &= k \ln(n) - 2\ln L, \quad HQIC = 2k \ln[\ln(n)] - 2\ln L, \\ k-s &= \sup_y [F_n(y) - F(y)] \end{aligned}$$

where k is the number of parameters in the statistical model, n is the sample size and $\ln L$ is the maximized value of the log-likelihood function under the considered model, $F_n(y)$ is the empirical distribution function and $F(y)$ denotes the cdf for each distribution.

The "best" distribution corresponds to the smallest values of $-2\ln L$, AIC , BIC , $CAIC$, $HQIC$, $K-S$ and the biggest value of p -value criteria.

TABLE 1. Shows the MLEs of the model parameters and its standard error (S.E) for data set 1.

Model	MLEs				S.E			
	\hat{a}	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	S.E (\hat{a})	S.E ($\hat{\theta}$)	S.E ($\hat{\alpha}$)	S.E ($\hat{\lambda}$)
TGQL	3.321	0.823	66.155	0.778	0.76754	0.166	0.579	0.494
TL	-	0.71	-	0.387	-	0.64212	-	0.364
QL	-	0.567	505.816	-	-	0.02093	0.076	-
L	-	0.868	-	-	-	0.13804	-	-

The following table gives the values of mesurments for the data set 1.

TABLE 2. Measurements for all models based on for the data set 1.

Model	-2ℓ	AIC	BIC	CAIC	HQIC	K-S	P-value
TGQL	210.	218.	219.02	217.85	222.05	0.087	0.634
	426	426	3	5	2	85	81
TL	328.	332.	332.28	332.74	334.38	0.247	0.000
	573	573	8	7	6	7	29
QL	387.	391.	391.64	392.10	393.74	0.294	0.000
	928	928	3	2	1	55	01
L	259.	261.	261.05	261.25	262.10	0.246	0.000
	197	197	5	4	4	67	31

Figures 6 and 7 provide the plots of estimated cumulative and estimated densities of the fitted TGQL, TL, QL and L models for the data set 1.

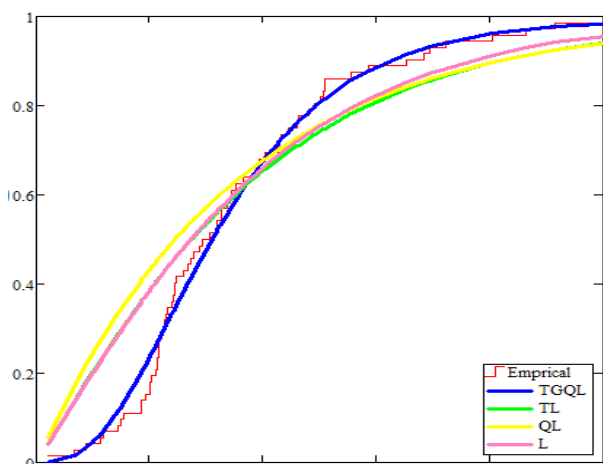


Fig. 6. Estimated cumulative densities of the models for data set 1.

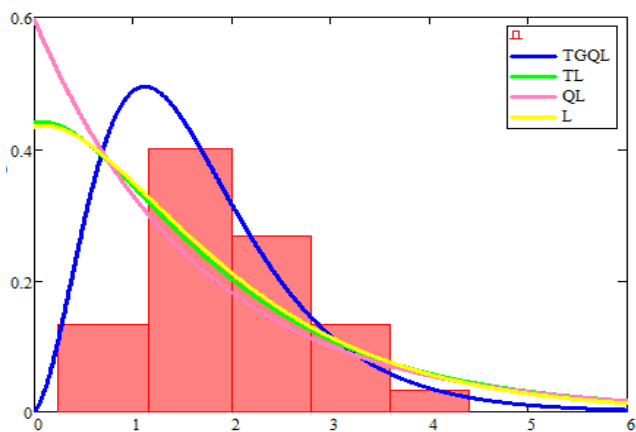


Fig. 7. Estimated densities of the models for data set 1.

TABLE 3. shows the MLEs of the model parameters and its standard error (S.E) (in parentheses) for data set 2.

Model	MLEs				S.E			
	\hat{a}	S.E (\hat{a})	S.E (\hat{a})	S.E (\hat{a})	S.E (\hat{a})	S.E ($\hat{\theta}$)	S.E ($\hat{\alpha}$)	S.E ($\hat{\lambda}$)
TGQL	2.136	1.471	0.355	0.105	2.15157	0.393	1.207	0.859
TL	-	0.738	-	0.402	-	0.4696	-	0.501
QL	-	0.598	649.763	-	-	0.0326	0.114	-
L	-	0.91	-	-	-	0.21385	-	-

The following table gives the values of measurements for the data set 2.

TABLE 4. Measurements for all models based on for the data set 2.

Model	-2ℓ	AIC	BIC	CAIC	HQIC	K-S	P-value
TGQL	76.82	84.82	82.729	86.421	86.61	0.0566	0.999
	1	1	1	1	4	9	98
TL	108.6	112.6	111.61	113.1	113.5	0.1898	0.229
	56	56	6	6	52	3	82
QL	162.2	166.2	165.23	166.72	167.1	0.2351	0.072
	82	82	6	6	78	9	38
L	101.7	103.7	103.23	103.90	104.2	0.1882	0.238
	61	61	8	3	09	3	25

Figures 8 and 9 provide the plots of estimated cumulative and estimated densities of the fitted TGQL, TL, QL and L models for the data set 2.

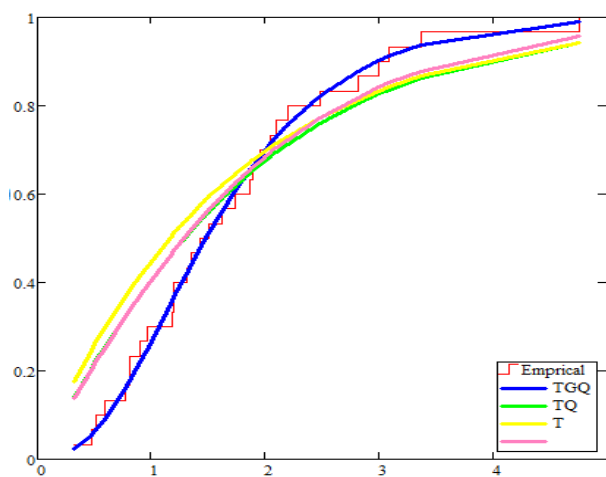


Fig. 8. Estimated cumulative densities of the models for data set 2.

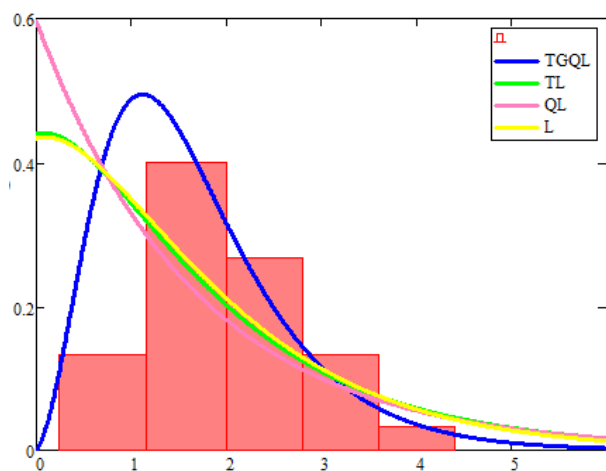


Fig. 9. Estimated densities of the models for data set 2.

The values in tables 2 and 4, indicate that the transmuted generalized quasi Lindley distribution is a strong competitor to other distributions used here for fitting data sets 1 and 2. A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 7 and 9). The fitted density for the transmuted generalized quasi

Lindley model is the closest to the empirical histogram of the other fitted models.

VII. CONCLUSIONS

In this article we introduced a new four-parameter model called transmuted generalized quasi Lindley distribution. Some of its structural properties including an expansion for the density function and explicit expression for moments, moment generating function, quantile function, order statistics and reliability are derived. Maximum likelihood method of estimation is implanted for estimating the model parameters. The observed Fisher information matrix is determined. Moreover, the new model (TGQL) provides a consistently better fit than other models namely, TL, QL and L in a real data set.

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